数理解析研究所講究録
第 1970 巻 2015 年 45-63

Topological Current in Fractional Chern Insulators

Tohru Koma

Department of Physics, Gakushuin University, Mejiro, Toshima-ku, Tokyo 171-8588, JAPAN

e-mail: tohru.koma@gakushuin.ac.jp

Abstract

We announce two theorems on fractional Chern insulators and sketch the proofs.[1] We consider interacting fermions in a magnetic field on a two-dimensional lattice with the periodic boundary conditions. In order to measure the Hall current, we apply an electric potential with a compact support. Then, due to the Lorentz force, the Hall current appears along the equipotential line. Introducing a local current operator at the edge of the potential, we derive the Hall conductance as a linear response coefficient. For a wide class of the models, we prove that if there exists a spectral gap above the degenerate ground state, then the Hall conductance of the ground state is fractionally quantized without averaging over the fluxes. This is an extension of the topological argument for the integrally quantized Hall conductance in noninteracting fermion systems on lattices.

1 Introduction

The quantum Hall effect [12, 11] is one of the most astonishing phenomena in condensed matter physics. In fact, the quantization of the Hall conductance is surprisingly insensitive to disorder and interactions [15]. This robustness of the quantization reflects the topological nature of the Hall conductance formula. Namely, the topological structure never changes for continuously deforming disorder or interactions.

The first discovery was that Thouless, Kohmoto, Nightingale and den Nijs [24] found that the Hall conductance is quantized to a nontrivial integer in a two-dimensional noninteracting electron system in a periodic potential and a magnetic field. Later, Kohmoto [13] realized that the integral quantization of the Hall conductance is due to the topological nature of the Hall conductance formula. Namely, the integer is nothing but the Chern number which is the winding number for the quantum mechanical $U(1)$ phases of the wavefunctions on the magnetic Brillouin zone which can be identified with a two-dimensional torus. More precisely, the $U(1)$ phases are nontrivially twisted on the torus, and the winding number is given by the number of times when the phases rotate on the $U(1)$ circle, and must be an integer.

Although realistic systems have disorder and interactions, they treated translationally invariant noninteracting systems only. Instead of resorting to the topological argument in usual differential geometry, the method of noncommutative geometry [4] was found to be useful to show the integral quantization of the Hall conductance for noninteracting systems which do not necessarily require translation invariance [3]. (See also [2, 1, 5, 17].) However, its extension to interacting systems still remains an open problem.

[1] The full detail is given by [18].

Another tricky approach which we will focus on the present paper was introduced by Niu, Thouless and Wu [23]. They considered a generic quantum Hall system which is allowed to have disorder and interactions. They imposed the twisted boundary conditions with angles ϕ_1 and ϕ_2 at the two boundaries in a two dimensional system, and assumed that the system exhibits a nonvanishing uniform spectral gap above a q-hold degenerate ground state. Under the assumptions, they showed that the Hall conductance averaged over the two angles is fractionally quantized. (See also [16].)

Clearly, their method is artificial, and their result does not implies that the Hall conductance for fixed twisted angles is fractionally quantized. But one can expect that the twisted boundary conditions do not affect the quantization of the Hall conductance. Quite recently, Hastings and Michalakis [8] treated an interacting lattice electron system which does not necessarily require translation invariance, and showed that the Hall conductance is quantized to an integer irrespective of the twisted angles under the assumptions that the system shows a nonvanishing spectral gap above the unique ground state.

In the present paper, we consider interacting fermions in a magnetic field on a two-dimensional lattice with the periodic boundary conditions. As is well known, in generic lattice systems, a constant electric field is known to be useless in order to measure the conductance because the absolutely continuous spectrum of the unperturbed Hamiltonian often changes to a pure point spectrum. Therefore, one cannot expect that there appears an electric flow due to the constant electric field.

In the present paper, we overcome the difficulty as follows: In order to measure the Hall current, we apply an electric potential with a compact support, instead of the linear electric potential or the time dependent vector potential which yield the constant electric field. Then, due to the Lorentz force, the Hall current appears along the equipotential line. Introducing a local current operator at the edge of the potential, we derive the Hall conductance as a linear response coefficient. For a wide class of the models, we prove that if there exists a spectral gap above the degenerate ground state, then the Hall conductance of the ground state is fractionally quantized without averaging over the fluxes. This is an extension of the topological argument for the integrally quantized Hall conductance in noninteracting fermion systems on lattices.

The present paper is organized as follows. In Sec. 2, we describe the model, and state our main Theorem 1. In Sec. 3, we define current operators. Using these expressions, we derive the Hall conductance formula as a linear response coefficient in Sec. 4. Following the idea by Niu, Thouless and Wu, we twist the phases of the hopping amplitudes of the model, and we present Theorem 2 below about the stability of the spectral gap above the degenerate ground state against twisting the boundary conditions in Sec. 5. In Sec. 6, the Hall conductance averaged over the phases is shown to be quantized to a fraction by using the standard topological argument. In Sec. 7, we consider deformation of the current operators, and prove that the statement of our main Theorem 1 is still valid for the deformed current operators so obtained. In other words, the fractional quantization of the Hall conductance is robust against such deformation of the current operators. In Sec. 8, we sketch the proof of the main Theorem 1.

2 Lattice Fermions in Two Dimensions

Let us describe the model which we will treat in the present paper. Consider a rectangular box, $\Lambda := [-L^{(1)}/2, L^{(1)}/2] \times [-L^{(2)}/2, L^{(2)}/2]$, which is a finite subset of the two-dimensional square lattice $\mathbb{Z}^2$. Here both of $L^{(1)}$ and $L^{(2)}$ are taken to be a positive even integer. We consider interacting fermions on the lattice Λ with the periodic boundary conditions. The Hamiltonian is given by

$$H_0^{(\Lambda)} = \sum_{x,y\in\Lambda} t_{x,y} c_x^\dagger c_y + \sum_{I\ge 1} \sum_{x_1,x_2,\ldots,x_I\in\Lambda} U_{x_1,x_2,\ldots,x_I} n_{x_1} n_{x_2} \cdots n_{x_I}, \tag{2.1}$$

where $c_x^\dagger, c_x$ are, respectively, the creation and annihilation fermion operators at the site $x \in \Lambda$, the hopping amplitudes $t_{x,y}$ are complex numbers which satisfy the Hermitian conditions, $t_{y,x} = t_{x,y}^*$, and the coupling constants $U_{x_1,x_2,\ldots,x_I}$ of the interactions are real; $n_x = c_x^\dagger c_x$ is the number operator of the fermion at the site $x \in \Lambda$. We assume that both of the hopping amplitudes and the the interactions are of finite range in the sense of the graph theoretic distance, and that the strengths are uniformly bounded with respect to the sites as $|t_{x,y}| \le t^{(0)}$, and $|U_{x_1,x_2,\ldots,x_I}| \le U^{(0)}$ with some finite constants, $t^{(0)}$ and $U^{(0)}$. Although our method can be extended to more general models, e.g. in higher dimensions and with internal degrees of freedom such as spin, we will treat the above model for simplicity. The essential properties which we require for applying our method to a model are charge conservation and a spectral gap above the ground state.

Clearly, the Hamiltonian $H_0^{(\Lambda)}$ of (2.1) commutes with the total number operator $\sum_{x\in\Lambda} n_x$ of the fermions for a finite volume $|\Lambda| < \infty$. We denote by $H_0^{(\Lambda,N)}$ the restriction of $H_0^{(\Lambda)}$ onto the eigenspace of the total number operator with the eigenvalue N.

We require the existence of a "uniform gap" above the sector of the ground state of the Hamiltonian $H_0^{(\Lambda,N)}$. Since we will take the infinite-volume limit $\Lambda \to \mathbb{Z}^2$, we keep the filling factor $\nu = N/|\Lambda|$ to be a nonzero value in the limit. The precise definition of the "uniform gap" is:

Definition 1. We say that there is a uniform gap above the sector of the ground state if the spectrum $\sigma(H_0^{(\Lambda,N)})$ of the Hamiltonian $H_0^{(\Lambda,N)}$ satisfies the following conditions: The ground state of the Hamiltonian $H_0^{(\Lambda,N)}$ is q-fold (quasi)degenerate in the sense that there are q eigenvalues, $E_{0,1}^{(N)}, \ldots, E_{0,q}^{(N)}$, in the sector of the ground state at the bottom of the spectrum of $H_0^{(\Lambda,N)}$ such that

$$\delta E := \max_{m,m'} \{ |E_{0,m}^{(N)} - E_{0,m'}^{(N)}| \} \to 0 \quad \text{as} \quad |\Lambda| \to \infty.$$

Further the distance between the spectrum, $\{E_{0,1}^{(N)}, \ldots, E_{0,q}^{(N)}\}$, of the ground state and the rest of the spectrum is larger than a positive constant ΔE which is independent of the volume $|\Lambda|$. Namely there is a spectral gap ΔE above the sector of the ground state.

We prove:

Theorem 1. *We assume that the ground state of the Hamiltonian $H_0^{(\Lambda,N)}$ is q-fold degenerate or quasi-degenerate, and there is a uniform gap above the sector of the ground state. Further, we assume that, even when changing the boundary conditions, there is no other infinite-volume ground state except the infinite-volume ground states which are derived from the q ground-state vectors of the Hamiltonian $H_0^{(\Lambda,N)}$. Then, the Hall conductance σ_{12} is fractionally quantized as*

$$\sigma_{12} = \frac{1}{2\pi}\frac{p}{q} \tag{2.2}$$

with some integer p in the infinite-volume limit.

The precise definition of the Hall conductance σ_{12} is given by (4.11) in Sec. 4 below.

Remark . (i) We can relax the assumption on the number of the ground states so that when twisting the phases of the hopping amplitudes at the boundaries, there appears no other infinite-volume ground state at energies lower than or close to the energies of the q-fold ground state, except the infinite-volume ground states which are derived from the q ground-state vectors of the Hamiltonian $H_0^{(\Lambda,N)}$.
(ii) The possibility of the spectral gap above the q-fold degenerate ground state in quantum Hall systems was treated by [14] in a mathematical manner. The numerical evidences of the existence of nontrivial fractional Chern insulators were shown in [22].

3 Local Current Operators

In order to measure the Hall current, we must define current operators by relying on the Ehrenfest's theorem in quantum mechanics.

3.1 Current operators for a single particle

Consider the time evolution of a wavepacket of the single particle on the infinite-volume lattice $\mathbb{Z}^2$. The single-body Hamiltonian $\mathcal{H}_0$ corresponding to the present system is given by

$$(\mathcal{H}_0\psi)(x) = \sum_{y\in\mathbb{Z}^2} t_{x,y}\psi(y) + U_x\psi(x)$$

for $\psi \in \ell^2(\mathbb{Z}^2)$. The Schrödinger equation which determines the time evolution of the wavepacket is given by

$$i\frac{d}{dt}\psi_t = \mathcal{H}_0\psi_t$$

for the wavefunction ψ_t at the time t. The expectation value of an observable a is $\langle a\rangle_t := \langle\psi_t, a\psi_t\rangle/\langle\psi_t,\psi_t\rangle$, where the inner product is defined by $\langle\varphi,\psi\rangle := \sum_{x\in\mathbb{Z}^2}\varphi(x)^*\psi(x)$ for $\varphi,\psi\in\ell^2(\mathbb{Z}^2)$.

The current due to the motion of the particle is determined by the velocity of the center of mass. The center of mass is nothing but the expectation value of the position operator. The position operator $X = (X^{(1)}, X^{(2)})$ is defined by $(X^{(j)}\psi)(x) = x^{(j)}\psi(x)$,

$j = 1, 2$, where we have written $x = (x^{(1)}, x^{(2)}) \in \mathbb{Z}^2$. The Ehrenfest's theorem for the position operator is expressed as

$$\frac{d}{dt}\langle X^{(j)}\rangle_t = \langle i[\mathcal{H}_0, X^{(j)}]\rangle_t,$$

where $[A, B]$ is the commutator of two observables, A and B. Therefore, the natural definition of the current operator must be $\mathcal{J}^{(j)} = i[\mathcal{H}_0, X^{(j)}]$. Clearly, this is an infinite sum. We want to decompose the current operator into the sum of local current operators which are much more useful us to obtain the Hall conductance formula below.

In order to define local current operators, we introduce an approximate position operator as

$$(X_\ell^{(j)}\psi)(x) = f_\ell^{(j)}(x)\psi(x),$$

where the function $f_\ell^{(j)}$ on $\mathbb{Z}^2$ is given by

$$f_\ell^{(j)}(x) := \sum_{k=1}^{2\ell} \theta^{(j)}(x; k - \ell) - \ell$$

with the step function,

$$\theta^{(j)}(x; k) := \begin{cases} 1, & x^{(j)} \geq k; \\ 0, & x^{(j)} < k. \end{cases}$$

Clearly, we have $\|(X^{(j)} - X_\ell^{(j)})\psi\| \to 0$ as $\ell \to \infty$ for $(X^{(j)}\psi) \in \ell^2(\mathbb{Z}^2)$. By replacing $X^{(j)}$ with $X_\ell^{(j)}$ in the expression of the current operator $\mathcal{J}^{(j)}$, we obtain the approximate current operator as

$$\mathcal{J}_\ell^{(j)} = i[\mathcal{H}_0, X_\ell^{(j)}] = \sum_{k=1}^{2\ell} i[\mathcal{H}_0, \theta^{(j)}(\cdots; k - \ell)].$$

The summand in the right-hand side is interpreted as the local current operator. Namely, the local current operator $J^{(j)}(k)$ across the k-th site in the j direction is given by the commutator of the Hamiltonian $\mathcal{H}_0$ and the step function $\theta^{(j)}(\cdots; k)$ as $J^{(j)}(k) = i[\mathcal{H}_0, \theta^{(j)}(\cdots; k)]$.

3.2 Current operators for many fermions

The step function is written

$$\theta^{(j)}(k) = \sum_{x \in \mathbb{Z}^2} \theta^{(j)}(x; k)\, n_x \tag{3.1}$$

in terms of the number operator n_x of the fermions. Therefore, the local current operator $J^{(j)}(k)$ is formally written

$$J^{(j)}(k) = i[H_0^{(\mathbb{Z}^2)}, \theta^{(j)}(k)] = i \sum_{\substack{x \in \mathbb{Z}^2 \\ x^{(j)} < k}} \sum_{\substack{y \in \mathbb{Z}^2 \\ y^{(j)} \geq k}} (t_{xy} c_x^\dagger c_y - t_{yx} c_y^\dagger c_x), \tag{3.2}$$

where we have used the expression (2.1) of the Hamiltonian, and we have written $x = (x^{(1)}, x^{(2)})$. Although this is clearly an infinite sum, the expression can be justified for a localized wavefunction of many fermions. For the present finite-volume lattice Λ, we can define the local current operator $J^{(j)}(k)$ as

$$J^{(j)}(k) := i \sum_{\substack{x\in\Lambda \\ x^{(j)}<k}} \sum_{\substack{y\in\Lambda \\ y^{(j)}\ge k}} (t_{xy}c_x^\dagger c_y - t_{yx}c_y^\dagger c_x)$$

with the periodic boundary conditions because the range of the hopping amplitudes is finite. In order to measure the Hall current, we introduce an approximate local current operator as

$$J_{\mathcal{N}}^{(1)}(k,\ell) := i \sum_{x^{(1)}<k} \sum_{\ell-\mathcal{N}\le x^{(2)}\le \ell+\mathcal{N}} \sum_{y^{(1)}\ge k} \sum_{\ell-\mathcal{N}\le y^{(2)}\le \ell+\mathcal{N}} (t_{xy}c_x^\dagger c_y - t_{yx}c_y^\dagger c_x), \tag{3.3}$$

where $\mathcal{N}$ is a positive integer and ℓ is an integer. Namely, $\mathcal{N}$ is the cutoff in the second direction.

4 Linear Response

In order to measure the Hall current, we apply an electric potential to the present system. For this purpose, we introduce a $2\mathcal{M} \times 2\mathcal{M}$ square box as

$$\Gamma_{\mathcal{M}}(k,\ell) := \{x = (x^{(1)}, x^{(2)}) \mid k - \mathcal{M} \le x^{(1)} \le k + \mathcal{M}, \ell \le x^{(2)} \le \ell + 2\mathcal{M}\}$$

for a positive integer $\mathcal{M}$, and the sum of the number operators n_x on the box as

$$\chi(\Gamma_{\mathcal{M}}(k,\ell)) := \sum_{x\in\Gamma_{\mathcal{M}}(k,\ell)} n_x. \tag{4.1}$$

The latter yields the unit voltage difference from the outside of the region. The current operator segment $J_{\mathcal{N}}^{(1)}(k,\ell)$ of (3.3) goes across the bottom side of the square box $\Gamma_{\mathcal{M}}(k,\ell)$. When we change the potential energy inside the box by using the potential operator $\chi(\Gamma_{\mathcal{M}}(k,\ell))$, the Hall current is expected to appear along the boundary of the box $\Gamma_{\mathcal{M}}(k,\ell)$ due to the Lorentz force. The Hall current is measured by the observable $J_{\mathcal{N}}^{(1)}(k,\ell)$. More precisely, we use a time-dependent electric potential. The Hamiltonian having the electric potential on the region $\Gamma_{\mathcal{M}}(k,\ell)$ is given by

$$H^{(\Lambda)}(t) := H_0^{(\Lambda)} + \lambda W(t) \tag{4.2}$$

with the perturbed Hamiltonian, $W(t) = e^{\eta t}\chi(\Gamma_{\mathcal{M}}(k,\ell))$. Here the voltage difference λ is a real parameter, and the adiabatic parameter η is a small positive number. We switch on the electric potential at the initial time $t = -T$ with a large positive T, and measure the Hall current at the final time $t = 0$.

The time-dependent Schrödinger equation is given by

$$i\frac{d}{dt}\Psi^{(N)}(t) = H^{(\Lambda)}(t)\Psi^{(N)}(t)$$

for the wavefunction $\Psi^{(N)}(t)$ for the N fermions. We denote the time evolution operator for the unperturbed Hamiltonian $H_0^{(\Lambda)}$ by

$$U_0^{(\Lambda)}(t,s) := \exp\left[-i(t-s)H_0^{(\Lambda)}\right] \quad \text{for } t,s \in \mathbb{R}. \tag{4.3}$$

We choose the initial vector $\Psi^{(N)}(-T)$ as $\Psi^{(N)}(-T) = U_0^{(\Lambda)}(-T,0)\Phi^{(N)}$ with a vector $\Phi^{(N)}$. Then, the final vector $\Psi^{(N)} = \Psi^{(N)}(t=0)$ is obtained as

$$\Psi^{(N)} = \Phi^{(N)} - i\lambda \int_{-T}^{0} ds\, U_0^{(\Lambda)}(0,s)W(s)U_0^{(\Lambda)}(s,0)\Phi^{(N)} + o(\lambda) \tag{4.4}$$

by using a perturbation theory [16], where $o(\lambda)$ denotes a vector $\Psi_R^{(N)}$ with the norm $\|\Psi_R^{(N)}\|$ satisfying $\|\Psi_R^{(N)}\|/\lambda \to 0$ as $\lambda \to 0$.

We denote the q ground-state vectors of the unperturbed Hamiltonian $H_0^{(\Lambda,N)}$ by $\Phi_{0,m}^{(N)}$ with the energy eigenvalue $E_{0,m}^{(N)}$, $m = 1,2,\ldots,q$. We choose the initial vector $\Phi^{(N)} = \Phi_{0,m}^{(N)}$ with the norm 1, and write the corresponding final vector at $t=0$ as $\Psi^{(N)} = \Psi_{0,m}^{(N)}$. Then, the ground-state expectation value of the approximate local current operator $J_{\mathcal{N}}^{(1)}(k,\ell)$ is given by

$$\left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle := \frac{1}{q}\sum_{m=1}^{q}\left\langle \Psi_{0,m}^{(N)}, J_{\mathcal{N}}^{(1)}(k,\ell)\Psi_{0,m}^{(N)}\right\rangle. \tag{4.5}$$

Let $P_0^{(\Lambda,N)}$ be the projection onto the sector of the ground state of $H_0^{(\Lambda,N)}$, and we denote the ground-state expectation by

$$\omega_0^{(\Lambda,N)}(\cdots) := \frac{1}{q}\,\mathrm{Tr}\,(\cdots)P_0^{(\Lambda,N)},$$

where q is the degeneracy of the ground state. Using the linear perturbation (4.4), the expectation value of (4.5) is decomposed into three parts as

$$\left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle = \left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle_0 + \left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle_1 + o(\lambda),$$

where $\left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle_0 = \omega_0^{(\Lambda,N)}\left(J_{\mathcal{N}}^{(1)}(k,\ell)\right)$, and

$$\left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle_1 = -i\lambda\int_{-T}^{0} ds\, \omega_0^{(\Lambda,N)}\left(J_{\mathcal{N}}^{(1)}(k,\ell)U_0^{(\Lambda)}(0,s)W(s)U_0^{(\Lambda)}(s,0)\right) + \text{c.c.} \tag{4.6}$$

The first term $\left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle_0$ is the persistent current which is usually vanishing. We are interested in the second term $\left\langle J_{\mathcal{N}}^{(1)}(k,\ell)\right\rangle_1$ which gives the linear response coefficient, i.e., the Hall conductance. We write

$$\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s) := U_0^{(\Lambda)}(0,s)\chi(\Gamma_{\mathcal{M}}(k,\ell))U_0^{(\Lambda)}(s,0). \tag{4.7}$$

Using the expression of the perturbed Hamiltonian $W(t)$, the contribution (4.6) of the expectation value of the local current is written

$$(4.8)\qquad \langle J_{\mathcal{N}}^{(1)}(k,\ell)\rangle_1 = i\lambda\int_{-T}^{0} ds\, e^{\eta s}\omega_0^{(\Lambda,N)}\left([\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s), J_{\mathcal{N}}^{(1)}(k,\ell)]\right).$$

Note that, by using integral by parts, we have

$$(4.9)\qquad \int_{-T}^{0} ds\, e^{\eta s}\omega_0^{(\Lambda,N)}\left([\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s), J_{\mathcal{N}}^{(1)}(k,\ell)]\right)$$
$$= \left[se^{\eta s}\omega_0^{(\Lambda,N)}\left([\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s), J_{\mathcal{N}}^{(1)}(k,\ell)]\right)\right]_{-T}^{0}$$
$$- \int_{-T}^{0} ds\, \eta s e^{\eta s}\omega_0^{(\Lambda,N)}\left([\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s), J_{\mathcal{N}}^{(1)}(k,\ell)]\right)$$
$$- \int_{-T}^{0} ds\, s e^{\eta s}\omega_0^{(\Lambda,N)}\left(\left[\frac{d}{ds}\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s), J_{\mathcal{N}}^{(1)}(k,\ell)\right]\right).$$

Clearly, the first term in the right-hand side is vanishing as $T \to \infty$. The second term is also vanishing as $\eta \to 0$ after taking the limit $T \to \infty$. The third term in the right-hand side is written

$$\int_{-T}^{0} ds\, s e^{\eta s}\omega_0^{(\Lambda,N)}\left(\left[J_{\mathcal{N}}^{(1)}(k,\ell), J^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s)\right]\right),$$

where

$$J^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s) := \frac{d}{ds}\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s) = U_0^{(\Lambda)}(0,s)i[H_0^{(\Lambda)}, \chi(\Gamma_{\mathcal{M}}(k,\ell))]U_0^{(\Lambda)}(s,0).$$

For getting the second equality in the right-hand side, we have used the definition (4.3) of the time evolution operator $U_0^{(\Lambda)}(t,s)$ and the definition (4.7) of $\chi^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s)$. Clearly, the above operator $J^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s)$ is interpreted as the current across the boundary of the region $\Gamma_{\mathcal{M}}(k,\ell)$ at the time s. From these observations, we define the Hall conductance for the finite lattice Λ as

$$(4.10)\qquad \tilde{\sigma}_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M}) := i\int_{-T}^{0} ds\, s e^{\eta s}\omega_0^{(\Lambda,N)}\left(\left[J_{\mathcal{N}}^{(1)}(k,\ell), J^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s)\right]\right)$$

because λ is the voltage difference. The Hall conductance in the infinite-volume limit is given by

$$(4.11)\qquad \sigma_{12} := \lim_{\mathcal{N}\to\infty}\lim_{\mathcal{M}\to\infty}\lim_{\eta\to 0}\lim_{T\to\infty}\lim_{\Lambda\nearrow\mathbb{Z}^2}\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M}).$$

5 Twisting the Phases

Consider $H_0^{(\mathbb{Z}^2)}(\phi_j,k) := \exp[-i\phi_j\theta^{(j)}(k)]H_0^{(\mathbb{Z}^2)}\exp[i\phi_j\theta^{(j)}(k)]$, for $\phi_j \in \mathbb{R}$, where $\theta^{(j)}(k)$ is the step function of (3.1). This transformation changes the hopping amplitudes as

$t_{xy} \to t_{xy}e^{i\phi_j}$ for $x^{(j)} < k$, $y^{(j)} \ge k$, and the opposite hopping amplitude t_{yx} is determined by the Hermitian condition $t_{yx} = t^*_{xy}$. The rest of the hopping amplitudes do not change. In both of the first and the second directions, we define the twist by

$$\begin{aligned} &H_0^{(\mathbb{Z}^2)}(\phi_1, k_1; \phi_2, k_2) \\ &\qquad := \exp\left[-i\left(\phi_1\theta^{(1)}(k_1) + \phi_2\theta^{(2)}(k_2)\right)\right] H_0^{(\mathbb{Z}^2)} \exp\left[i\left(\phi_1\theta^{(1)}(k_1) + \phi_2\theta^{(2)}(k_2)\right)\right]. \end{aligned}$$

This transformation is justified for localized wavefunctions.

Relying on this rule, we can twist the phases of the hopping amplitudes of the finite-volume Hamiltonian $H_0^{(\Lambda)}$. We denote by $H_0^{(\Lambda)}(\phi_1, k_1; \phi_2, k_2)$ the corresponding finite-volume Hamiltonian on Λ. When both of k_1 and k_2 are placed at the boundaries of the lattice Λ, the boundary conditions of the system become the usual twisted boundary conditions. We write $H_0^{(\Lambda)}(\phi_1; \phi_2)$ for the Hamiltonian with the twisted boundary conditions for short. Further, one has

$$H_0^{(\mathbb{Z}^2)}(\phi_j, k-1) = \exp\left[-i\phi_j \sum_{\substack{x\in\mathbb{Z}^2 \\ x^{(j)}=k-1}} n_x\right] H_0^{(\mathbb{Z}^2)}(\phi_j, k) \exp\left[i\phi_j \sum_{\substack{x\in\mathbb{Z}^2 \\ x^{(j)}=k-1}} n_x\right].$$

Thus, one can change the position of the twisted hopping amplitudes by using the unitary transformation which is local in the j-th direction.

Similarly, we can consider

$$\begin{aligned} &J^{(j)}(k; \phi_1, k_1; \phi_2, k_2) \\ &\qquad := \exp\left[-i\left(\phi_1\theta^{(1)}(k_1) + \phi_2\theta^{(2)}(k_2)\right)\right] J^{(j)}(k) \exp\left[i\left(\phi_1\theta^{(1)}(k_1) + \phi_2\theta^{(2)}(k_2)\right)\right] \end{aligned}$$

for the local current operator $J^{(j)}(k)$. Therefore, in the same way as in the case of the Hamiltonian, one can change the position of the twisted phases as

$$\begin{aligned} &J^{(j)}(k; \phi_1, k_1 - 1; \phi_2, k_2) \\ &\qquad = \exp\left[-i\phi_1 \sum_{\substack{x\in\mathbb{Z}^2 \\ x^{(1)}=k_1-1}} n_x\right] J^{(j)}(k; \phi_1, k_1; \phi_2, k_2) \exp\left[i\phi_1 \sum_{\substack{x\in\mathbb{Z}^2 \\ x^{(1)}=k_1-1}} n_x\right]. \end{aligned}$$

Further, one has

$$\begin{aligned} \frac{\partial}{\partial\phi_j} H_0^{(\mathbb{Z}^2)}(\phi_j, k) &= \exp[-i\phi_j\theta^{(j)}(k)] i[H_0^{(\mathbb{Z}^2)}, \theta^{(j)}(k)] \exp[i\phi_j\theta^{(j)}(k)] \\ &= \exp[-i\phi_j\theta^{(j)}(k)] J^{(j)}(k) \exp[i\phi_j\theta^{(j)}(k)]. \end{aligned}$$

Namely, we can obtain the local current by differentiating the Hamiltonian. Thus, one has

$$\frac{\partial}{\partial\phi_j} H_0^{(\Lambda)}(\phi_1, k_1; \phi_2, k_2) = J^{(j)}(k_j; \phi_1, k_1; \phi_2, k_2), \quad j = 1, 2, \tag{5.1}$$

for the finite-volume Hamiltonian $H_0^{(\Lambda)}$.

Theorem 2. *Under the same assumptions as in Theorem 1, we have that the ground state of the Hamiltonian $H_0^{(\Lambda,N)}(\phi_1,k_1;\phi_2,k_2)$ having the twisted phases ϕ_1 and ϕ_2 at the positions k_1 and k_2 in the first and the second directions, respectively, has the same q-fold (quasi)degeneracy as that of the Hamiltonian $H_0^{(\Lambda,N)}$ without the twisted phases. Further, there exists a uniform spectral gap above the sector of the ground state for any ϕ_1 and ϕ_2, and there exists a positive lower bound of the spectral gap such that the bound is independent of the phases ϕ_1,ϕ_2.*

The proof is given in [18].

Remark. As mentioned in Sec. 1, the method of twisting the phases at the boundaries was first introduced by [23] for the quantum Hall systems. Later, the method was extended to strongly correlated quantum systems such as quantum spin systems in [9], in order to measure the local topological properties such as a local singlet pair of two spins.

6 A Topological Invariant

Consider the Hamiltonian $H_0^{(\Lambda,N)}(\phi_1,k;\phi_2,\ell)$. Relying on Theorem 2, we denote the q vectors of the ground state by $\Phi_{0,m}^{(N)}(\phi_1,k;\phi_2,\ell)$ with the energy eigenvalue $E_{0,m}^{(N)}(\phi_1;\phi_2)$, $m=1,2,\ldots,q$, and denote the excited-state vectors $\Phi_n^{(N)}(\phi_1,k;\phi_2,\ell)$ with the energy eigenvalue $E_n^{(N)}(\phi_1;\phi_2)$, $n\ge 1$. We write $\phi=(\phi_1,\phi_2)$, and $\tilde{\phi}=(\phi_1,k;\phi_2,\ell)$ for short. We define

$$(6.1)\quad \hat{\sigma}_{12}^{(\Lambda,N)}(\tilde{\phi}):=\frac{i}{q}\sum_{m=1}^{q}\sum_{n\ge 1}\left[\frac{\langle\Phi_{0,m}^{(N)}(\tilde{\phi}),J^{(1)}(k;\tilde{\phi})\Phi_n^{(N)}(\tilde{\phi})\rangle\langle\Phi_n^{(N)}(\tilde{\phi}),J^{(2)}(\ell;\tilde{\phi})\Phi_{0,m}^{(N)}(\tilde{\phi})\rangle}{(E_{0,m}^{(N)}(\phi)-E_n^{(N)}(\phi))^2}\right.$$
$$\left.-\frac{\langle\Phi_{0,m}^{(N)}(\tilde{\phi}),J^{(2)}(\ell;\tilde{\phi})\Phi_n^{(N)}(\tilde{\phi})\rangle\langle\Phi_n^{(N)}(\tilde{\phi}),J^{(1)}(k;\tilde{\phi})\Phi_{0,m}^{(N)}(\tilde{\phi})\rangle}{(E_{0,m}^{(N)}(\phi)-E_n^{(N)}(\phi))^2}\right].$$

We write $\Phi_{0,m}^{(N)}(\phi)$ for the vectors of the ground state of the Hamiltonian $H_0^{(\Lambda)}(\phi_1;\phi_2)$ with the twisted boundary conditions with the angles ϕ_1 and ϕ_2, and $\Phi_n^{(N)}(\phi)$ for the vectors of the excited states. We define

$$(6.2)\quad \hat{\sigma}_{12}^{(\Lambda,N)}(\phi):=\frac{i}{q}\sum_{m=1}^{q}\sum_{n\ge 1}\left[\frac{\langle\Phi_{0,m}^{(N)}(\phi),J^{(1)}(k)\Phi_n^{(N)}(\phi)\rangle\langle\Phi_n^{(N)}(\phi),J^{(2)}(\ell)\Phi_{0,m}^{(N)}(\phi)\rangle}{(E_{0,m}^{(N)}(\phi)-E_n^{(N)}(\phi))^2}\right.$$
$$\left.-\frac{\langle\Phi_{0,m}^{(N)}(\phi),J^{(2)}(\ell)\Phi_n^{(N)}(\phi)\rangle\langle\Phi_n^{(N)}(\phi),J^{(1)}(k)\Phi_{0,m}^{(N)}(\phi)\rangle}{(E_{0,m}^{(N)}(\phi)-E_n^{(N)}(\phi))^2}\right].$$

This is the standard form of the Hall conductance for the degenerate ground state with the twisted boundary conditions. Since we can change the position of the twisted hopping amplitudes by using the unitary transformation as shown in the preceding section, this conductance $\hat{\sigma}_{12}^{(\Lambda,N)}(\phi)$ is equal to the above $\hat{\sigma}_{12}^{(\Lambda,N)}(\tilde{\phi})$ of (6.1).

Niu, Thouless and Wu [23] showed the following: When averaging the conductance $\hat{\sigma}_{12}^{(\Lambda,N)}(\phi)$ over the phases ϕ_1 and ϕ_2, the averaged Hall conductance is fractionally quantized [16] as in (6.4) below.

Using a contour integral, the projection $P_0^{(\Lambda,N)}(\tilde{\phi})$ onto the sector of the ground state is written

$$P_0^{(\Lambda,N)}(\tilde{\phi}) = \frac{1}{2\pi i}\oint dz \frac{1}{z - H_0^{(\Lambda,N)}(\tilde{\phi})}$$

because the spectral gap exists above the sector of the ground state as we proved in Theorem 2. Note that, for $j = 1, 2$, one has

$$P_{0,j}^{(\Lambda,N)}(\tilde{\phi}) := \frac{\partial}{\partial \phi_j} P_0^{(\Lambda,N)}(\tilde{\phi}) = \frac{1}{2\pi i}\oint dz \frac{1}{z - H_0^{(\Lambda,N)}(\tilde{\phi})} J^{(j)}(k;\tilde{\phi}) \frac{1}{z - H_0^{(\Lambda,N)}(\tilde{\phi})},$$

where we have used (5.1). Using these, we have

$$\hat{\sigma}_{12}^{(\Lambda,N)}(\tilde{\phi}) = \frac{i}{q} \operatorname{Tr} P_0^{(\Lambda,N)}(\tilde{\phi}) \left[P_{0,1}^{(\Lambda,N)}(\tilde{\phi}), P_{0,2}^{(\Lambda,N)}(\tilde{\phi}) \right]. \tag{6.3}$$

Remark. In the infinite-volume limit, $\Lambda \nearrow \mathbb{Z}^2$ and $N \nearrow \infty$, one formally has

$$\hat{\sigma}_{12}^{(\mathbb{Z}^2,\infty)} = -\frac{i}{q} \operatorname{Tr} P_0^{(\mathbb{Z}^2,\infty)} \left[\left[P_0^{(\mathbb{Z}^2,\infty)}, \theta^{(1)}(k_1) \right], \left[P_0^{(\mathbb{Z}^2,\infty)}, \theta^{(2)}(k_2) \right] \right],$$

where we have used $P_{0,j}^{(\mathbb{Z}^2,\infty)} = i\left[P_0^{(\mathbb{Z}^2,\infty)}, \theta^{(j)}(k_j)\right]$ for $j = 1, 2$, which are derived from (3.2). This expression of the Hall conductance $\hat{\sigma}_{12}^{(\mathbb{Z}^2,\infty)}$ has the same form as that for formally applying the method of noncommutative geometry [4, 3, 1] to an interacting fermion system. Actually, the expression can be obtained by replacing the single-fermion step function[2] with that for many fermions. But, justifying the expression in a mathematical rigorous manner remains a challenge because the many-body step function $\theta^{(j)}(k_j)$ is expressed in terms of the infinite sum of the number operators.

The following proposition is essentially due to Kato [10]. See also Proposition 5.1 in [16].

Proposition 3. *There exist orthonormal vectors $\hat{\Phi}_{0,m}^{(N)}(\tilde{\phi})$, $m = 1, 2, \ldots, q$ such that the sector of the ground state of $H_0^{(\Lambda,N)}(\tilde{\phi})$ is spanned by the q vectors $\hat{\Phi}_{0,m}^{(N)}(\tilde{\phi})$, and that all the vectors $\hat{\Phi}_{0,m}^{(N)}(\tilde{\phi})$ are infinitely differentiable with respect to the phase parameters $\phi \in [0, 2\pi] \times [0, 2\pi]$.*

Relying on this proposition, one has [16]

$$\frac{1}{(2\pi)^2} \iint_{[0,2\pi]\times[0,2\pi]} d\phi_1 d\phi_2 \, \hat{\sigma}_{12}^{(\Lambda,N)}(\tilde{\phi}) = \frac{1}{2\pi}\frac{p}{q} \tag{6.4}$$

with an integer p. Since $\hat{\sigma}_{12}^{(\Lambda,N)}(\phi) = \hat{\sigma}_{12}^{(\Lambda,N)}(\tilde{\phi})$ as mentioned above, we obtain

$$\frac{1}{(2\pi)^2} \iint_{[0,2\pi]\times[0,2\pi]} d\phi_1 d\phi_2 \, \hat{\sigma}_{12}^{(\Lambda,N)}(\phi) = \frac{1}{2\pi}\frac{p}{q} \tag{6.5}$$

for the Hall conductance $\hat{\sigma}_{12}^{(\Lambda,N)}(\phi)$ with the simple twisted boundary conditions.

[2]See, e.g., [2, 5, 17].

7 Topological Currents

We write $H_0^{(\Lambda)}(\tilde{\phi})$ for the Hamiltonian $H_0^{(\Lambda)}(\phi_1, k; \phi_2, \ell)$ in Sec. 5. Consider a further deformation of the Hamiltonian $H_0^{(\Lambda)}(\tilde{\phi})$ as $H_0^{(\Lambda)}(\tilde{\phi}, \tilde{\alpha}) := \exp[-i\alpha\phi_1 n_x] H_0^{(\Lambda)}(\tilde{\phi}) \exp[i\alpha\phi_1 n_x]$, where we have written $\tilde{\alpha} = (\alpha, x)$ for short. Then, the relation (5.1) is modified as

$$\frac{\partial}{\partial\phi_1} H_0^{(\Lambda)}(\tilde{\phi}, \tilde{\alpha}) = \exp[-i\alpha\phi_1 n_x] \left\{ J^{(1)}(k; \tilde{\phi}) + i\alpha[H_0^{(\Lambda)}(\tilde{\phi}), n_x] \right\} \exp[i\alpha\phi_1 n_x].$$

We write $J^{(1)}(k; \tilde{\phi}, \tilde{\alpha})$ for this right-hand side. Namely, we have

$$\frac{\partial}{\partial\phi_1} H_0^{(\Lambda)}(\tilde{\phi}, \tilde{\alpha}) = J^{(1)}(k; \tilde{\phi}, \tilde{\alpha}). \tag{7.1}$$

Clearly, this operator $J^{(1)}(k; \tilde{\phi}, \tilde{\alpha})$ is the local current operator which is obtained by twisting the phases of

$$J^{(1)}(k, \tilde{\alpha}) := J^{(1)}(k) + i\alpha[H_0^{(\Lambda)}, n_x] \tag{7.2}$$

with the angles ϕ_1 and ϕ_2. In the infinite volume, the operator is formally written

$$J^{(1)}(k, \tilde{\alpha}) = i[H_0^{(\mathbb{Z}^2)}, \theta^{(1)}(k)] + i\alpha[H_0^{(\mathbb{Z}^2)}, n_x] = i[H_0^{(\mathbb{Z}^2)}, \theta^{(1)}(k) + \alpha n_x].$$

That is to say, the step potential is slightly deformed with α at the site x.

In the same way, the projection operator onto the sector of the ground state is transformed as

$$P_0^{(\Lambda,N)}(\tilde{\phi}, \tilde{\alpha}) := \exp[-i\alpha\phi_1 n_x] P_0^{(\Lambda,N)}(\tilde{\phi}) \exp[i\alpha\phi_1 n_x] = \frac{1}{2\pi i} \oint \frac{dz}{z - H_0^{(\Lambda,N)}(\tilde{\phi}, \tilde{\alpha})}.$$

We write

$$P_{0,j}^{(\Lambda,N)}(\tilde{\phi}, \tilde{\alpha}) := \frac{\partial}{\partial\phi_j} P_0^{(\Lambda,N)}(\tilde{\phi}, \tilde{\alpha})$$

for $j = 1, 2$. Then we have:

Lemma 4.

$$\begin{aligned} &\iint_{[0,2\pi]\times[0,2\pi]} d\phi_1 d\phi_2 \ \mathrm{Tr}\ P_0^{(\Lambda,N)}(\tilde{\phi}, \tilde{\alpha}) \left[P_{0,1}^{(\Lambda,N)}(\tilde{\phi}, \tilde{\alpha}), P_{0,2}^{(\Lambda,N)}(\tilde{\phi}, \tilde{\alpha}) \right] \\ &= \iint_{[0,2\pi]\times[0,2\pi]} d\phi_1 d\phi_2 \ \mathrm{Tr}\ P_0^{(\Lambda,N)}(\tilde{\phi}) \left[P_{0,1}^{(\Lambda,N)}(\tilde{\phi}), P_{0,2}^{(\Lambda,N)}(\tilde{\phi}) \right]. \end{aligned}$$

The proof is given in [18]. This lemma implies that the topological invariant, the Chern number, does not change for the local deformation of the phases of the Hamiltonian. But the local current operator in the first direction changes from (3.2) to (7.2). Even when starting from this deformed local current operator $J^{(1)}(k, \tilde{\alpha})$, our argument holds in the same way. In consequence, we can prove the same statement as in Theorem 1 for the deformed local current operator. Since we can repeatedly apply local deformations, we can define more generic local currents in the following way.

Consider generic functions $\vartheta^{(j)}(x;k,\ell)$, $j=1,2$, which satisfy $\vartheta^{(1)}(x;k,\ell)=\theta^{(1)}(x;k)$, $\vartheta^{(2)}(x;k,\ell)=\theta^{(2)}(x,\ell)$ for $\mathrm{dist}(x,(k,\ell))\ge R_0$ with a large positive constant R_0. Namely, the functions $\vartheta^{(j)}(x;k,\ell)$ coincide with the step functions at the large distances. At short distances, the functions $\vartheta^{(j)}(x;k,\ell)$ can take any real values. We define $\vartheta^{(j)}(k,\ell):=\sum_{x\in\mathbb{Z}^2}\vartheta^{(j)}(x;k,\ell)\,n_x$. Then, the twisted Hamiltonian is formally given by

$$H_0^{(\mathbb{Z}^2)}(\phi):=\exp\bigl[-\sum_{j=1,2}i\phi_j\vartheta^{(j)}(k,\ell)\bigr]H_0^{(\mathbb{Z}^2)}\exp\bigl[\sum_{j=1,2}i\phi_j\vartheta^{(j)}(k,\ell)\bigr].$$

The generic local current operators are $\mathfrak{J}^{(j)}(k,\ell):=i[H_0^{(\mathbb{Z}^2)},\vartheta^{(j)}(k,\ell)]$. The potential operator of (4.1) can be also replaced by a generic electric potential such that, except the boundary ribbon region, the potential takes the unit inside the finite large region and zero outside region. Namely, the potential coincides with the characteristic function of the finite region at the large distances from the boundary of the region, while the potential takes any values at the short distances from the boundary of the region.

8 Sketch of the Proof of Theorem 1

8.1 Estimates of the current-current correlations

The conductance (4.10) is written

$$\begin{aligned}
&\tilde\sigma_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M})\\
&=\frac{i}{q}\int_{-T}^{0}ds\;se^{\eta s}\sum_{m=1}\sum_{n\ge1}\Bigl[\langle\Phi_{0,m}^{(N)},J_{\mathcal{N}}^{(1)}(k,\ell)\Phi_n^{(N)}\rangle\langle\Phi_n^{(N)},J(\Gamma_{\mathcal{M}}(k,\ell))\Phi_{0,m}^{(N)}\rangle e^{i(E_n^{(N)}-E_{0,m}^{(N)})s}\\
&-\langle\Phi_{0,m}^{(N)},J(\Gamma_{\mathcal{M}}(k,\ell))\Phi_n^{(N)}\rangle\langle\Phi_n^{(N)},J_{\mathcal{N}}^{(1)}(k,\ell)\Phi_{0,m}^{(N)}\rangle e^{-i(E_n^{(N)}-E_{0,m}^{(N)})s}\Bigr]
\end{aligned}$$

in terms of the ground-state vectors $\Phi_{0,m}^{(N)}$ with the eigenvalue $E_{0,m}^{(N)}$, $m=1,2,\dots,q$, and the excited-state vectors $\Phi_n^{(N)}$ with the eigenvalue $E_n^{(N)}$, $n\ge1$. Here, we have dropped the twisted phase dependence for short. Consider the limit

$$\tilde\sigma_{12}^{(\Lambda,N)}(0,\infty,\mathcal{N},\mathcal{M}):=\lim_{\eta\to0}\lim_{T\to\infty}\tilde\sigma_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M}).$$

Integrating with respect to s, one has

$$\begin{aligned}
\tilde\sigma_{12}^{(\Lambda,N)}(0,\infty,\mathcal{N},\mathcal{M})=\frac{i}{q}\sum_{m=1}^{q}\sum_{n\ge1}\Biggl[&\frac{\langle\Phi_{0,m}^{(N)},J_{\mathcal{N}}^{(1)}(k,\ell)\Phi_n^{(N)}\rangle\langle\Phi_n^{(N)},J(\Gamma_{\mathcal{M}}(k,\ell))\Phi_{0,m}^{(N)}\rangle}{(E_{0,m}^{(N)}-E_n^{(N)})^2}\\
&-\frac{\langle\Phi_{0,m}^{(N)},J(\Gamma_{\mathcal{M}}(k,\ell))\Phi_n^{(N)}\rangle\langle\Phi_n^{(N)},J_{\mathcal{N}}^{(1)}(k,\ell)\Phi_{0,m}^{(N)}\rangle}{(E_{0,m}^{(N)}-E_n^{(N)})^2}\Biggr],
\end{aligned}$$

where we have written $J(\Gamma_{\mathcal{M}}(k,\ell)) := J^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);0)$ for short. We can obtain

$$(8.1)\quad \left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M}) - \tilde{\sigma}_{12}^{(\Lambda,N)}(0,\infty,\mathcal{N},\mathcal{M})\right| \\ \le 2\left[\frac{2\Delta E+\eta}{\Delta E^4}\eta + \frac{1+T\Delta E}{\Delta E^2}e^{-\eta T}\right]\left\|J_{\mathcal{N}}^{(1)}(k,\ell)\right\|\,\left\|J(\Gamma_{\mathcal{M}}(k,\ell))\right\|.$$

Both of the current operators, $J_{\mathcal{N}}^{(1)}(k,\ell)$ and $J(\Gamma_{\mathcal{M}}(k,\ell))$, are written as a sum of local operators. We want to show that the dominant contributions to the conductance in the double sum are given by local operators in the neighborhood of the point (k,ℓ). In other words, if the distance between two local operators in the sums is sufficiently large, then the corresponding contribution are negligible.

We define

$$\langle\!\langle A;B\rangle\!\rangle := \frac{1}{q}\sum_{q=1}^{q}\sum_{n\ge 1}\langle\Phi_{0,m}^{(N)},A\Phi_n^{(N)}\rangle\frac{1}{(E_{0,m}^{(N)}-E_n^{(N)})^2}\langle\Phi_n^{(N)},B\Phi_{0,m}^{(N)}\rangle$$

for local observables A and B.

Proposition 5. *The following bound holds:*

$$|\langle\!\langle A;B\rangle\!\rangle| \le Ce^{-\kappa r},$$

where $r = \mathrm{dist}(\mathrm{supp}\,A, \mathrm{supp}\,B)$*, and C and κ are positive constants.*

Proof. The above correlation is written

$$\begin{aligned}\langle\!\langle A;B\rangle\!\rangle &= \frac{1}{q}\sum_{m=1}^{q}\sum_{n\ge 1}\int_0^\infty ds\; s\langle\Phi_{0,m}^{(N)},A\Phi_n^{(N)}\rangle\exp[-(E_n^{(N)}-E_{0,m}^{(N)})s]\langle\Phi_n^{(N)},\tilde{B}\Phi_{0,m}^{(N)}\rangle\\ &= \frac{1}{q}\sum_{m=1}^{q}\sum_{n\ge 1}\int_0^{cr} ds\; s\langle\Phi_{0,m}^{(N)},A\Phi_n^{(N)}\rangle\exp[-(E_n^{(N)}-E_{0,m}^{(N)})s]\langle\Phi_n^{(N)},\tilde{B}\Phi_{0,m}^{(N)}\rangle\\ &\quad+\frac{1}{q}\sum_{m=1}^{q}\sum_{n\ge 1}\int_{cr}^\infty ds\; s\langle\Phi_{0,m}^{(N)},A\Phi_n^{(N)}\rangle\exp[-(E_n^{(N)}-E_{0,m}^{(N)})s]\langle\Phi_n^{(N)},\tilde{B}\Phi_{0,m}^{(N)}\rangle,\end{aligned}$$

where c is a positive constant. Clearly, the second sum leads to the desired bound because of the spectral gap above the ground state. The first sum is written

$$\int_0^{cr} ds\; s\,\omega_0^{(\Lambda,N)}(A\tilde{B}^{(\Lambda)}(is)),$$

where $\tilde{B} := B - P_0^{(\Lambda,N)}BP_0^{(\Lambda,N)}$ and $\tilde{B}^{(\Lambda)}(z) := e^{iH_0^{(\Lambda)}z}\tilde{B}e^{-iH_0^{(\Lambda)}z}$ for $z\in\mathbb{C}$. Therefore, it is sufficient to estimate $\omega_0^{(\Lambda,N)}(A\tilde{B}(ib))$ for $0\le b\le cr$. By definition, one has

$$\omega_0^{(\Lambda,N)}(A\tilde{B}^{(\Lambda)}(ib)) = \omega_0^{(\Lambda,N)}(AB^{(\Lambda)}(ib)) - \omega_0^{(\Lambda,N)}(AP_0^{(\Lambda,N)}B^{(\Lambda)}(ib))$$

where we have written $B^{(\Lambda)}(z) := e^{iH_0^{(\Lambda)}z}Be^{-iH_0^{(\Lambda)}z}$ for $z \in \mathbb{C}$. For a large distance $r = \mathrm{dist}(\mathrm{supp}\, A, \mathrm{supp}\, B)$, we can prove the exponential clustering

$$\omega_0^{(\Lambda,N)}(AB^{(\Lambda)}(ib)) \sim \omega_0^{(\Lambda,N)}(AP_0^{(\Lambda,N)}B^{(\Lambda)}(ib)) + \mathcal{O}(e^{-\kappa r})$$

for a small b because of the spectral gap above the ground state, by using the Lieb-Robinson bounds. (See Theorem 2 in [21].) □

Using Proposition 5, we have

$$\lim_{\mathcal{N}\to\infty}\lim_{\mathcal{M}\to\infty}\lim_{\Lambda\nearrow\mathbb{Z}^2}\left|\tilde{\sigma}_{12}^{(\Lambda,N)}(0,\infty,\mathcal{N},\mathcal{M},\phi) - \hat{\sigma}_{12}^{(\Lambda,N)}(\phi)\right| = 0 \tag{8.2}$$

for the Hall conductance $\hat{\sigma}_{12}^{(\Lambda,N)}(\phi)$ of (6.2) with the twisted boundary phase ϕ.

8.2 Twisted Phase Dependence of the Hall Conductance

We write $\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M},\phi)$ for the conductance of (4.10) in the case of the twisted boundary conditions. We write $A^{(\Lambda)}(t) := e^{itH_0^{(\Lambda)}}Ae^{-itH_0^{(\Lambda)}}$ for the time evolution for a local operator A. Let Ω be a subset of Λ, and write $H_0^{(\Omega)}$ for the Hamiltonian $H_0^{(\Lambda)}$ restricted to the subset Ω. Let A be a local observable with supp $A \subset \Omega$, and define $A^{(\Omega)}(t) := e^{itH_0^{(\Omega)}}Ae^{-itH_0^{(\Omega)}}$. This is the time evolution of A on the region Ω. Then, one has

$$\left\|A^{(\Lambda)}(t) - A^{(\Omega)}(t)\right\| \le \int_0^t ds\, \left\|[(H_0^{(\Lambda)} - H_0^{(\Omega)}), A^{(\Omega)}(t-s)]\right\| \tag{8.3}$$

for $t \ge 0$. When the distance between the supports of two observables $(H_0^{(\Lambda)} - H_0^{(\Omega)})$ and A is sufficiently large, the norm of the commutator in the integrand in the right-hand side becomes very small for a finite t. This inequality can be proved by using the Lieb-Robinson bounds [20].

We define

$$\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M},\Omega,\phi) := i\int_{-T}^0 ds\ se^{\eta s}\omega_0^{(\Lambda,N)}\big([J_{\mathcal{N}}^{(1)}(k,\ell), J^{(\Omega)}(\Gamma_{\mathcal{M}}(k,\ell);s)];\phi\big). \tag{8.4}$$

This is given by replacing the current operator $J^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s)$ by $J^{(\Omega)}(\Gamma_{\mathcal{M}}(k,\ell);s) := e^{itH_0^{(\Omega)}}J(\Gamma_{\mathcal{M}}(k,\ell))e^{-itH_0^{(\Omega)}}$ in the expression (4.10) of the conductance $\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M},\phi)$ in the case of the twisted boundary conditions. Note that

$$\begin{aligned}\tilde{\sigma}_{12}^{(\Lambda,N)}&(\eta,T,\mathcal{N},\mathcal{M},\phi) - \tilde{\sigma}_{12}^{(\Lambda,N)}(\eta,T,\mathcal{N},\mathcal{M},\Omega,\phi)\\ &= i\int_{-T}^0 ds\ se^{\eta s}\omega_0^{(\Lambda,N)}\big([J_{\mathcal{N}}^{(1)}(k,\ell), J^{(\Lambda)}(\Gamma_{\mathcal{M}}(k,\ell);s) - J^{(\Omega)}(\Gamma_{\mathcal{M}}(k,\ell);s)];\phi\big).\end{aligned} \tag{8.5}$$

Using the above bound (8.3), we can prove that the difference between the two conductances becomes very small for a large lattice Λ. Actually, we choose the region Ω so that $\mathrm{dist}(\Omega, \partial\Lambda) = \mathcal{O}(L)$ and $\mathrm{dist}(\partial\Omega, \Gamma_{\mathcal{M}}(k,\ell)) = \mathcal{O}(L)$, where $\partial\Lambda$ and $\partial\Omega$ denote the boundary of the regions Λ and Ω, respectively, and $L = \min\{L^{(1)}, L^{(2)}\}$. Then, we have

$$\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \phi) - \tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \Omega, \phi) \to 0 \quad \text{as } \Lambda \nearrow \mathbb{Z}^2 \tag{8.6}$$

by using the Lieb-Robinson bounds. (See also [19, 7, 21].)

In the right-hand side of (8.4), the support of the commutator of the two current operators is a finite subset of Λ and apart from the boundary of Λ. Therefore, we can expect that the effect of the twisted boundary condition is exponentially small in the size L in the approximated Hall conductance (8.4). Actually, we can prove the inequality (8.7) below.

To begin with, we note that the expectation value of a local operator A for the ground state of the Hamiltonian $H_0^{(\Lambda,N)}(\phi)$ with the twisted boundary conditions is written $\omega_0^{(\Lambda,N)}(A;\phi) = q^{-1}\mathrm{Tr}\ AP_0^{(\Lambda,N)}(\phi)$, where $P_0^{(\Lambda,N)}(\phi)$ is the projection onto the sector of the ground state. Using contour integral, one has

$$P_0^{(\Lambda,N)}(\phi) = \frac{1}{2\pi i}\oint dz \frac{1}{z - H_0^{(\Lambda,N)}(\phi)}.$$

Further, by relying on the existence of the spectral gap above the sector of the ground-state, one obtains

$$\frac{\partial}{\partial\phi_j}P_0^{(\Lambda,N)}(\phi) = \frac{1}{2\pi i}\oint dz \frac{1}{z - H_0^{(\Lambda,N)}(\phi)}B_j(\phi)\frac{1}{z - H_0^{(\Lambda,N)}(\phi)},$$

where we have written

$$B_j(\phi) := \frac{\partial}{\partial\phi_j}H_0^{(\Lambda)}(\phi).$$

From this, we have

$$\frac{\partial}{\partial\phi_j}\mathrm{Tr}\ AP_0^{(\Lambda,N)}(\phi) = \frac{1}{2\pi i}\oint dz\ \mathrm{Tr}\ A\frac{1}{z - H_0^{(\Lambda,N)}(\phi)}B_j(\phi)\frac{1}{z - H_0^{(\Lambda,N)}(\phi)}.$$

Integrating both side in the case of $j = 1$, we obtain

$$\begin{aligned}
&\mathrm{Tr}\ AP_0^{(\Lambda,N)}(\phi_1,\phi_2) - \mathrm{Tr}\ AP_0^{(\Lambda,N)}(0,\phi_2) \\
&\qquad = \int_0^{\phi_1} d\phi' \frac{1}{2\pi i}\oint dz\ \mathrm{Tr}\ A\frac{1}{z - H_0^{(\Lambda,N)}(\boldsymbol{\phi}')}B_1(\boldsymbol{\phi}')\frac{1}{z - H_0^{(\Lambda,N)}(\boldsymbol{\phi}')},
\end{aligned}$$

where we have written $\boldsymbol{\phi}' = (\phi', \phi_2)$. Therefore,

$$\begin{aligned}
&\omega_0^{(\Lambda,N)}(A;\phi) - \omega_0^{(\Lambda,N)}(A;0,\phi_2) \\
&\qquad = \int_0^{\phi_1} d\phi' \frac{1}{q}\sum_{m=1}^{q}\sum_{n\ge 1}\frac{\langle\Phi_{0,m}^{(N)}(\boldsymbol{\phi}'), A\Phi_n^{(N)}(\boldsymbol{\phi}')\rangle\langle\Phi_n^{(N)}(\boldsymbol{\phi}'), B_1(\boldsymbol{\phi}')\Phi_{0,m}^{(N)}(\boldsymbol{\phi}')\rangle}{E_{0,m}^{(N)}(\boldsymbol{\phi}') - E_n^{(N)}(\boldsymbol{\phi}')} \\
&\qquad\qquad + (A \leftrightarrow B_1(\boldsymbol{\phi}')).
\end{aligned}$$

This right-hand side can be evaluated in the same way as in the preceding Sec. 8.1. Thus, the difference of the two conductances with the different phases is exponentially small in the linear size $L = \min\{L^{(1)}, L^{(2)}\}$ of the lattice-Λ as

$$\left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \Omega, \phi_1, \phi_2) - \tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \Omega, 0, \phi_2)\right| \le \mathcal{C}(\eta, T) \times \mathcal{O}(e^{-\mathrm{Const}.L}),$$

where $\mathcal{C}(\eta, T)$ is a constant which depends on η and T only. In the same way, we obtain

$$\left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \Omega, \phi_1, \phi_2) - \tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \Omega, 0, 0)\right| \le \mathcal{C}(\eta, T) \times \mathcal{O}(e^{-\mathrm{Const}.L}). \tag{8.7}$$

This is the desired inequality.

8.3 Proof of Theorem 1

Now, we shall give the proof of Theorem 1. From (8.6) and (8.7), the difference between the two conductances with the twisted phases, $\boldsymbol{\phi} = (\phi_1, \phi_2)$ and $(0, 0)$, is vanishing in the infinite-volume limit as

$$\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \phi_1, \phi_2) - \tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, 0, 0) \to 0 \quad \text{as} \quad \Lambda \nearrow \mathbb{Z}^2 \tag{8.8}$$

By using the formula (6.5) for the averaged Hall conductance, we have

$$\left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, 0, 0) - \frac{1}{2\pi}\frac{p}{q}\right| \le \mathrm{Const.} \max_{\boldsymbol{\phi}} \left\{\left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, 0, 0) - \hat{\sigma}_{12}^{(\Lambda,N)}(\boldsymbol{\phi})\right|\right\}.$$

Therefore, in order to prove Theorem 1, it is sufficient to show that the integrand in the right-hand side is vanishing in the multiple limit in (4.11). The integrand is estimated as

$$\begin{aligned}
&\left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, 0, 0) - \hat{\sigma}_{12}^{(\Lambda,N)}(\boldsymbol{\phi})\right| \\
&\quad \le \left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, 0, 0) - \tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \phi_1, \phi_2)\right| \\
&\quad + \left|\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, \phi_1, \phi_2) - \tilde{\sigma}_{12}^{(\Lambda,N)}(0, \infty, \mathcal{N}, \mathcal{M}, \phi_1, \phi_2)\right| \\
&\qquad + \left|\tilde{\sigma}_{12}^{(\Lambda,N)}(0, \infty, \mathcal{N}, \mathcal{M}, \phi_1, \phi_2) - \hat{\sigma}_{12}^{(\Lambda,N)}(\boldsymbol{\phi})\right|.
\end{aligned}$$

As shown in (8.8) above, the first term in the right-hand side is vanishing in the infinite-volume limit $\Lambda \nearrow \mathbb{Z}^2$. Relying on the estimate (8.1), we can show that the second term is vanishing in the double limit $\eta \to 0$ and $T \to \infty$ after taking the infinite-volume limit. Finally, the result (8.2) implies that the third term in the right-hand side is vanishing in the limit. Since the above Hall conductance $\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M}, 0, 0)$ with the vanishing phases is equal to the Hall conductance $\tilde{\sigma}_{12}^{(\Lambda,N)}(\eta, T, \mathcal{N}, \mathcal{M})$ in the case of the periodic boundary condition in the right-hand side of (4.11), we have proved the fractional quantization (2.2) for the Hall conductance σ_{12} in the limit.

Acknowledgement: It is a pleasure to thank the following people for discussions and correspondence: Hosho Katsura and Mahito Kohmoto.

References

[1] Aizenman, M., Graf, G. M.: Localization Bounds for an Electron Gas. J. Phys. A**31**, 6783–6806 (1998)

[2] Avron, J. E., Seiler, R., Simon, B.: Charge Deficiency, Charge Transport and Comparison of Dimensions. Commun. Math. Phys. **159**, 399–422 (1994)

[3] Bellissard, J., Van Elst, A., Schulz-Baldes, H.: The Noncommutative Geometry of the Quantum Hall Effect. J. Math. Phys. **35**, 5373–5451 (1994)

[4] Connes, A.: *Noncommutative Geometry*, Academic Press, San Diego, 1994

[5] Elgart, A., Schlein, B.: Adiabatic Charge Transport and the Kubo Formula for Landau Type Hamiltonians. Comm. Pure Appl. Math. **57**, 590–615 (2004)

[6] Haegeman, J., Michalakis, S., Nachtergaele, B., Osborne, T. J., Schuch, N., Verstraete, F.: Elementary excitations in gapped quantum spin systems. Phys. Rev. Lett. **111** (2013), 080401.

[7] Hastings, M. B.: Lieb-Schultz-Mattis in higher dimensions. Phys. Rev. B**69** (2004), 104431; Hastings, M. B., Koma, T.: Spactral gap and exponential decay of correlations. Commun. Math. Phys. **265** (2006), 781–804.

[8] Hastings, M. B., Michalakis, S.: Quantization of Hall Conductance For Interacting Electrons on a Torus. Commun. Math. Phys. **334** (2015), 433–471.

[9] Hatsugai, Y.: Quantized Berry phases as a local order parameters of a quantum liquid. J. Phys. Soc. Jpn. **75** (2006), 123601; Hirano, T., Katsura, H., Hatsugai, Y.: Topological classification of gapped spin chains: Quantized Berry phase as a local order parameter. Phys. Rev. B **77** (2008), 094431; Degeneracy and consistency condition for Berry phases: Gap closing under a local gauge twist. Phys. Rev. B **78** (2008), 054431.

[10] Kato, T.: Perturbation Theory for Linear Operators, 2nd ed. Springer, Berlin, Heidelberg, New York, 1980.

[11] Kawaji, S., Wakabayashi, J.: Temperature Dependence of Transverse and Hall Conductivities of Silicon MOS Inversion Layers under Strong Magnetic Fields. In: *Physics in High Magnetic Fields*. Chikazumi, S., Miura, N. (eds), pp. 284–287, Springer, Berlin, Heidelberg, New York, 1981

[12] von Klitzing, K., Dorda, G., Pepper, M.: New Method for High Accuracy Determination of the Fine Structure Constant Based on Quantized Hall Resistance. Phys. Rev. Lett. **45**, 494–497 (1980)

[13] Kohmoto, M.: Topological Invariant and the Quantization of the Hall Conductance. Ann. Phys. **160**, 343–354 (1985)

[14] Koma, T.: Spectral Gaps of Quantum Hall Systems with Interactions. J. Stat. Phys. **99**, 313–381 (2000)

[15] Koma, T.: Insensitivity of Quantized Hall Conductance to Disorder and Interactions. J. Stat. Phys. **99**, 383–459 (2000)

[16] Koma, T.: Revisiting the charge transport in quantum Hall systems. Rev. Math. Phys. **16** (2004), 1115–1189.

[17] Koma, T.: Widths of the Hall Conductance Plateaus. *J. Stat. Phys.* **130** (2008), 843-934.

[18] Koma, T.: Topological Current in Fractional Chern Insulators, Preprint, arXiv:1504.01243.

[19] Lieb, E. H., Robinson, D. W.: The finite group velocity of quantum spin systems. Commun. Math. Phys. **28** (1972), 251–257.

[20] Nachtergaele, B., Ogata, Y., Sims, R.: Propagation of correlations in quantum lattice systems. J. Stat. Phys. **124** (2006), 1–13.

[21] Nachtergaele, B., Sims, R.: Lieb-Robinson bounds and the exponential clustering theorem. Commun. Math. Phys. **265** (2006), 119–130.

[22] Neupert, T., Santos, L., Chamon, C., Mudry, C.: Fractional quantum Hall states at zero magnetic field. Phys. Rev. Lett. **106** (2011), 236804; Regnault, N., Bernevig, B. A.: Fractional Chern insulator. Phys. Rev. X **1** (2011), 021014; Sheng, D. N., Gu, Z.-C., Sun, K., Sheng, L.: Fractional quantum Hall effect in the absence of Landau levels. Nature Commun. **2** (2011), 389; Wu, Y.-L., Bernevig, B. A., Regnault, N.: Zoology of fractional Chern insulators. Phys. Rev. B **85** (2012), 075116.

[23] Niu, Q., Thouless, D. J., Wu, Y. S.: Quantized Hall conductance as a topological invariant. Phys. Rev. B**31** (1985), 3372–3377.

[24] Thouless, D. J., Kohmoto, M., Nightingale, M. P., den Nijs, M.: Quantized Hall Conductance in a Two-Dimensional Periodic Potential. Phys. Rev. Lett. **49**, 405–408 (1982)