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Transformation of Herglotz functions and KdV equation

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Abstract

It is expected to solve the KdV equation

$$\partial_t q = \partial_x^3 q - 6q\partial_x q$$

starting from various oscilating functions such as quasi-periodic functions, derivative of Brownian motion, etc. Althogh a KdV flow can be constructed on a space containing some quasi-periodic functions and rapidly decreasing functions, the space is quite restrictive. In this note an atempt to extend the domain of the flow is tried by interpreting the Darboux transformation in terms of the Weyl-Titchmarsh functions and iterating the transformation.

1 Weyl-Titchmarsh functions

Let q be a measurable real valued function on $\mathbb{R}_+ = [0, \infty)$ and consider a 1D Schrödinger operator on $\mathbb{R}_+$

$$L = L^q = -\partial_x^2 + q$$

with Dirichlet boundary condition at 0. Assume L^q is uniquely defined as a self-adjoint extension of a symmetric operator with domain of smooth functions having compact supports in $\mathbb{R}_+$. For such a function the boundedness is sufficient. Let

$$\lambda_0 = \inf \ \mathrm{sp} L^q,$$

and assume $\lambda_0 > -\infty$. Then, the condition on q implies the boundary ∞ is of the limit point type, and it is known that for any $\lambda \in \mathbb{C} \setminus [\lambda_0, \infty)$ there exists uniquely $f = f_q(x, \lambda)$ modulo constant multiple satisfying

$$Lf = \lambda f, \text{ and } f \in L^2(\mathbb{R}_+), \quad f \neq 0.$$

The **Weyl-Titchmarsh function** is defined by

$$m(\lambda) = m(\lambda, q) = \frac{f_q'(0,\lambda)}{f_q(0,\lambda)}.$$

$m(\lambda)$ is holomorphic on $\mathbb{C} \setminus [\lambda_0, \infty)$ and maps $\mathbb{C}_+$ (the upper half plane) into $\mathbb{C}_+$. Such a function is called a **Herglotz function**. Any Herglotz function m has a representation:

$$m(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \sigma(d\xi)$$

with $\alpha \in \mathbb{R}$, $\beta \geq 0$ and a measure σ on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{1}{1+\xi^2} \sigma(d\xi) < \infty.$$

σ is called as **spectral measure** of m. For $m(\lambda, q)$ it holds that

$$\text{supp}\sigma = \text{sp}L^q \subset [\lambda_0, \infty), \quad \beta = 0,$$

and

$$m(\lambda, q) = \alpha + \int_{\lambda_0}^{\infty} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1+\xi^2} \right) \sigma(d\xi). \tag{1}$$

The measure σ describes the nature of the spectrum of L^q completely.

On the other hand, Gelfand-Levitan theorem states that q can be recovered from its Weyl-Titchmarsh function m uniquely. Generally any Weyl-Titchmarsh function m of L^q has an asymptotics:

$$m(\lambda) = i\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}\right), \tag{2}$$

as $\lambda \to \infty$ in a suitable sense. Here $i\sqrt{\lambda}$ is the Weyl-Titchmarsh function for the free potential $q = 0$, and if a Herglotz function m is close to $i\sqrt{\lambda}$ like (2), then it is known that there exists a potential q with its Weyl-Titchmarsh function as m. In this way, between potentials and Herglotz functions there is a one-to-one correspondence, and in some cases it is useful to replace potentials q by their Weyl-Titchmarsh functions.

2 Darboux transformation

Darboux [1] introduced the following transformation of potentials to investigate the eigenvalues of L^q. For $\lambda_1 \in \mathbb{C} \setminus [\lambda_0, \infty)$ set

$$(D_{\lambda_1} q)(x) = q(x) - 2\partial_x^2 \log f_q(x, \lambda_1),$$

and let us call it as **Darboux transformation** of q. $D_{\lambda_1}q$ is real valued for $\lambda_1 < \lambda_0$, and

$$f_{D_{\lambda_1}q}(x,\lambda) = f_q'(x,\lambda) - \frac{f_q'(x,\lambda_1)}{f_q(x,\lambda_1)} f_q(x,\lambda), \quad ({}' = \partial_x)$$

satisfies

$$L^{D_{\lambda_1}q} f_{D_{\lambda_1}q} = \lambda f_{D_{\lambda_1}q}.$$

Since most of the cases we have

$$f_{D_{\lambda_1}q} \in L^2(\mathbb{R}_+),$$

it holds that

$$m(\lambda, D_{\lambda_1}q) = \frac{f'_{D_{\lambda_1}q}(0,\lambda)}{f_{D_{\lambda_1}q}(0,\lambda)} = -\frac{\lambda - \lambda_1}{m(\lambda,q) - m(\lambda_1,q)} - m(\lambda_1, q). \tag{3}$$

If $m(\lambda, q)$ satisfies the property (2), then so does $m(\lambda, D_{\lambda_1}q)$. An identity

$$\frac{m(\lambda,q) - m(\lambda_1,q)}{\lambda - \lambda_1} = \int_{\lambda_0}^{\infty} \frac{1}{\xi - \lambda} \frac{\sigma(d\xi)}{\xi - \lambda_1}$$

shows this function is of Herglotz, hence so is the right hand side of (3), which makes it natural to define Darboux transformation in the space of Herglotz functions. For any Herglotz function m let

$$(\Delta_\zeta m)(\lambda) = -\frac{\lambda - \zeta}{m(\lambda) - m(\zeta)} - m(\zeta),$$

and call it as Darboux transformation of Herglotz function m. This has meaning if $\zeta \in \mathbb{C}\backslash\mathbb{R}$, and sometimes for some real numbers. The following is easily verified.

Lemma 1 *For any ζ_1, ζ_2 it holds that $\Delta_{\zeta_1}\Delta_{\zeta_2} = \Delta_{\zeta_2}\Delta_{\zeta_1}$.*

Generally, $\Delta_\zeta m$ is no longer a Herglotz function, even if so is m. However, as we have already seen, if the spectral measure σ of a Herglotz function m has a finite $\lambda_0 = \inf \operatorname{supp}\sigma$, then $\Delta_\zeta m$ is of Herglotz for $\zeta < \lambda_0$. Moreover we have

Lemma 2 *Let m be of Herglotz. Then, $\Delta_\zeta \Delta_{\bar{\zeta}} m$ is of Herglotz as well for any $\zeta \in \mathbb{C}\backslash\mathbb{R}$.*

Proof. Set

$$f(\lambda) = \frac{\lambda - \zeta}{\dfrac{\lambda - \overline{\zeta}}{m(\lambda) - \overline{m(\zeta)}} - \dfrac{\zeta - \overline{\zeta}}{m(\zeta) - \overline{m(\zeta)}}}.$$

Then

$$\begin{aligned}\frac{m(\lambda) - \overline{m(\zeta)}}{\lambda - \overline{\zeta}} &= \beta + \frac{1}{\lambda - \overline{\zeta}} \int_{-\infty}^{\infty} \left(\frac{1}{\xi - \lambda} - \frac{1}{\xi - \overline{\zeta}} \right) \sigma(d\xi) \\ &= \beta + \int_{-\infty}^{\infty} \frac{1}{(\xi - \lambda)(\xi - \overline{\zeta})} \sigma(d\xi) = \beta + \int_{-\infty}^{\infty} \frac{\xi - \zeta}{\xi - \lambda} \sigma_\zeta(d\xi),\end{aligned}$$

with $\sigma_\zeta(d\xi) = |\xi - \zeta|^{-2} \sigma(d\xi)$, and

$$\frac{\lambda - \overline{\zeta}}{m(\lambda) - \overline{m(\zeta)}} - \frac{\zeta - \overline{\zeta}}{m(\zeta) - \overline{m(\zeta)}} = \frac{(\zeta - \lambda) \displaystyle\int_{-\infty}^{\infty} \frac{1}{\xi - \lambda} \sigma_\zeta(d\xi)}{\left(\beta + \displaystyle\int_{-\infty}^{\infty} \frac{\xi - \zeta}{\xi - \lambda} \sigma_\zeta(d\xi) \right) (\beta + \gamma)},$$

where $\gamma = \displaystyle\int_{-\infty}^{\infty} \sigma_\zeta(d\xi)$, hence

$$f(\lambda) = -\frac{\left(\beta + \displaystyle\int_{-\infty}^{\infty} \frac{\xi - \zeta}{\xi - \lambda} \sigma_\zeta(d\xi) \right) (\beta + \gamma)}{\displaystyle\int_{-\infty}^{\infty} \frac{1}{\xi - \lambda} \sigma_\zeta(d\xi)}.$$

Note

$$-\frac{\beta + \displaystyle\int_{-\infty}^{\infty} \frac{\xi - \zeta}{\xi - \lambda} \sigma_\zeta(d\xi)}{\displaystyle\int_{-\infty}^{\infty} \frac{1}{\xi - \lambda} \sigma_\zeta(d\xi)} = -\frac{\beta + \gamma}{\displaystyle\int_{-\infty}^{\infty} \frac{1}{\xi - \lambda} \sigma_\zeta(d\xi)} - \lambda + \zeta.$$

The above first term is of Herglotz, hence, for some $\left(\widehat{\alpha}, \widehat{\beta}, \widehat{\sigma}\right)$

$$-\frac{1}{\displaystyle\int_{-\infty}^{\infty} \frac{1}{\xi - \lambda} \sigma_\zeta(d\xi)} = \widehat{\alpha} + \widehat{\beta}\lambda + \int_{-\infty}^{\infty} \left(\frac{1}{\xi - \lambda} - \frac{\xi}{1 + \xi^2} \right) \widehat{\sigma}(d\xi)$$

is valid. Since σ_ζ is a finite measure, we easily see

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} \frac{-\lambda}{\xi - \lambda} \sigma_\zeta(d\xi) = \int_{-\infty}^{\infty} \sigma_\zeta(d\xi) = \gamma,$$

and $\widehat{\beta} = \gamma^{-1}$, hence

$$-\frac{\beta+\gamma}{\int_{-\infty}^{\infty} \frac{1}{\xi-\lambda}\sigma_{\zeta}(d\xi)} - \lambda = \widehat{\alpha}(\beta+\gamma) + \frac{\beta}{\gamma}\lambda + (\beta+\gamma)\int_{-\infty}^{\infty}\left(\frac{1}{\xi-\lambda} - \frac{\xi}{1+\xi^2}\right)\widehat{\sigma}(d\xi).$$

Consequently we have

$$\begin{aligned}
&\left(\Delta_{\zeta}\Delta_{\overline{\zeta}} m\right)(\lambda) \\
&= f(\lambda) - \frac{\zeta-\overline{\zeta}}{m(\zeta)-\overline{m(\zeta)}} + \overline{m(\zeta)} \\
&= \widehat{\alpha}(\beta+\gamma)^2 + \frac{\beta}{\gamma}(\beta+\gamma)\lambda + (\beta+\gamma)\int_{-\infty}^{\infty}\left(\frac{1}{\xi-\lambda} - \frac{\xi}{1+\xi^2}\right)\widehat{\sigma}(d\xi) \\
&\quad - \frac{\operatorname{Im}\zeta}{\beta+\gamma} + (\beta+\gamma)\zeta + \alpha + \beta\overline{\zeta} + \int_{-\infty}^{\infty}\left(\frac{1}{\xi-\overline{\zeta}} - \frac{\xi}{1+\xi^2}\right)\sigma(d\xi),
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Im}\left(\Delta_{\zeta}\Delta_{\overline{\zeta}} m\right)(\lambda) \\
&= \frac{\beta}{\gamma}(\beta+\gamma)\operatorname{Im}\lambda + (\beta+\gamma)\int_{-\infty}^{\infty}\frac{1}{|\xi-\lambda|^2}\widehat{\sigma}(d\xi) + \gamma\operatorname{Im}\zeta - \int_{-\infty}^{\infty}\frac{\operatorname{Im}\zeta}{|\xi-\overline{\zeta}|^2}\sigma(d\xi) \\
&= \frac{\beta}{\gamma}(\beta+\gamma)\operatorname{Im}\lambda + (\beta+\gamma)\int_{-\infty}^{\infty}\frac{1}{|\xi-\lambda|^2}\widehat{\sigma}(d\xi) \geq 0,
\end{aligned}$$

which proves the lemma. ■

Remark 1 *For a potential q and real numbers λ_j satisfying $\lambda_j < \lambda_0$ $(1 \leq j \leq n)$ the n-fold iteration $D_{\lambda_1}D_{\lambda_2}\cdots D_{\lambda_n}q$ can be described as follows. For smooth functions f_j $(1 \leq j \leq n)$ let*

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_{n-1}(x) & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_{n-1}'(x) & f_n'(x) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ f_1^{(n-2)}(x) & f_2^{(n-2)}(x) & \cdots & f_{n-1}^{(n-2)}(x) & f_n^{(n-2)}(x) \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_{n-1}^{(n-1)}(x) & f_n^{(n-1)}(x) \end{vmatrix}.$$

Then, setting $f_j(x) = f_q(x, \lambda_j)$ and $f = f_q(x, \lambda)$

$$\begin{cases} \left(D_{\lambda_1}D_{\lambda_2}\cdots D_{\lambda_n}q\right)(x) = q(x) - 2\partial_x^2 \log W(f_1, f_2, \cdots, f_m) \\ f_{D_{\lambda_1}D_{\lambda_2}\cdots D_{\lambda_n}q}(x, \lambda) = \dfrac{W(f_1, f_2, \cdots, f_m, f)}{W(f_1, f_2, \cdots, f_m)} \end{cases}.$$

3 KdV flow

For potential q defined on $\mathbb{R}$ one can define a Weyl-Titchmarsh function on $R_- = (-\infty, 0]$ similarly, and we denote it by $m_-(\lambda, q)$. The previous one is denoted by $m_+(\lambda, q)$. If

$$m_+(\xi + i0, q) = -\overline{m_-(\xi + i0, q)} \quad \text{for a.e.} \quad \xi \in F$$

holds for a measurable set F on $\mathbb{R}$ with positive Lebesgue measure, q is called as **reflectionless** on F. For $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $\lambda_0 < \lambda_1$ define the underlying space

$$\Omega_{\lambda_0,\lambda_1} = \{q;\ \mathrm{sp}L^q \subset [\lambda_0, \infty) \text{ and } q \text{ is reflectionless on } [\lambda_1, \infty)\},$$

where L^q is a Schrödinger operator defined on $L^2(\mathbb{R})$. For $q \in \Omega_{\lambda_0,\lambda_1}$ the Weyl-Titchmarsh function m_+ is given by

$$m_+(-z^2 + \lambda_1, q) = -z - \int_{-\sqrt{\lambda_1-\lambda_0}}^{\sqrt{\lambda_1-\lambda_0}} \frac{\sigma(d\zeta)}{\zeta - z}, \quad \text{with} \int_{-\sqrt{\lambda_1-\lambda_0}}^{\sqrt{\lambda_1-\lambda_0}} \frac{\sigma(d\zeta)}{\lambda_1 - \lambda_0 - \zeta^2} \leq 1.$$

Moreover, it is known by Lundina [4], Marchenko [5] that $q \in \Omega_{\lambda_0,\lambda_1}$ is holmorphic on a strip

$$\left\{z \in \mathbb{C};\quad |\mathrm{Im}\, z| < \sqrt{\lambda_1 - \lambda_0}^{-1}\right\}$$

with uniform bound

$$|q(z) - \lambda_1| \leq 2(\lambda_1 - \lambda_0)\left(1 - \sqrt{\lambda_1 - \lambda_0}\,|\mathrm{Im}\, z|\right)^{-2}.$$

Therefore, $\Omega_{\lambda_0,\lambda_1}$ turns to be compact. Let

$$\begin{cases} \Gamma = \{g;\ g = e^h,\ h \text{ is holomorphic on } \mathbb{B}_{\sqrt{\lambda_1-\lambda_0}}\} \\ \Gamma_{\mathrm{real}} = \{g \in \Gamma;\ g(x) \in \mathbb{R} \text{ for any } x \in \mathbb{B}_{\sqrt{\lambda_1-\lambda_0}} \cap \mathbb{R}\} \end{cases},$$

where

$$\mathbb{B}_r = \{z \in \mathbb{C};\quad |z| \leq r\}.$$

Then Γ and Γ_{real} become Abelian groups by usual multiplication. In [2],[3] it was proved that there exists a smooth flow $\{K(g)\}_{g \in \Gamma_{\mathrm{real}}}$ on $\Omega_{\lambda_0,\lambda_1}$ satisfies

$$\left(K\left(e^{-tz}\right) q\right)(x) = q(x + t),$$

and

$$q(t, x) = \left(K\left(e^{4tz^3}\right) q\right)(x)$$

solves the KdV equation. Actually $K(g)$ can be defined for $g \in \Gamma$ and the resulting potential $K(g)q$ takes complex values on $\mathbb{R}$ even if $q \in \Omega_{\lambda_0,\lambda_1}$.

On the other hand, if we choose

$$r_\zeta(z) = 1 - \frac{z}{\zeta} \in \Gamma \quad \text{for} \quad |\zeta| > \sqrt{\lambda_1 - \lambda_0},$$

then $r_\zeta r_{\overline{\zeta}} \in \Gamma_{\text{real}}$ and it is related to the Darboux transformation as

$$q \in \Omega_{\lambda_0,\lambda_1} \Longrightarrow K\left(r_\zeta r_{\overline{\zeta}}\right) q = D_\zeta D_{\overline{\zeta}} q.$$

Therefore, noticing for $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that $\zeta^{-1} + \overline{\zeta}^{-1} = t$

$$e^{-tz} = \lim_{n\to\infty} \left(1 - \frac{z}{n\zeta}\right)^n \left(1 - \frac{z}{n\overline{\zeta}}\right)^n = \lim_{n\to\infty} \left(r_{n\zeta} r_{n\overline{\zeta}}\right)^n (z),$$

and, ω such that $\omega^3 = 1$, $(\omega \neq 1)$

$$1 - \frac{z^3}{n\zeta} = \left(r_{(n\zeta)^{1/3}} r_{(n\zeta)^{1/3}\omega} r_{(n\zeta)^{1/3}\omega^2}\right)(z),$$

for $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that $\zeta^{-1} + \overline{\zeta}^{-1} = -4t$ we have

$$\begin{aligned} e^{4tz^3} &= \lim_{n\to\infty} \left(1 - \frac{z^3}{n\zeta}\right)^n \left(1 - \frac{z^3}{n\overline{\zeta}}\right)^n \\ &= \lim_{n\to\infty} \left(r_{(n\zeta)^{1/3}} r_{\left(n\overline{\zeta}\right)^{1/3}} r_{(n\zeta)^{1/3}\omega} r_{\left(n\overline{\zeta}\right)^{1/3}\omega} r_{(n\zeta)^{1/3}\omega^2} r_{\left(n\overline{\zeta}\right)^{1/3}\omega^2}\right)^n (z). \end{aligned}$$

Summing up the argument we obtain

Theorem 1 *For $q \in \Omega_{\lambda_0,\lambda_1}$ and $\zeta \in \mathbb{C}\backslash\mathbb{R}$ such that $\zeta^{-1} + \overline{\zeta}^{-1} = -4t$ the limit*

$$\lim_{n\to\infty} \left(\Delta_{(n\zeta)^{1/3}} \Delta_{\left(n\overline{\zeta}\right)^{1/3}} \Delta_{(n\zeta)^{1/3}\omega} \Delta_{\left(n\overline{\zeta}\right)^{1/3}\omega} \Delta_{(n\zeta)^{1/3}\omega^2} \Delta_{\left(n\overline{\zeta}\right)^{1/3}\omega^2}\right)^n m_+(\lambda, q) \qquad (4)$$

exists finitely and the associated $q(t,x)$ yields a solution to the KdV equation starting from q.

4 Open problem

In order to construct solutions to the KdV equation starting from more general functions Theorem 1 suggests the followings:

(1) Determine the class of Herglotz functions such that (4) converges.

(2) Characterize the class of two Herglotz functions (m_+, m_-) such that the limits (m_+^t, m_-^t) yields potentials $\left(q_+(t,x)|_{x\in\mathbb{R}_+}, q_-(t,x)|_{x\in\mathbb{R}_-}\right)$ for which

$$q(t,x) = \begin{cases} q_+(t,x) & \text{for } x \in \mathbb{R}_+ \\ q_-(t,x) & \text{for } x \in \mathbb{R}_- \end{cases}$$

solves the KdV equation.

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