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# Integrated density of states for the Schrödinger operators with random $\delta$ magnetic fields

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**Abstract**

We consider the Schrödinger operator with magnetic fields given as the sum of randomly distributed $\delta$ functions with random coefficients. We give a brief review about the recent results on the integrated density of states (IDS) $N(\lambda)$ for this model, particularly about (i) the Lifshitz tail (the exponential decay of $N(\lambda)$ near the bottom of the spectrum), and the asymptotics of the Laplace transform $\mathcal{L}(t)$ of the density of states $dN$ as $t \to 0$ in the case of the Poisson configuration.

## 1 Introduction

We shall consider the magnetic Schrödinger operator on the Euclidean plane $\mathbb{R}^2$

$$H_\omega = (-i\nabla - A_\omega)^2 .$$

Here $A_\omega = (A_{\omega,1}, A_{\omega,2})$ is the magnetic vector potential, and the corresponding magnetic field $B_\omega$ is given by

$$B_\omega = \operatorname{curl} A_\omega = \partial_{x_1} A_{\omega,2} - \partial_{x_2} A_{\omega,1}. \tag{1}$$

We assume

$$B_\omega = \sum_{\gamma\in\Gamma_\omega} 2\pi\alpha_\gamma(\omega)\delta_\gamma, \tag{2}$$

where $\omega$ is a random parameter belonging to some probability space $\Omega$, $\Gamma_\omega$ is a discrete set without accumulation point in $\mathbb{R}^2$, $\alpha_\gamma(\omega)$ is a real number satisfying $0 \le \alpha_\gamma(\omega) < 1$, and $\delta_\gamma$ is the Dirac delta function supported on the point $\gamma$. It is known that there exists a vector potential $A_\omega \in C^\infty(\mathbb{R}^2 \setminus \Gamma_\omega; \mathbb{R}^2) \cap L^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ satisfying (1) and (2) for any given $\Gamma_\omega$ and $\alpha_\gamma(\omega)$ (see [Ge-St, section 4]).

We introduce the integrated density of states (IDS) as follows. For a bounded open set $\mathcal{D}$ in $\mathbb{R}^2$, let $H^N_{\omega,\mathcal{D}}$ be the operator $H_\omega$ restricted to the region $\mathcal{D}$ with the Neumann boundary conditions. For $\lambda \in \mathbb{R}$, let $N^N_{\omega,\mathcal{D}}(\lambda)$ be the number of the eigenvalues of $H^N_{\omega,\mathcal{D}}$ less than or equal to $\lambda$ counted with multiplicity. We define the IDS $N(\lambda)$ by

$$N(\lambda) = \lim_{\mathcal{D}\to\mathbb{R}^2} \frac{N^N_{\omega,\mathcal{D}}(\lambda)}{|\mathcal{D}|}, \tag{3}$$

where $|\mathcal{D}|$ is the Lebesgue measure of $\mathcal{D}$. Under some stationarity condition on $B_\omega$ and regularity condition on the boundary of $\mathcal{D}$, it is well-known that the limit (3) exists for almost every $\lambda$ and independent of the random parameter $\omega$ almost surely.

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There are numerous study about the *Lifshitz tail* for the random Schrödinger operators, that is, the exponential decay of $N(\lambda)$ as $\lambda$ tends to the boundary of the essential spectrum (see e.g. [Ca-La, Ki, St] and references therein). The Lifshitz tail for the Schrödinger operators with random magnetic fields is proved by some authors (see e.g. [Gh, Na1, Na2, Ue1, Mi-No1] and references therein). There is also a detailed Japanese review [Ue2] about the Lifshitz tail. In the present paper, we briefly report on the following subjects; (i) the Lifshitz tail for our $H_\omega$, (ii) the stochastic representation of the Laplace transform $\mathcal{L}(t)$ of the density of states $dN$ (J. L. Borg's result), (iii) the behavior of $\mathcal{L}(t)$ as $t \to 0$ for the case $\Gamma_\omega$ is the Poisson configuration. In most cases we give only an idea of the proof, and the detail will be given in our forthcoming paper [Mi-No2] or elsewhere.

## 2 Lifshitz tail

The Lifshitz tail for the random $\delta$ magnetic field is already established in authors' earlier proceedings paper [Mi-No1]. Here we report our recent progress given in [Mi-No2]. Before stating our assumptions, we prepare some notations. For $S \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$, and $r > 0$, let $S + x = \{s + x \mid s \in S\}$ and $rS = \{rs \mid s \in S\}$. For $k \geq 0$, let

$$Q_k = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid -k - \frac{1}{2} \leq x_j < k + \frac{1}{2} \ (j = 1, 2)\right\},$$

which is a square with edge length $2k + 1$ centered at the origin. Especially $Q_0$ is a unit square centered at the origin. The boundary of a set $S$ is denoted by $\partial S$. The open ball of radius $r$ centered at $x$ is denoted by $B_x(r)$, that is,

$$B_x(r) = \{y \in \mathbb{R}^2 \mid |y - x| < r\}.$$

**Assumption 2.1.** *Let $(\Omega, \mathbb{P})$ be a probability space, $\Gamma_\omega$ a discrete set in $\mathbb{R}^2$ dependent on $\omega \in \Omega$ without accumulation points in $\mathbb{R}^2$, and $\alpha(\omega) = \{\alpha_\gamma(\omega)\}_{\gamma \in \Gamma_\omega}$ a sequence of real numbers with $0 \leq \alpha_\gamma(\omega) < 1$ dependent on $\omega \in \Omega$. For a Borel set $E$ in $\mathbb{R}^2$, put*

$$\Phi_\omega(E) = \sum_{\gamma \in \Gamma_\omega \cap E} \alpha_\gamma(\omega).$$

*We assume the following conditions (i)-(vi).*

*(i) For any Borel set $E$ in $\mathbb{R}^2$, the random variable $\Phi(E) : \omega \mapsto \Phi_\omega(E)$ is measurable with respect to $\omega \in \Omega$.*

*(ii) For any finite distinct points $\{n_j\}_{j=1}^J$ with $n_j \in \mathbb{Z}^2$, and for any Borel sets $\{E_j\}_{j=1}^J$ with $E_j \subset n_j + Q_0$, the random variables $\{\Phi(E_j)\}_{j=1}^J$ are independent.*

*(iii) For any Borel set $E \subset Q_0$, the random variables $\{\Phi(E + n)\}_{n \in \mathbb{Z}^2}$ are identically distributed.*

*(iv) The mathematical expectation $\mathbf{E}[\Phi(Q_0)]$ is positive and finite. The variance $\mathbf{V}[\Phi(Q_0)]$ is finite.*

*(v)* $\Phi_\omega(\partial Q_0) = 0$ *almost surely.*

*(vi) One of the following two conditions (a) or (b) holds.*

*(a) There exists a positive constant* $c$ *with* $0 < c \leq 1$ *independent of* $\omega$ *such that the probability of the event 'both of the following two conditions (4) and (5) hold' is positive for any* $\epsilon > 0$.

$$\Phi_\omega(Q_0) = \sum_{\gamma \in \Gamma_\omega \cap Q_0} \alpha_\gamma < \epsilon, \tag{4}$$

$$B_\gamma(c\sqrt{\alpha_\gamma}) \cap B_{\gamma'}(c\sqrt{\alpha_{\gamma'}}) = \emptyset, \quad B_\gamma(c\sqrt{\alpha_\gamma}) \cap \partial Q_0 = \emptyset$$
$$\text{for every } \gamma, \gamma' \in \Gamma_\omega \cap Q_0 \text{ with } \gamma \neq \gamma'. \tag{5}$$

*(b) The probability of the event*

$$\sum_{\gamma \in \Gamma_\omega \cap Q_0} \sqrt{\alpha_\gamma} < \epsilon \tag{6}$$

*is positive for any* $\epsilon > 0$.

The assumptions (i)-(v) mean the $\mathbb{Z}^2$-stationarity of the random magnetic field $B_\omega$. The assumption (vi) is improved compared with authors' former result [Mi-No1]. It accepts the case the number of the lattice points in $Q_0$ is unlimited (in [Mi-No1], the authors have assumed there is only one lattice point in $Q_0$ with positive probability). The assumption (4) means the magnetic flux through $Q_0$ can be arbitrarily small, and (5) means the points $\Gamma_\omega$ are separated farther than a constant multiple of the magnetic length $\sqrt{\alpha_\gamma}$ as the flux tends to 0. The assumption (6) is independent of the positions of the points $\Gamma_\omega$, but the restriction on the flux is stronger than (4), since $0 \leq \alpha_\gamma \leq \sqrt{\alpha_\gamma} \leq 1$. If the number of $\Gamma_\omega \cap Q_0$ is bounded by a constant independent of $\omega$, then (4) implies (6) by the Schwarz inequality.

Let $L_\omega = (-i\nabla - A_\omega)^2$ with the domain $D(L_\omega) = C_0^\infty(\mathbb{R}^2 \setminus \Gamma_\omega)$, then $L_\omega$ is a non-negative operator. We denote the Friedrichs extension of $L_\omega$ by $H_\omega$, which is a self-adjoint operator on $L^2(\mathbb{R}^2)$. Our result is as follows.

**Theorem 2.2.** *Suppose Assumption 2.1 holds. Then,*

*(i)* $\sigma(H_\omega) = [0, \infty)$ *almost surely.*

*(ii) There exist positive constants* $C$ *and* $E_0$ *independent of* $\omega$ *and* $E$, *such that*

$$N(E) \leq e^{-\frac{C}{E}}$$

*for any* $E$ *with* $0 < E < E_0$.

For the proof of (i), we construct the approximating eigenfunctions of $H_\omega$ for every $\lambda > 0$, that is, the sequence $\{u_n\}_{n=1}^\infty \subset D(H_\omega)$ such that $\|u_n\| = 1$ and $\|(H_\omega - \lambda)u_n\| \to 0$. This method is rather standard (see e.g. [Ki-Ma]), but here we have to take care the operator domain of $H_\omega$ depends on $\omega$; if $u \in D(H_\omega)$ and $0 < \alpha_\gamma(\omega) < 1$, then $u(\gamma) = 0$.

The proof of (ii) is essentially the same as in our earlier work [Mi-No1], summarized as follows. We use the Laptev-Weidl inequality [La-We]

$$\int_{\mathbb{R}^2} |(-i\nabla - A_\alpha)\, u|^2\, dx \geq (\min(\alpha, 1-\alpha))^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2} dx \tag{7}$$

for every $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, where $0 < \alpha < 1$ and $A_\alpha$ is the vector potential for the single solenoid

$$A_\alpha(x) = \alpha \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right). \tag{8}$$

By (7) and the diamagnetic inequality $|(-i\nabla - A)u| \geq |\nabla |u||$, we can construct a random scalar potential $V_\omega$ such that

$$\int_{\mathcal{D}} |(-i\nabla - A_\omega)\, u|^2\, dx \geq \frac{1}{2} \int_{\mathcal{D}} \left( |\nabla |u||^2 + V_\omega |u|^2 \right) dx$$

for every square region $\mathcal{D}$ and $u \in Q(H_{\omega,\mathcal{D}}^N)$ (the form domain of $H_{\omega,\mathcal{D}}^N$). By this inequality and the min-max principle, we know that the lowest eigenvalue of $H_{\omega,\mathcal{D}}^N$ is bounded from below by the lowest eigenvalue of $\frac{1}{2}(-\Delta_{\mathcal{D}}^N + V_\omega)$, where $\Delta_{\mathcal{D}}^N$ is the Neumann Laplacian on $\mathcal{D}$. This estimate is enough to reduce the proof of (ii) to the case of the random scalar potential (a similar argument is used in Nakamura's papers [Na1, Na2]). For the detail, see [Mi-No2].

# 3 James L. Borg's result

Here we review some results obtained in the Ph. D. Thesis of James L. Borg [Bo]. Let $\Gamma$ be a discrete set in $\mathbb{R}^2$ without accumulation points in $\mathbb{R}^2$ . Let $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ be a sequence of real numbers satisfying $0 < \alpha_\gamma < 1$. Let $A$ be the vector potential satisfying

$$B = \operatorname{curl} A = \sum_{\gamma\in\Gamma} 2\pi\alpha_\gamma \delta_\gamma,$$

and let

$$L(A) = (-i\nabla - A)^2, \quad D(L(A)) = C_0^\infty(\mathbb{R}^2 \setminus \Gamma).$$

Let $H(A)$ be the Friedrichs extension of $L(A)$.

The first result is about the Feynman-Kac-Itô formula

$$\begin{aligned} & e^{-tH(A)}(x, x') \\ = \; & \frac{1}{4\pi t} \exp\left( -\frac{|x - x'|^2}{4t} \right) \mathbb{E}_{0,x,t,x'} \left[ \exp\left( -i \int_0^t A(w_s) \cdot dw \right) \right], \end{aligned} \tag{9}$$

where the left hand side denotes the integral kernel of the heat semigroup $e^{-tH(A)}$, $\mathbb{E}_{0,x,t,x'}$ the expectation with respect to the Brownian bridge process starting from $x$ at time 0 and ending at $x'$ at time $t$, and $w$ a sample path of the Brownian bridge process. The integral is interpreted in the sense of Itô stochastic integral. It is well-known that the formula (9) holds if the vector potential $A$ is locally square integrable, but our vector potential does not satisfy this condition, since it behaves like $O(|x-\gamma|^{-1})$ around $x=\gamma\in\Gamma$ (see (12) and (13) below). Borg proves that (9) also holds in this case, if we choose the Friedrichs extension as the self-adjoint realization.

In order to formulate the result, we introduce some terminology in the theory of the Wiener process. Let $x\in\mathbb{R}^2$ and $w$ a sample path of the Wiener process starting from $x$. For a Borel set $S$ in $\mathbb{R}^2$, let $T_S$ be the *hitting time*

$$T_S=\inf\{t>0\mid w_t\in S\}.$$

We call $S$ a *polar set* if

$$\mathbb{P}_x(T_S<\infty)=0\quad\text{for every } x\in\mathbb{R}^2,$$

where $\mathbb{P}_x$ denotes the probability with respect to the Wiener process starting from $x$.

**Theorem 3.1** ([Bo, Theorem 3.1.1]). *Let $S$ be a polar set in $\mathbb{R}^2$ and suppose the vector potential $A$ satisfies $A\in L^2_{\mathrm{loc}}(\mathbb{R}^2\setminus S;\mathbb{R}^2)$ and $\nabla\cdot A=0$ in $\mathbb{R}^2\setminus S$. Let $H(A)$ be the Friedrichs extension of $L(A)=(-i\nabla-A)^2$ with the domain $D(L(A))=C_0^\infty(\mathbb{R}^2\setminus S)$. Then, the formula (9) holds for any $t>0$.*

The main idea of the proof consists of approximating the singular vector potential $A$ by a sequence of bounded vector potentials, obtained by truncating the singularities of $A$. The formula (9) immediately implies the *diamagnetic inequality*

$$|e^{-tH(A)}(x,x')|\le e^{-tH(0)}(x,x'). \tag{10}$$

From this point of view the choice of the Friedrichs extension is natural; since the diamagnetic inequality (10) never holds if we choose another self-adjoint extensions.

Borg also obtains an interesting representation for the IDS by using the formula (9), and the result also holds in our case. For simplicity, we assume there exists $C>0$ such that

$$\#\left(\Gamma\cap\{|x|\le R\}\right)\le CR^2 \tag{11}$$

for every $R>1$. Then, the vector potential $A$ is explicitly given by

$$A(x)=(\operatorname{Im}\zeta(x),\operatorname{Re}\zeta(x)), \tag{12}$$

$$\zeta(x)=\frac{\alpha_0}{x}+\sum_{\gamma\in\Gamma\setminus\{0\}}\alpha_\gamma\left(\frac{1}{x-\gamma}+\frac{1}{\gamma}+\frac{x^2}{\gamma^2}\right), \tag{13}$$

where we identify $x=(x_1,x_2)$ with the complex number $x_1+ix_2$ in the right hand side (we sometimes use this convention also in the sequel). Then, for the Brownian bridge process $w_t=(w_{1,t},w_{2,t})$ whose starting point and ending point are the same point $x$, we can formally calculate as

$$\begin{aligned}&\int_0^t A(w_s)\cdot dw_s\\ =\;&\sum_{\gamma\in\Gamma}\alpha_\gamma\int_0^t\frac{-(w_{2,s}-\gamma_2)dw_1+(w_{1,s}-\gamma_1)dw_2}{|w_s-\gamma|^2}\\ =\;&\sum_{\gamma\in\Gamma}\alpha_\gamma\Theta_{t,\gamma}(w),\end{aligned} \tag{14}$$

where $\Theta_{t,\gamma}(w)$ is the winding angle of the path $w$ around the point $\gamma$, that is,

$$\Theta_{t,\gamma}(w) = \arg(w_t - \gamma) - \arg(w_0 - \gamma).$$

The formula (14) is rigorously justified by using the Itô-formula and the fact $\Delta \arg(x - \gamma) = 0$. Notice that only a finite number of $\Theta_{t,\gamma}(w)$ take non-zero values in the sum (14), since $\Theta_{t,\gamma}(w) = 0$ for $|\gamma| > \max_{0 \le s \le t} |w_s|$. Thus the formula (9) for $x = x'$ is rewritten as

$$e^{-tH(A)}(x,x) = \frac{1}{4\pi t}\mathbb{E}_{0,x,t,x}\left[\exp\left(-i\sum_{\gamma\in\Gamma}\alpha_\gamma\Theta_{t,\gamma}(w)\right)\right]. \tag{15}$$

Let us return to the case of the random magnetic field. We can prove that (11) almost surely holds under Assumption 2.1, so we can apply the above argument for our operator $H_\omega$. Moreover, if $B_\omega$ has the $\mathbb{Z}^2$-stationarity, it is well-known that the Laplace transform $\mathcal{L}(t)$ of the density of state $dN$ is represented as

$$\mathcal{L}(t) = \int_0^\infty e^{-t\lambda}dN(\lambda) = \int_{Q_0} e^{-tH_\omega}(x,x)dx$$

for almost every $\omega$, by the Ergodic theorem. Substituting (15) into this formula, we obtain the representation of $\mathcal{L}(t)$ via the winding number of the Brownian bridge.

**Theorem 3.2** ([Bo, Theorem 4.4.1]). *Under Assumption 2.1, we have*

$$\mathcal{L}(t) = \frac{1}{4\pi t}\int_{Q_0}\mathbb{E}_{0,x,t,x}\left[\exp\left(-i\sum_{\gamma\in\Gamma}\alpha_\gamma(\omega)\Theta_{t,\gamma}(w)\right)\right]dx. \tag{16}$$

Another interesting result of Borg is kind of trace formula, formulated as follows.[3]

**Theorem 3.3** ([Bo, Theorem 3.3.2]). *Let $0 < \alpha < 1$. Let $A_\alpha$ be the vector potential for the single solenoid given in (8). Then,*

$$\lim_{\Lambda\to\mathbb{R}^2}\int_\Lambda \left(e^{-tH(A_\alpha)}(x,x) - e^{-tH(0)}(x,x)\right)dx = -\frac{\alpha(1-\alpha)}{2}. \tag{17}$$

Borg calls (17) the *depletion of states*. The formula (17) can be interpreted as a kind of the diamagnetic inequality, since it means the eigenvalues of $H(0)$ are *raised* in some averaged sense by the the Aharonov-Bohm magnetic potential $A_\alpha$. Borg gives two proofs of (17). (i) By adding the harmonic oscillator potential $\omega_0 x^2$ to both operators $H(A_\alpha)$ and $H(0)$, calculating two traces, and taking the limit $\omega_0 \to 0$. (ii) By using the known probability distribution of the winding angle, and calculate the left hand side of (17) directly. Both methods rely on the formula (15).

[3]Originally Borg considers the difference of the traces of the two Dirichlet realizations. The above result can be proved by almost the same argument.

# 4 IDS for the Poisson model

Let us consider the case $\Gamma_\omega$ is the Poisson configuration and $\alpha_\gamma(\omega)$ is a constant sequence, that is, we assume the following.

**Assumption 4.1.** *(i) For any Borel set $E$ in $\mathbb{R}^2$, the random variable $\#(E \cap \Gamma_\omega)$ is measurable with respect to $\omega \in \Omega$, where $\#S$ is the number of elements in a set $S$.*

*(ii) For any disjoint Borel sets $\{E_j\}_{j=1}^n$ in $\mathbb{R}^2$, the random variables $\{\#(E_j \cap \Gamma_\omega)\}_{j=1}^n$ are independent.*

*(iii) There exists a positive constant $\rho$ (called the intensity) such that*

$$\mathbb{P}(\#(E \cap \Gamma_\omega) = k) = \frac{(\rho|E|)^k}{k!} e^{-\rho|E|} \quad (k = 0, 1, 2, \ldots)$$

*for any Borel set $E$ with $|E| < \infty$, where $|E|$ is the Lebesgue measure of $E$.*

*(iv) There exists a constant $\alpha$ with $0 < \alpha < 1$ such that $\alpha_\gamma(\omega) = \alpha$ for every $\gamma \in \Gamma_\omega$.*

Especially, Assumption 4.1 implies Assumption 2.1, so the Lifshitz tail holds in this case. By the Tauberian theorem, this fact implies the Laplace transform

$$\mathcal{L}(t) = \int_0^\infty e^{-t\lambda} dN(\lambda)$$

of the density of states $dN$ decays exponentially as $t \to \infty$. Moreover, recently we found the asymptotic behavior of $\mathcal{L}(t)$ as $t \to 0$ up to the constant term.

**Proposition 4.2.** *Suppose Assumption 4.1 holds.*

*(i) The Laplace transform $\mathcal{L}(t)$ of $dN$ is represented as*

$$\mathcal{L}(t) = \frac{1}{4\pi t} \mathbb{E}_{0,0,t,0} \left[ \exp \left( \rho \int_{\mathbb{R}^2} (\exp(-i\alpha\Theta_{t,\gamma}(w)) - 1)\, d\gamma \right) \right], \tag{18}$$

*where $d\gamma$ denotes the Lebesgue measure with respect to $\gamma \in \mathbb{R}^2$.*

*(ii) The asymptotics of $\mathcal{L}(t)$ as $t \to 0$ is given by*

$$\mathcal{L}(t) = \frac{1}{4\pi t} - \frac{\rho\alpha(1-\alpha)}{2} + O(t). \tag{19}$$

We remark that formulas similar to (18) are found in various contexts; see e.g. [Do-Va], [Na3].

*Outline of Proof.* (i) Since the system has the $\mathbb{R}^2$-stationarity, $\mathcal{L}(t)$ is represented as

$$\mathcal{L}(t) = \frac{1}{4\pi t} \mathbb{E}_P \left[ \mathbb{E}_{0,0,t,0} \left[ \exp \left( -i\alpha \sum_{\gamma \in \Gamma_\omega} \Theta_{t,\gamma}(w) \right) \right] \right], \tag{20}$$

where $\mathbb{E}_P$ denotes the expectation with respect to the probability space of the Poisson configuration. We approximate the Poisson configuration on $\mathbb{R}^2$ by the Poisson configuration on the finite square $\Lambda$ centered at the origin. The probability space $\Omega^\Lambda$ for the Poisson configuration on $\Lambda$ is the disjoint sum of the space of $k$-point configuration $\Lambda^k$, that is,

$$\Omega^\Lambda = \sum_{k=0}^{\infty} \Lambda^k,$$

where $\Lambda^k$ is the direct product of $k$-$\Lambda$'s ($\Lambda^0$ is a one-point set). The probability on $\Omega^\Lambda$ is given by

$$\mathbb{P}(\Lambda^0) = e^{-\rho|\Lambda|}$$

and for $k \geq 1$

$$d\mathbb{P}|_{\Lambda^k} = \frac{\rho^k}{k!} e^{-\rho|\Lambda|} d\gamma_1 \ldots d\gamma_k,$$

where $\gamma = (\gamma_1, \ldots, \gamma_k) \in \Lambda^k$ and $d\gamma_1 \ldots d\gamma_k$ is the Lebesgue measure on $\Lambda^k$. So

$$\begin{aligned}
&\mathbb{E}_P \left[ \exp\left( -i\alpha \sum_{\gamma \in \Gamma_\omega} \Theta_{t,\gamma}(w) \right) \right] \\
&= \lim_{\Lambda \to \mathbb{R}^2} \mathbb{E}_{\Omega^\Lambda} \left[ \exp\left( -i\alpha \sum_{\gamma \in \Gamma_\omega} \Theta_{t,\gamma}(w) \right) \right] \\
&= \lim_{\Lambda \to \mathbb{R}^2} \sum_{k=0}^{\infty} \frac{\rho^k}{k!} e^{-\rho|\Lambda|} \int_{\Lambda^k} \exp\left( -i\alpha \sum_{j=1}^{k} \Theta_{t,\gamma_j}(w) \right) d\gamma_1 \ldots d\gamma_k \\
&= \lim_{\Lambda \to \mathbb{R}^2} e^{-\rho|\Lambda|} \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \left( \int_\Lambda \exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) d\gamma \right)^k \\
&= \lim_{\Lambda \to \mathbb{R}^2} \exp\left( \rho \int_\Lambda \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma \right) \\
&= \exp\left( \rho \int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma \right).
\end{aligned}$$

Notice that the last integral converges since

$$\Theta_{t,\gamma}(w) = 0 \quad \text{for} \quad |\gamma| > \max_{0 \leq s \leq t} |w_s|. \tag{21}$$

Thus, changing the order of the expectation in (20), we have (18).

(ii) We consider the Taylor expansion of the first exponential function in (18)

$$\begin{aligned}
&\exp\left( \rho \int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma \right) \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \rho \int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma \right)^n.
\end{aligned} \tag{22}$$

By (21), we have

$$\left|\rho \int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma\right|^n \leq \left(2\pi\rho \max_{0\leq s\leq t} |w_s|^2\right)^n. \tag{23}$$

Here we review some formulas on the Brownian motion. The representation of the Brownian bridge by the Brownian motion [Na3, (3.1)]

$$w_s = b_s - (s/t)b_t \quad (0 \leq s \leq t), \tag{24}$$

where $b_t = (b_{1,t}, b_{2,t})$ is the 2-dimensional Brownian motion starting from 0 at time 0. The Doob inequality [Fu, Teiri 3.11]

$$\mathbb{E}\left[\max_{0\leq s\leq t} |b_{j,s}|^2\right]^{1/2} \leq 2\mathbb{E}[|b_{j,t}|^2]^{1/2} \quad (j = 1, 2). \tag{25}$$

The moments of the Brownian motion [Fu, section 2.3]

$$\mathbb{E}[|b_{j,t}|^{2p}] = (2p-1)!!t^p \quad (j = 1, 2, \quad p = 0, 1, 2, \ldots), \tag{26}$$

where $(2p-1)!! = (2p)!/(2^p p!)$. By using (23), (24), (25) and (26), we can prove that there exist positive constants $C$ and $t_0$ independent of $\rho$, $t$ and $n$ such that

$$\mathbb{E}_{0,0,t,0}\left[\frac{1}{n!}\left|\rho \int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma\right|^n\right] \leq C(\rho t)^n$$

for $0 \leq t \leq t_0$ and $n = 1, 2, \ldots$. So the substitution of the expansion (22) into (18) is justified for sufficiently small $t$. The $n$-th term in the resulting expansion is

$$\begin{aligned}
&\frac{\rho^n}{4\pi t n!}\mathbb{E}_{0,0,t,0}\left[\left(\int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma\right)^n\right] \\
= \;& \frac{\rho^n}{4\pi t n!}\mathbb{E}_{0,0,1,0}\left[\left(\int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{1,\sqrt{t}\gamma'}(\sqrt{t}w')\right) - 1\right) t d\gamma'\right)^n\right] \\
= \;& \frac{(\rho t)^n}{4\pi t n!}\mathbb{E}_{0,0,1,0}\left[\left(\int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{1,\gamma'}(w')\right) - 1\right) d\gamma'\right)^n\right],
\end{aligned}$$

where we used the change of variable $\gamma = \sqrt{t}\gamma'$, $w = \sqrt{t}w'$, and the scaling property of the 2-dimensional Brownian motion. Thus the $n$-th term is proportional to $t^{-1+n}$. Moreover, the constant term ($n = 1$) is calculated as follows.

$$\begin{aligned}
&\frac{\rho}{4\pi t}\mathbb{E}_{0,0,t,0}\left[\int_{\mathbb{R}^2} \left(\exp\left(-i\alpha\Theta_{t,\gamma}(w)\right) - 1\right) d\gamma\right] \\
= \;& \frac{\rho}{4\pi t}\int_{\mathbb{R}^2} \mathbb{E}_{0,x,t,x}\left[\left(\exp\left(-i\alpha\Theta_{t,0}(w)\right) - 1\right)\right] dx \\
= \;& \rho \int_{\mathbb{R}^2} \left(e^{-tH(A_\alpha)}(x,x) - e^{-tH(0)}(x,x)\right) dx \\
= \;& -\frac{\rho\alpha(1-\alpha)}{2},
\end{aligned}$$

where we used Theorem 3.3 in the last equality. □

Proposition 4.2 (ii) means the energy is *raised* in some averaged sense by randomly distributed Aharonov-Bohm solenoids; one solenoid causes the decrease of 'heat trace' by $\alpha(1-\alpha)/2$, so the solenoids distributed with intensity $\rho$ causes the decrease of 'heat trace' by $\rho\alpha(1-\alpha)/2$ in spatial average. This consideration suggests us the formula (19) could be generalized to $\mathcal{L}(t)$ for other stationary $\delta$ magnetic fields. We will argue this subject in the future work.

Of course, the most interesting problem is to obtain the detailed asymptotics of $\mathcal{L}(t)$ as $t \to \infty$, which is equivalent to obtain the optimal decaying order of the Lifshitz tail (actually, Borg seems to consider this problem in the case $\Gamma_\omega = \mathbb{Z}^2$, but not to succeed yet). In the case of the random *scalar* potential of the Poisson type, this subject is well-studied with the aid of the large deviation theory (see [Do-Va, Na3, Ue2]). In our case, it seems to require deep knowledge about the winding angle of the Brownian bridge, which is also interesting subject in itself. We hope the formulas (16) and (18) would give us an opportunity to develop the study of these subjects.

## References

[Bo] J. L. Borg, Random Schrödinger Operators with Magnetic Vortices, thesis at University College Dublin.

[Ca-La] R. Carmona and J. Lacroix, *Spectral theory of random Schrödinger operators, Probability and its Applications*, Birkhäuser Boston, Inc., Boston, MA, 1990.

[Do-Va] M. D. Donsker and S. R. S. Varadhan, Asymptotics for the Wiener Sausage, *Comm. Pure and Applied Math.* vol. XXVIII (1975), 525–565.

[Fu] N. Funaki, 確率微分方程式 *(Stochastic differential equations)* (Japanese), 岩波書店 (Iwanami), 1997.

[Ge-St] V. A. Geyler and P. Šťovíček, Zero modes in a system of Aharonov-Bohm fluxes, *Rev. Math. Phys.* **16** (2004), no. 7, 851–907.

[Gh] F. Ghribi, Internal Lifshits tails for random magnetic Schrödinger operators, *J. Funct. Anal.* **248** (2007), no. 2, 387–427.

[Ki] W. Kirsch, An invitation to random Schrödinger operators, With an appendix by Frédéric Klopp, Panor. Synthèses, 25, *Random Schrödinger operators*, 1–119, Soc. Math. France, Paris, 2008.

[Ki-Ma] W. Kirsch, and F. Martinelli, On the spectrum of Schrödinger operators with a random potential, *Commun. Math. Phys.* **85** (1982), 329–350.

[La-We] A. Laptev and T. Weidl, Hardy inequalities for magnetic Dirichlet forms, Mathematical results in quantum mechanics (Prague, 1998), *Oper. Theory Adv. Appl.*, **108** (1999), 299–305.

[Mi-No1] T. Mine and Y. Nomura, Schrödinger operators with random $\delta$ magnetic fields, in *Spectral and scattering theory and related topics*, RIMS KôKyûroku **1607** (2008), 1–8.

[Mi-No2] T. Mine and Y. Nomura, Schrödinger operators with random $\delta$ magnetic fields, submitted to *Ann. Henri Poincare.*

[Na1] S. Nakamura, Lifshitz tail for Schrödinger operator with random magnetic field, *Comm. Math. Phys.* **214** (2000), no. 3, 565–572.

[Na2] S. Nakamura, Lifshitz tail for 2D discrete Schrödinger operator with random magnetic field, *Ann. Henri Poincaré* **1** (2000), no. 5, 823–835.

[Na3] S. Nakao, On the spectral distribution of the Schrödinger operator with random potential, *Japan J. Math.* vol. 3, No. 1 (1977), 111–139.

[St] P. Stollmann, *Caught by disorder, Bound states in random media*, Progress in Mathematical Physics, 20, Birkhäuser Boston, Inc., Boston, MA, 2001.

[Ue1] N. Ueki, Simple examples of Lifschitz tails in Gaussian random magnetic fields, *Ann. Henri Poincaré* **1** (2000), no. 3, 473–498.

[Ue2] N. Ueki, 確率解析とランダムシュレディンガー作用素 (Stochastic analysis and random Schrödinger operators) (Japanese), 数学 *(Mathematics)*, vol. 66, no. 4 (2014), 岩波書店 (Iwanami), 392–412.