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Limit of Sine_{β} and Sch_{τ} processes

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Abstract

We discuss a simple generalization of the results by Allez-Dumaz [1] to study the behavior of Sine_{β}- and Sch_{τ}-proceses as $\beta\to 0, \infty$.

1 Introduction

We first explain the background of this problem. Let $H:=-\frac{d^{2}}{dt^{2}}+V$ be a Schrödinger operator on the real line and let $H_{L}:=H|_{[0,L]}$ be its Dirichlet realization on $[0, L]$. Let $\{E_{n}(L)\}_{n\geq 1}$ be the eigenvalues of H_{L} in the increasing order. Fix the reference energy $E_{0}>0$ arbitrary. To study the local distribution of $E_{n}(L)$'s near E_{0}, we consider the following point process :

$$\xi_{L}:=\sum_{n\geq n(L)}\delta_{L(\sqrt{E_{n}(L)}-\sqrt{E_{0}})} \tag{1.1}$$

where $n(L):=\min\{n|E_{n}(L)>0\}$; we only consider the positive eigenvalues. We take $\sqrt{E_{n}(L)}$ instead of $E_{n}(L)$ to unfold the eigenvalues with respect to the density of states. In [2, 4], we studied the behavior of ξ_{L} as L tends to infinity in the following two cases, some part of which can be regarded as a continuum analogue of [3].

(1) (**decaying potential**) We take $V(t):=a(t)F(X_{t})$, where $a\in C^{\infty}(\mathbf{R})$, $a(-t)=a(t)$, a is non-increasing for $t\geq 0$, and $a(t)=t^{-\alpha}(1+o(1))$, $t\to\infty$, $\alpha>0$, and M is a torus, $F\in C^{\infty}(M)$, F is non-constant with $\langle F\rangle:=\int_{M}F(x)dx=0$. We sometimes need to work under the following condition.

(A) The subsequence $\{L_j\}_{j=1}^{\infty}$ satisfies $L_j \overset{j\to\infty}{\to} \infty$ and

$$\sqrt{E_0}L_j = m_j\pi + \gamma + o(1), \quad j \to \infty$$

$m_j \in \mathbf{N}$, $\gamma \in [0, \pi)$.

Theorem 1.1
(1) [2] Let $\alpha > \frac{1}{2}$ and assume (A). Then we have a probability measure μ_γ on $[0, \pi]$ such that $\xi_\infty \overset{d}{=} \lim_{j\to\infty} \xi_{L_j}$ satisfies

$$\mathbf{E}[e^{-\xi_\infty(f)}] = \int_0^\pi d\mu_\gamma(\theta) \exp\left(- \sum_{n\in\mathbf{Z}} f(n\pi - \theta) \right).$$

(2) [4] Let $\alpha = \frac{1}{2}$. Then $\xi_\infty \overset{d}{=} \lim_{L\to\infty} \xi_L$ is the Sine$_\beta$-process $\zeta_{Sine,\beta}$ [5] with $\beta = \beta(E_0) := 8E_0/C(E_0)$. $C(E)$ is defined in Theorem 1.2.

(2) (**decaying coupling constant**) We let the potential $V(t) := \lambda_L$ constant but $\lambda_L = L^{-\alpha}$, $\alpha > 0$ is size dependent.

Theorem 1.2 *[4]*
(1) Assume (A) and $\alpha > \frac{1}{2}$. Then $\xi_\infty \overset{d}{=} \lim_{j\to\infty} \xi_{L_j}$ satisfies

$$\mathbf{E}[e^{-\xi_\infty(f)}] = \exp\left(- \sum_{n\in\mathbf{Z}} f(n\pi - \gamma) \right).$$

(2) Assume (A) and $\alpha = \frac{1}{2}$. Then $\xi_\infty \overset{d}{=} \lim_{j\to\infty} \xi_{L_j}$ satisfies

$$\mathbf{E}[e^{-\xi_\infty(f)}] = \mathbf{E}\left[\exp\left(- \sum_{n\in\mathbf{Z}} f(\Psi_1^{-1}(2n\pi - 2\gamma)) \right) \right]$$

where $\Psi_t(c)$ is a strictly-increasing function valued process such that for any $c_1, c_2, \cdots, c_m$, $\Psi_t(c_1), \cdots, \Psi_t(c_m)$ jointly satisfy the following SDE.

$$d\Psi_t(c_j) = \left(2c_j - Re\, \frac{i}{2E_0} \langle F g_{\sqrt{E_0}} \rangle \right) dt + \frac{1}{\sqrt{E_0}} \left\{ \sqrt{\frac{C(E_0)}{2}} Re\, \left(e^{i\Psi_t(c_j)} dZ_t \right) + \sqrt{C(0)} dB_t \right\} \quad (1.2)$$

$j = 1, 2, \cdots, m$, where Z_t is a complex Brownian motion independent of a Brownian motion B_t and

$$g_{\sqrt{E_0}} := (L + 2i\sqrt{E_0})^{-1}F, \quad g := L^{-1}(F - \langle F \rangle),$$
$$C(E_0) := \int_M |\nabla g_{\sqrt{E_0}}|^2 dx, \quad C(0) := \int_M |\nabla g|^2 dx.$$

This SDE is the same as that satisfied by the phase function of "Sch$_\tau$" process [3] up to scaling. Thus we abuse the notation and call ξ_∞ Sch$_\tau$ - process and denote it by ζ_{Sch}.

To summarize both cases, for the extended case $\alpha > \frac{1}{2}$, ξ_L converges to a version of clock process, while for the critical case $\alpha = \frac{1}{2}$, ξ_L converges to those originating from the random matrix theory. If $\alpha < \frac{1}{2}$, we have no results but believe that ξ_∞ is a Poisson process.

The purpose of this note is state the behavior of the limiting point processes for the critical case ($\alpha = \frac{1}{2}$) as $\beta \to 0, \infty$. For Sine$_\beta$ - process, we have

Theorem 1.3
(1) $\zeta_{Sine,\beta} \to \zeta_{clock}$ as $\beta \to \infty$, where ζ_{clock} is a clock process satisfying

$$\mathbf{E}[e^{-\zeta_{clock}(f)}] = \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(- \sum_{n \in \mathbf{Z}} f(2n\pi + \theta) \right).$$

(2) (Allez-Dumaz [1]) $\zeta_{Sine,\beta} \to Poisson(d\lambda/2\pi)$ as $\beta \to 0$, where $Poisson(\mu)$ is the Poisson point process with intensity measure μ.

Since $\beta(E_0)$ is strictly monotone increasing function of E_0 and since $\lim_{E_0 \downarrow 0} \beta(E_0) = 0$, $\lim_{E_0 \uparrow \infty} \beta(E_0) = \infty$, Theorem 1.3 is reasonable in view of Theorem 1.1.

Sch$_\tau$-process is not stationary but invariant under the shift of 2π, so that we need some modification. Let $U := unif[0, 2\pi]$ be a uniform distribution on $[0, 2\pi]$ independent of ζ_{Sch}. Writing $\zeta_{Sch} =: \sum_j \delta_{\lambda_j}$, let

$$\tilde{\zeta}_{Sch,\beta} := \sum_j \delta_{\tilde{\lambda}_j}, \quad \tilde{\lambda}_j := 2\lambda_j + U.$$

We used the terminology $\tilde{\zeta}_{Sch,\beta}$ instead of $\tilde{\zeta}_{Sch}$ because the law of that turns out to depend only on $\beta = \beta(E_0)$. The set of atoms of $\tilde{\zeta}_{Sch,\beta}$ is equal to Sch_τ^* (translation invariant "version" of Sch_τ) in [3] with $\tau = \frac{4}{\beta}$.

Theorem 1.4
(1) $\tilde{\zeta}_{Sch,\beta} \to \zeta_{clock}$ *as* $\beta \to \infty$.
(2) $\tilde{\zeta}_{Sch,\beta} \to Poisson(d\lambda/2\pi)$ *as* $\beta \to 0$.

Remark 1.1 *Suppose* $f \in C[0,\infty)$ *is a non-increasing function with* $f(0) > 0$, $f \geq 0$, $\int_0^\infty f(t)dt = 1$ *and* $\lim_{t\to\infty} f(t) = 0$. *Let* $\alpha_t^f(\lambda)$ *be the solution to*

$$d\alpha_t^f(\lambda) = \lambda \frac{\beta}{4} f\left(\frac{\beta}{4}t\right) dt + Re[(e^{i\alpha_t^f(\lambda)} - 1)dZ_t], \quad \alpha_0^f(\lambda) = 0$$

and let $\zeta_{f,\beta}$ *be the point process whose counting function is given by* $N_f[0,\lambda] = \alpha_\lambda^f(\infty)/2\pi$. *Sine*$_\beta$ *- process is a special case where* $f(t) = e^{-t}$. *It is straightforward to extend the result in [1] to show that*

$$\zeta_{f,\beta} \to Poisson(d\lambda/2\pi), \quad \beta \to 0.$$

We can also show Theorem 1.4(2) by using this convergence.

The idea of proof of Theorem 1.4(2) is due to the work by Allez-Dumaz [1] which is outlined in Section 2. Theorem 1.3(1), 1.4(1) is proved in Section 3. But the idea of that is suggested by B. Valkó.

2 High temperature limit

We outline the proof of $\beta \to 0$ limit. Let $A := \{\lambda \in \mathbf{R} \mid \Psi_1(\lambda) \in 2\pi\mathbf{Z}\}$. By examining the SDE (1.2) satisfied by $\Psi_t(\lambda)$, $A + \theta = \{\lambda \in \mathbf{R} \mid \Psi_1(\lambda) \in 2\pi\mathbf{Z} + \theta\}$. Hence the set of atoms of $\tilde{\xi}_{\infty,\beta}$ satisfies

$$\{2\lambda_j\} + U = \{\lambda \in \mathbf{R} \mid \Psi_1(\lambda) \in 2\pi\mathbf{Z} + U'\}$$

where $U' = U - 2\gamma$ is an uniform distribution on $[0, 2\pi]$. Let

$$\alpha_t(\lambda) := \Psi_t(\lambda) - \Psi_t(0).$$

We shall show below that the point process ζ_β whose set of atoms is equal to

$$S := \{\lambda \in \mathbf{R} | \alpha_\lambda(1) \in 2\pi\mathbf{Z}\} = \{\lambda | \Psi_1(\lambda) \in 2\pi\mathbf{Z} + \Psi_1(0)\}$$

converges to $Poisson(d\lambda/2\pi)$ as $\beta \to 0$ from which Theorem 1.4(2) follows. The proof of $\zeta_\beta \overset{\beta\to 0}{\to} Poisson(d\lambda/2\pi)$ consists of following two steps.

Step 1 : $\zeta_\beta[\lambda_1, \lambda_2] \overset{\beta\to 0}{\to} Poisson((\lambda_2 - \lambda_1)/2\pi)$ where $Poisson(\lambda)$ obeys the Poisson law with parameter $\lambda \in \mathbf{R}$.

Step 2 : If $\lambda_1 < \lambda_2 < \lambda_3$, the limits of $\zeta_\beta[\lambda_1, \lambda_2]$, $\zeta_\beta[\lambda_2, \lambda_3]$ are independent.

Step 1 : By (1.2), $\alpha(\lambda)$ satisfies

$$d\alpha_t(\lambda) = \lambda dt + \frac{2}{\sqrt{\beta}} Re\left[(e^{i\alpha_t(\lambda)} - 1)dZ_t\right], \quad t \in [0, 1],$$

so that by the time change $t = cs$, $c = \frac{\beta}{4}$, we have

$$d\alpha_s(\lambda) = \frac{\beta}{4}\lambda ds + Re\left[(e^{i\alpha_s(\lambda)} - 1)dZ_s\right], \quad s \in \left[0, \frac{4}{\beta}\right].$$

For fixed $\lambda \in \mathbf{R}$, let $\lambda \in \mathbf{R}$

$$\zeta_k := \inf\{t \geq 0 | \alpha_t(\lambda) \geq 2k\pi\}$$

be the k-th jump time of $\left\lfloor \frac{\alpha_t(\lambda)}{2\pi} \right\rfloor$. Note that $\left\lfloor \frac{\alpha_t(\lambda)}{2\pi} \right\rfloor$ is non-decreasing w.r.t. t. By analyzing the SDE satisfied by $\log \tan \frac{\alpha_t(\lambda)}{4}$, we can show that, under $\alpha_0(\lambda) \overset{\beta\to 0}{\to} 0$, $\zeta_1 / \frac{8\pi}{\beta\lambda}$ converges to the exponential distribution of parameter 1. Thus letting $\mu_\lambda^\beta[0, t]$ be the empirical measure of scaled $\{\zeta_k\}$

$$\mu_\lambda^\beta[0, t] := \sum_{k \geq 1} \delta_{\zeta_k}\left[0, \frac{8\pi}{\beta} t\right], \quad t \leq \frac{1}{2\pi},$$

we have

$$\mu_\lambda^\beta \to \mathcal{P}_\lambda := Poisson\left(\lambda 1_{[0, \frac{1}{2\pi}]} dt\right), \quad \beta \to 0.$$

Moreover, using the fact that $\alpha_t(\lambda') - \alpha_t(\lambda) \overset{d}{=} \alpha_t(\lambda' - \lambda)$, we have

$$\zeta_\beta[\lambda, \lambda'] \overset{d}{=} \mu_{\lambda' - \lambda}^\beta\left[0, \frac{1}{2\pi}\right] \overset{\beta\to 0}{\to} Poisson\left(\frac{\lambda' - \lambda}{2\pi}\right).$$

Step 2 : Let $0 < \lambda < \lambda' < \lambda''$. Since $\alpha_t(\lambda)$, $\alpha_t(\lambda')$, $\alpha_t(\lambda'') - \alpha_t(\lambda')$ are driven by the same Brownian motion, Z_t, the limits $\mathcal{P}_\lambda$, $\mathcal{P}_{\lambda'}$, $\mathcal{P}_{\lambda''-\lambda'}$ of μ_λ^β, $\mu_{\lambda'}^\beta$, $\mu_{\lambda''-\lambda'}^\beta$ are jointly realized as Poisson point processes under the same filtration. Let A_λ, $A_{\lambda'}$, $A_{\lambda''-\lambda'}$ be the corresponding set of atoms. We show that

$$A_\lambda \subset A_{\lambda'}, \quad A_{\lambda'} \cap A_{\lambda''-\lambda'} = \emptyset$$

which shows the independence of $\mathcal{P}_\lambda$ and $\mathcal{P}_{\lambda''-\lambda'}$ which, in turn, shows the independence of $\zeta_\beta[0, \lambda]$ and $\zeta_\beta[\lambda'', \lambda']$.

3 Low temperature limit

We study the $\beta \to \infty$ limit. We prove Theorem 1.3(1) only ; the proof of Theorem 1.4(1) is easier. The Laplace transform of Sine_β-process has the following representation [2].

$$\mathbf{E}[e^{-\zeta_{Sine,\beta}(f)}] = \int_0^{2\pi} \frac{d\theta}{2\pi} \mathbf{E}\left[\exp\left(-\sum_{n\in\mathbf{Z}} f\left((\Psi_1^{(\beta)})^{-1}(2n\pi+\theta)\right)\right)\right]$$

where $\Psi_t^{(\beta)}(\lambda)$ is increasing function valued process and the unique solution of the following SDE.

$$d\Psi_t^{(\beta)}(\lambda) = \lambda dt + \frac{2}{\sqrt{\beta t}} Re\left[\left(e^{i\Psi_t^{(\beta)}(\lambda)} - 1\right) dZ_t\right], \quad \Psi_0^{(\beta)}(\lambda) = 0.$$

By [2], Lemma 3.1, it suffices to show

$$\Psi_1^{(\beta)}(\lambda) \overset{\beta\uparrow\infty}{\to} \lambda, \quad a.s. \tag{3.1}$$

By using the estimate in [2], Lemma 6.4

$$\mathbf{E}[|\Psi_t^{(\beta)}(\lambda)|] \le Ct,$$

where the positive constant C is bounded w.r.t. β, we have

$$\mathbf{E}[|\Psi_t^{(\beta)}(\lambda) - \lambda t|^2] = \frac{4}{\beta}\int_0^t \mathbf{E}[|e^{i\Psi_t^{(\beta)}(\lambda)} - 1|^2]\frac{2ds}{s} \overset{\beta\uparrow\infty}{\to} 0$$

which yields (3.1) for some subsequence.

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References

[1] Allez, R., Dumaz, L., : From sine kernel to Poisson statistics, arXiv. 1407.5402.

[2] Kotani, S., Nakano, F., : Level statistics for the one-dimensional Schrödinger operators with random decaying potential, Interdisciplinary Mathematical Sciences Vol. 17 (2014), 343-373.

[3] Kritchevski, E., Valkó, B., and Virág, B., : The scaling limit of the critical one-dimensional random Schrödinger operators, Comm. Math. Phys. **314**(2012), 775-806.

[4] Nakano, F : Level statistics for one-dimensional Schrödinger operators and Gaussian beta ensemble, J. Stat. Phys. **156**(2014), 66-93.

[5] Valkó, B. and Virág, V. : Continuum limits of random matrices and the Brownian carousel, Invent. Math. **177**(2009), 463-508.

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