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Some Numerical Results of Rising Bubble Problems

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1 Introduction

We consider two-fluid flow problems, where each fluid is governed by the Navier-Stokes equations and the surface tension proportional to the curvature acts on the interface. The domain which each fluid occupies is unknown, and the interface moves with the velocity of the particle on it. While numerical solution of one-fluid flow problems governed by the Navier-Stokes equations has been successfully established from the point of stability and convergence, it is not an easy task to construct numerical schemes solving the two-fluid flow problems. To the best of our knowledge there are no numerical schemes whose solutions are proved to converge to the exact one and there is very little discussion even for the stability of schemes [1]. Recently we have developed an energy-stable Lagrange-Galerkin finite element scheme for the two-fluid flow problems [8]. The scheme is an extension of the energy-stable finite element scheme proposed by us [6, 7] to the Lagrange-Galerkin method. In this report we present some numerical results of rising bubble problems solved by it.

2 Two-fluid flow problems

Let Ω be a bounded domain in $\mathbf{R}^2$ with piecewise smooth boundary Γ, and $(0,T)$ be a time interval. The domain Ω is occupied by $m+1$ immiscible incompressible viscous fluids. Each fluid k, whose density and viscosity are ρ_k and μ_k, occupies an unknown domain $\Omega_k(t)$ at time t. Fluid $k(=1,\cdots,m)$ is surrounded by fluid 0, and the surface tension acts on the interface $\Gamma_k(t)$. Let the coefficient of the surface tension be σ_k. $\Gamma_k(t)$ is expressed as a closed curve,

$$\Gamma_k(t)=\{\chi_k(s,t);\ s\in[0,1)\},$$

where

$$\chi_k:[0,1)\times(0,T)\rightarrow\mathbb{R}^2,\ \chi(1,t)=\chi(0,t)\quad(t\in(0,T))$$

is a function to be determined. $\Omega_k(t)$, $k=1,\cdots,m$, is the interior of $\Gamma_k(t)$, and fluid 0 occupies

$$\Omega_0(t)=\Omega\setminus\bigcup\{\bar{\Omega}_k(t);\ k=1,\cdots,m\}.$$

Unknown functions (u,p), velocity and pressure,

$$u : \Omega \times (0,T) \to \mathbb{R}^2, \quad p : \Omega \times (0,T) \to \mathbb{R}$$

and χ_k satisfy the system of equations,

$$\rho_k \left\{ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right\} - \nabla \Big[2\mu_k D(u) \Big] + \nabla p = \rho_k f, \quad x \in \Omega_k(t), \ t \in (0,T) \tag{1a}$$

$$\nabla \cdot u = 0, \quad x \in \Omega_k(t), \ t \in (0,T) \tag{1b}$$

$$[u] = 0, \ [- pn + 2\mu D(u) n] = \sigma_k \kappa n, \quad x \in \Gamma_k(t), \ t \in (0,T) \tag{1c}$$

$$\frac{\partial \chi_k}{\partial t} = u(\chi_k, t), \quad s \in [0,1), \ t \in (0,T) \tag{1d}$$

$$u \cdot n = 0, \ D(u) n \parallel n, \ x \in \Gamma, \ t \in (0,T) \tag{1e}$$

$$u = u^0, \quad x \in \Omega, \ t = 0 \tag{1f}$$

$$\chi_k = \chi_k^0, \quad s \in [0,1), \ t = 0, \tag{1g}$$

where $k = 0, \cdots, m$ in (1a) and (1b), $k = 1, \cdots, m$ in (1c), (1d) and (1g), and

$$f : \Omega \times (0,T) \to \mathbb{R}^2, \quad u^0 : \Omega \to \mathbb{R}^2, \quad \chi_k^0 : [0,1) \to \mathbb{R}^2$$

are given functions; f is an acceleration, u^0 is an initial velocity, χ_k^0 is a function showing the initial interface position. $[\cdot]$ means the difference of the values approached from both sides to the interface, κ is the curvature of the interface, and n is the unit normal. On the boundary of Ω the slip boundary condition (1e) is imposed.

Lagrange-Galerkin method has nice features for the approximation of material derivative terms [2, 3, 4, 5]. Since the basic idea is to approximate the particle movement along characteristic curves, the method is robust for high Reyonolds number problems. Recently developed energy-stable Lagrange-Galerkin scheme for two-fluid flow problems is an extension of the energy-stable finite element scheme [6, 7] to the Lagrange-Galerkin method. It has the following advanteges. For the details refer to [8].

- It is stable in the sense of energy if an integral of the square of approximate curvature of the interface remains bounded.
- Since the resultant matrix is symmetric, we can use efficient solvers for symmetric system of linear equations, e.g., MINRES.
- Since we use the interface-tracking method, we can distribute much more nodes on the interface than the level-set method.
- When it is applied to incompressible viscous one-fluid flow problems, the stability and convergence is assured.
- Since the main computation part is similar to that of the Stokes problem, the computation is light.

We apply this scheme to rising bubble problems to analyse the effect of the viscosity and the coefficient of surface tension on the behavior of bubbles.

3 Numerical results

3.1 Example 1

Let $m = 1$ and set

$$\begin{aligned}
&\Omega = (0,1) \times (0,4),\\
&\Omega_1 = \{(x_1, x_2);\ (x_1 - a)^2 + (x_2 - 2a)^2 < a^2\},\ a = \frac{1}{5},\\
&\rho_0 = 100,\ \mu_0 = 0.05, 0.5, 5.0,\ \rho_1 = 0.1,\ \mu_1 = 1.0,\\
&f = (0, -1)^T,\ \sigma_1 = 2.0.
\end{aligned}$$

When the viscosity μ_0 of fluid 0 varies, we observe the change of the bubble movement depending on μ_0. The mesh for the computation is shown in the left of Fig. 1. We set the time increment $\Delta t = 1/16$. Figs. 2, 3 and 4 show the time histories of the interfaces and streamlines when $\mu_0 = 5.0, 0.5, 0.05$. When $\mu_0 = 5.0$, that is, the viscosity is large, the rising speed of the bubble is slow and any wakes are hardly visible after the bubble. When $\mu_0 = 0.5$, that is, the viscosity decreases, the rising speed of the bubble increases and there appear large wakes after the bubble. When $\mu_0 = 0.05$, that is, the viscosity is small, the rising speed of the bubble becomes high and there appears oscillation when the bubble rises up. The wake has a pattern similar to the Kármán vortex in the flow past a circular cylinder.

3.2 Example 2

Let $m = 1$ and set

$$\begin{aligned}
&\Omega = (0,1) \times (0,2),\\
&\Omega_1 = \{(x_1, x_2);\ (x_1 - a)^2 + (x_2 - 2a)^2 < a^2\},\ a = \frac{1}{5},\\
&\rho_0 = 100,\ \mu_0 = 0.05,\ \rho_1 = 0.1,\ \mu_1 = 1.0,\\
&f = (0, -1)^T,\ \sigma_1 = 2.0, 4.0.
\end{aligned}$$

When the coefficient of surface tension σ_1 varies, we observe the change of the bubble shape depending on σ_1. The mesh for the computation is shown in the right of Fig. 1. We set the time increment $\Delta t = 1/32$. Figs. 5 and 6 show the time histories of the interfaces and streamlines when $\sigma_1 = 2.0$ and 4.0. When σ_1 increase from 2.0 to 4.0, the shapes of the bubbles become more round and the change of the bubble shapes in the time history becomes smaller.

4 Concluding remarks

We have analyzed numerically the effect of the viscosity and the coefficient of surface tension on the behavior of rising bubbles. Our scheme is an interface-tracking method and the required memory is small. In Example 1 the number of elements

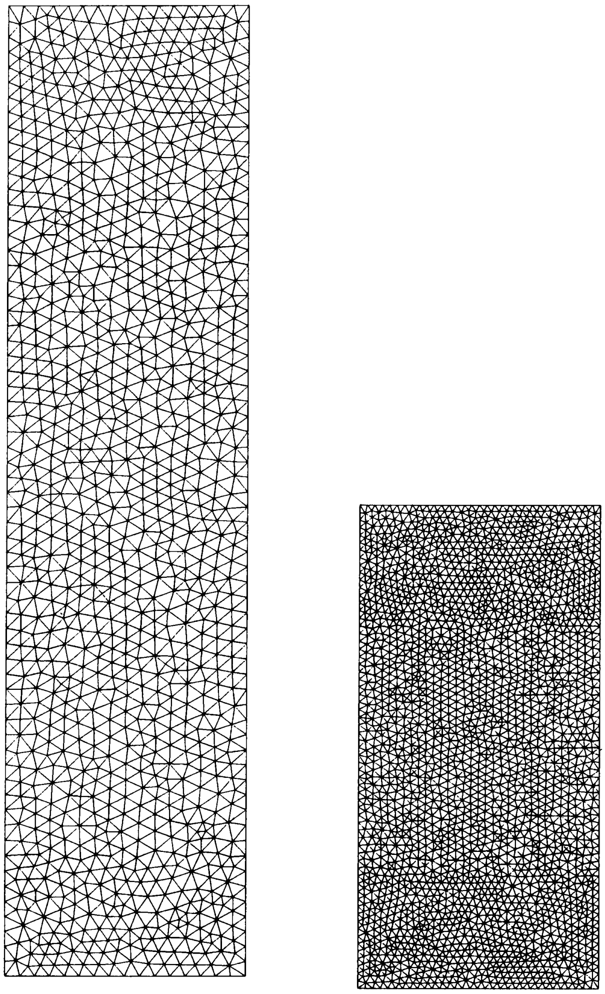

Figure 1: Meshes for Example 1(left) and Example2(right).

N_e is 2,174 and the number of degrees of freedom DOF(u,p) of the velocity and pressure is 10,186. In Example 2 N_e is 4,564 and DOF(u,p) is 21,021. DOF(u,p) is equal to the size of the system of linear equations solved at each time step. The P2/P1/P0 finite element spaces are used for the approximation of u, p and ρ.

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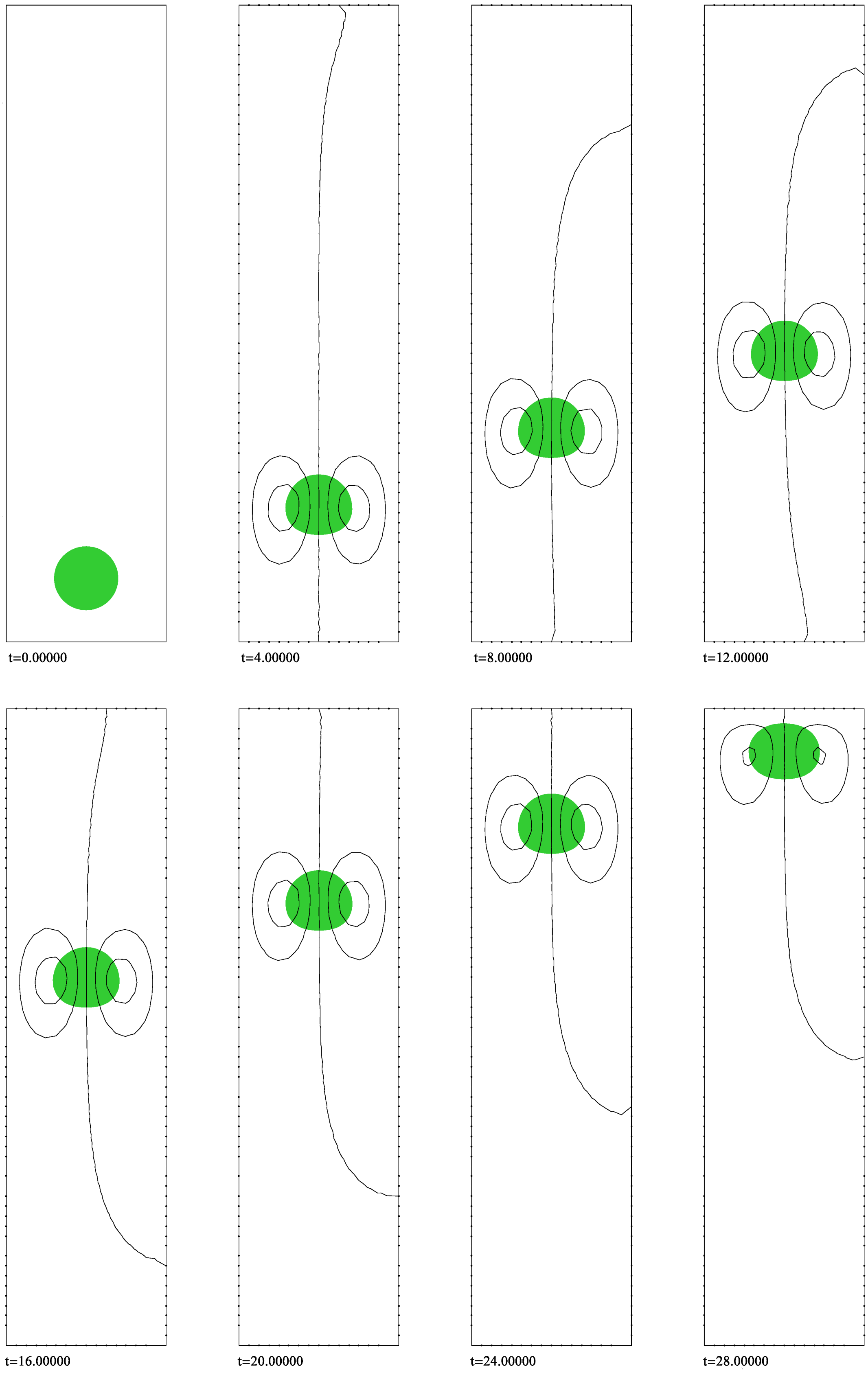

Figure 2: Interfaces and streamlines, $\mu_0 = 5.0$, $t = 0, 4, \cdots, 28$.

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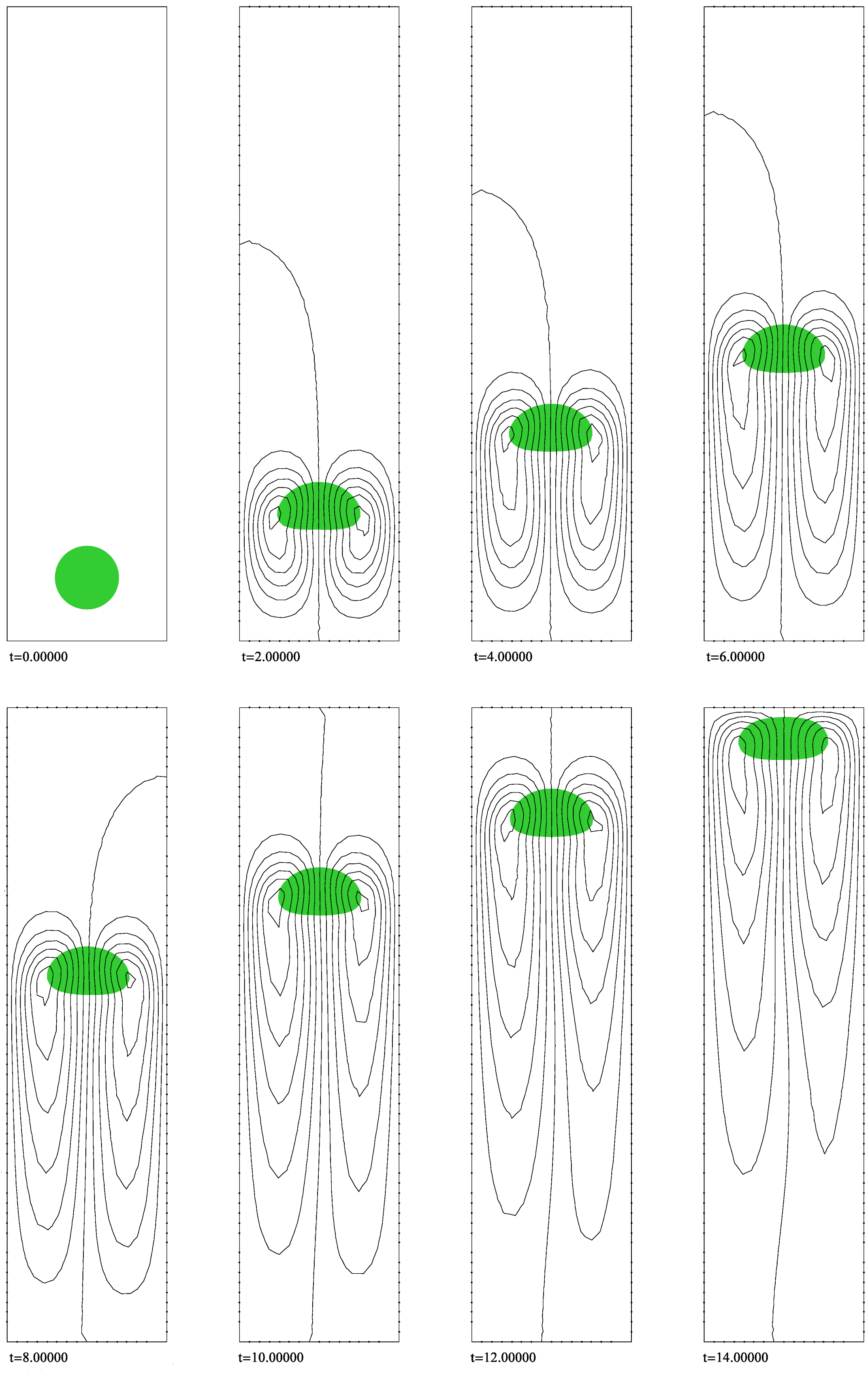

Figure 3: Interfaces and streamlines, $\mu_0 = 0.5$, $t = 0, 2, \cdots, 14$.

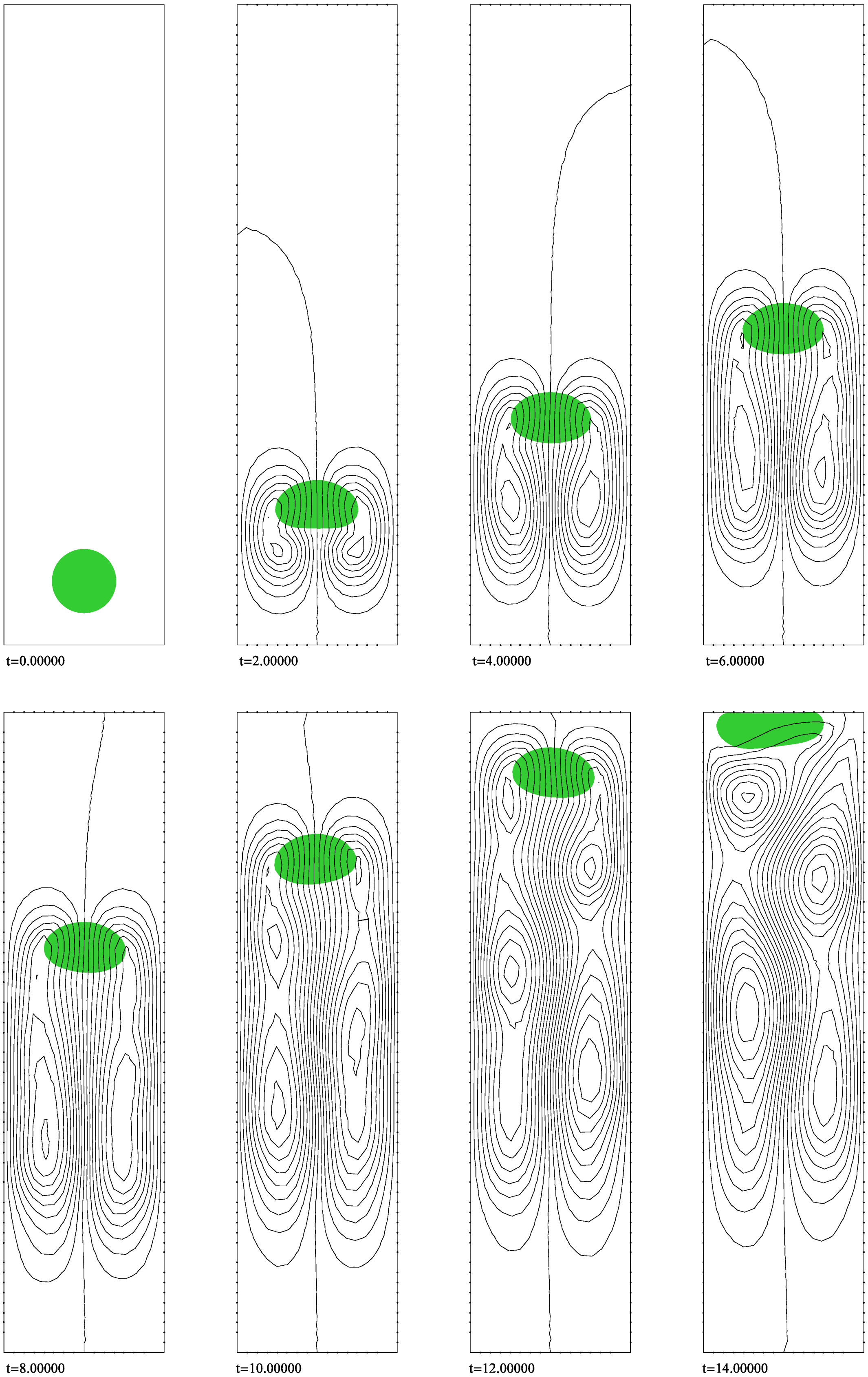

Figure 4: Interfaces and streamlines, $\mu_0 = 0.05,\ t = 0, 2, \cdots, 14.$

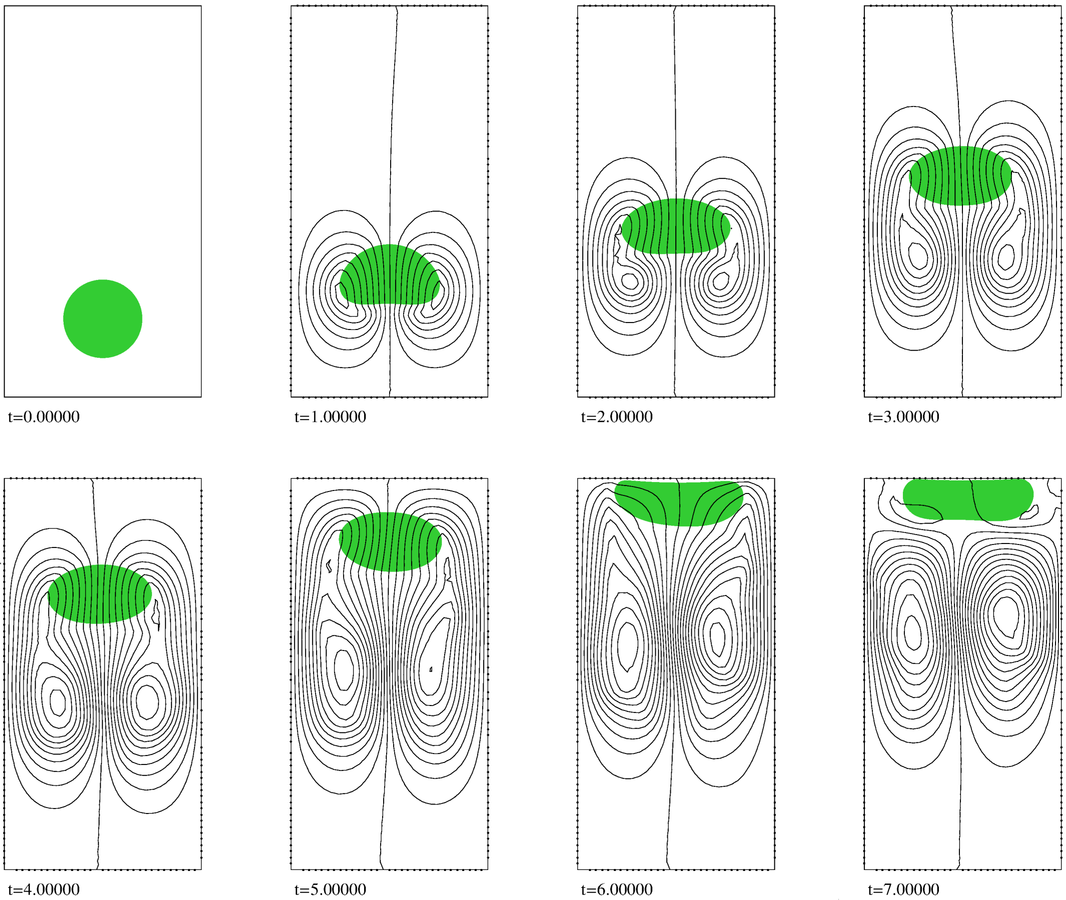

Figure 5: Interfaces and streamlines, $\sigma_1 = 2.0$, $t = 0, 1, \cdots, 7$.

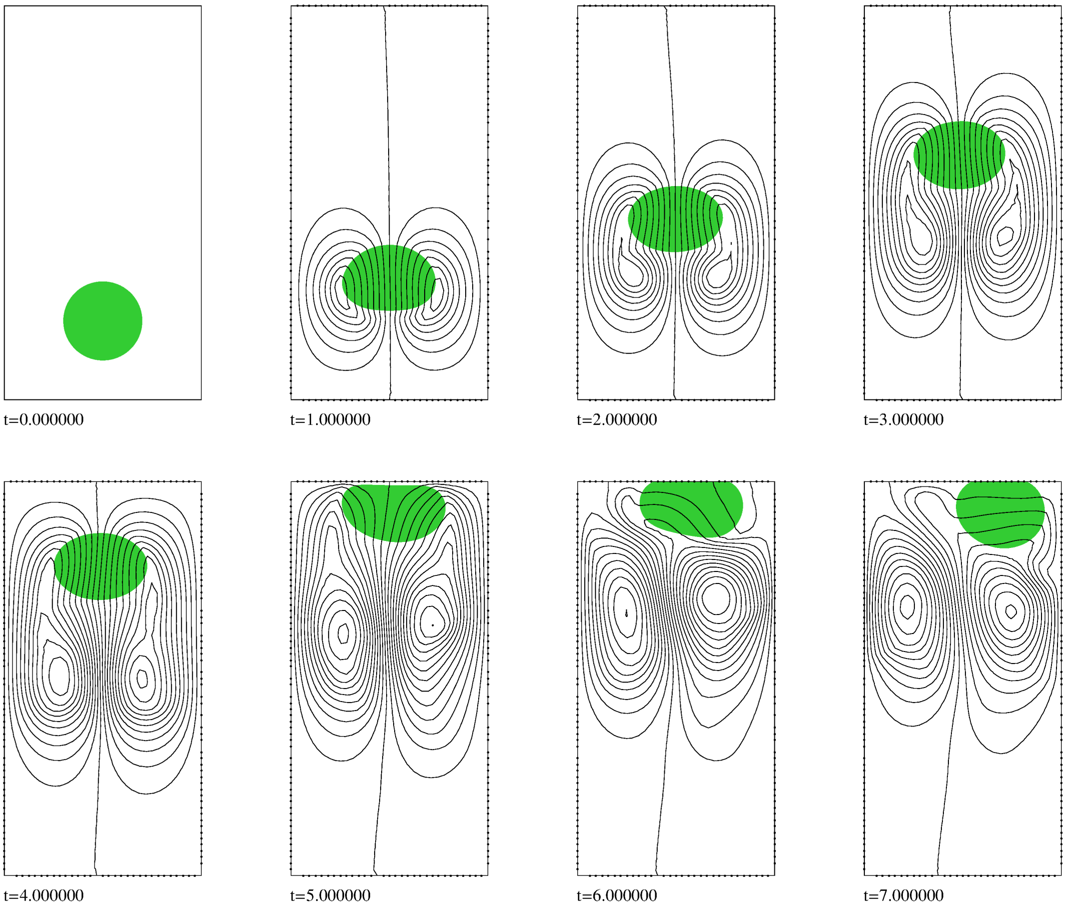

Figure 6: Interfaces and streamlines, $\sigma_1 = 4.0$, $t = 0, 1, \cdots, 7$.