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# Global Existence of $L^2$ Solutions of the Zakharov Equations with Additive Noises

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## 1 Introduction

We consider the almost sure global solvability of the Cauchy Problem for one dimensional Zakharov equations with additive noises:

$$idu = \left(-\partial_x^2 u + nu\right)dt + \Phi_1 dW_1, \tag{1}$$

$$d(\partial_t n) = \partial_x^2\left(n + |u|^2\right)dt + \Phi_2 dW_2, \tag{2}$$

$$t > 0, \quad x \in \mathbf{R},$$

$$\left(u, n, \partial_t n\right)(0) = (u_0(x), n_0(x), n_1(x)), \tag{3}$$

where $u : [0,\infty) \times \mathbf{R} \to \mathbf{C}$ is the slowly varying envelope of the electiric field, $n : [0,\infty) \times \mathbf{R} \to \mathbf{R}$ is the deviation of the ion density from the mean background density, and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Here, $W_j = \sum_{k=1}^{\infty} \beta_k^{(j)} e_k$, $j = 1, 2$, a sequence $\{e_k\}$ is the CONS in $L^2(\mathbf{R})$, $\{\beta_k^{(1)}\}$, $\{\beta_k^{(2)}\}$ are mutually independent complex and real Brownian motions associated with filtration $\{\mathcal{F}_t\}_{t\geq 0}$, respectively, and $\Phi_j : L^2 \to H^{s_j}$ are Hilbert-Schmidt operators for some $s_j \in \mathbf{R}$, $j = 1, 2$. Equations (1) - (2) without additive noises are the mathematical model which describes the Langmuir turbulence in a plasma.

The Zakharov equations with additive noises have its origin in geophysics. The system of equations (1) and (2) describes the geophysical phenomenon called NEIAL (Naturally Enhanced Ion-Acoustic Lines), in which one can observe spectrum lines of electro-magnetic waves generated by the ion-acoustic turbulence in plasmas of the ionosphere about 300 km above ground (see [8]). Because of thermal fluctuations, generally, the spectra scattered from the ionosphere are braod and noisy. This is why the random effect should be introduced into the system. These fluctuations contain, among others, featurs associated with ion-acoustic waves driven by random motions within the plasma. In this case, $\Phi_1 dW_1$ is a noise caused by the Cherenkov emission, and $\Phi_2 dW_2$ is a noise caused by fluctuations of background ion density.

If the external forcing terms vanish, we have two conservation laws for (1) and (2), which play an important role for the proof of the global existence of solutions in the deterministic case.

**Conservation Laws** ($\Phi_j = 0,\ j = 1, 2$)

$$\begin{aligned}&\text{(Mass Conservation)}\\ &\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t > 0,\\ &\text{(Energy Conservation)}\\ &E(u, n, \partial_t n)(t) = E(u_0, n_0, n_1), \quad t > 0,\end{aligned}$$

where

$$\begin{aligned}E(u(t), n(t), \partial_t n(t)) &= \|\nabla u\|_{L^2}^2\\ &+ \frac{1}{2}(\|n\|_{L^2}^2 + \|(-\Delta)^{-1/2}\partial_t n\|_{L^2}^2)\\ &+ \int_{\mathbf{R}} n|u|^2\, dx.\end{aligned}$$

In the present paper, we consider the *almost sure* global existence of solutions for (1)-(3) under proper assumptions on covariance operators $\Phi_j$.

**Remark 1.1** *Let $I$ be the identity operator and let $\varphi$ be a cut-off function in space. If $\Phi_j = \varphi I$, then $\Phi_j dW_j$ is called the localized space-time white noise. In this case,*

$$\begin{aligned}&\Phi_j = \varphi I : L^2 \to H^s, \textit{Hilbert-Schmidt}\\ &\iff s < -1/2.\end{aligned}$$

*The Itô integral makes sense in infinite dimensions if the covariance operators $\Phi_j \Phi_j^*$ are trace class or equivalently $\Phi_j$ are Hilbert-Schmidt. We note $\Phi_j$ are often called the covariance operators, though the covariance operators originally mean $\Phi_j \Phi_j^*$. In the context of the stochastic one dimensional Zakharov, in order to handle localized space-time white noises, we need to construct the solutions $(u, n) \in H^{s_1} \times H^{s_2}$ for $s_1 < -1/2$ and $s_2 < 1/2$ (note that for the solution $n(t)$ of ion-acoustic wave part, one can gain an additional regularity of one derivative from the external forcing term).*

# 2 Main Theorem

Before stating the main theorem, we introduce the Fourier restriction spaces. For $s$, $b \in \mathbf{R}$, we define spaces $X^{s,b}$ and $Y^{s,b}$ as follows.

$$X^{s,b} = \{u \in \mathcal{S}'(\mathbf{R}^2);\ \|u\|_{X^{s,b}} = \|(1+\xi^2)^{s/2}(1+|\tau-\xi^2|)^b\hat{u}\|_{L^2(\mathbf{R}^2)} < \infty\},$$
$$Y^{s,b} = \{u \in \mathcal{S}'(\mathbf{R}^2);\ \|u\|_{Y^{s,b}_{\pm}} = \|(1+\xi^2)^{s/2}\big(1+|\tau \pm |\xi||\big)^b\hat{u}\|_{L^2(\mathbf{R}^2)} < \infty\},$$

where $\hat{u}$ denotes the Fourier transform in space and time of $u$. Let $\psi \in C^\infty(\mathbf{R}\backslash\{0\})$ be a time cut-off function such that $\psi(t) = 1$ $(0 < t \leq 1)$, $\psi(t) = 0$ $(t < 0, t \geq 2)$. We put $\psi_T(t) = \psi(t/T)$ for $T > 0$. We denote $[0, \infty)$ by $\mathbf{R}_+$.

The main thoerem in the present paper is the following.

**Theorem 2.1** *Let $\varepsilon$ be an arbitrary positive number. Assume that*

$$\text{(HS)} \qquad \Phi_1 : L^2 \to H^\varepsilon,\ \Phi_2 : L^2 \to H^{-3/2};\ \textit{Hilbert-Schmidt.}$$

*Then, for any $(u_0, n_0, n_1) \in L^2 \times H^{-1/2} \times H^{-3/2}$, there exist unique global solutions $(u, n)$ of (1)-(3) a.s. such that*

$$(u, n, \partial_t n) \in C(\mathbf{R}_+; L^2 \times H^{-1/2} \times H^{-3/2}), \tag{4}$$
$$\psi_T u \in X^{0,1/3}, \quad \psi_T n \in Y^{-1/2,1/3}, \quad \psi_T\partial_t n \in Y^{-3/2,1/3}, \quad T > 0. \tag{5}$$

The mass conservation law yields the *a priori* estimate of the Schrödinger part, but we do not have an *a priori* estimate of the acoustic wave part. This is because the energy conservation law is not available in our case. The proof of Theorem 2.1 follows from the argument by [6], which is applied to the deterministic Zakharov equations (for the wave-Schrödinger equations, see [1]).

**Remark 2.1** *(i) The path of Brownian motion $\beta(t)$ barely fails to belong to $H^{1/2}(0,T)$ for any $T > 0$. Therefore, when $b \geq 1/2$, even for $s < 0$, we can not expect that $\psi_T u$ belongs to $X^{s,b}$, where $u$ is a solution of* (1). *This is one of the difficulties to apply the Foureir restriction method to the stochastic nonlinear dispersive equations.*

*(ii) It is not known if one can choose $\varepsilon = 0$ in Theorem 2.1. The fact that $\psi(t)\beta(t) \in B^{1/2}_{2,\infty}(\mathbf{R})$ might be helpful (see Roynette [11] for the regularity of path of the Brownian motion and see de Bouard, Debussche and Tsutsumi [4] and Oh [10] for the Fourier restriction norms of the Besov type).*

Now we show an example of covariance operators $\Phi_j$ satisfying (HS) in Theorem 2.1.

**Example 2.1** Covariance operators $\Phi_1$ and $\Phi_2$ are defined as follows:

$$\Phi_1 = \varphi(-\partial_x^2)^{s/2} \quad (s < -(1/2 + \varepsilon)), \qquad \Phi_2 = \varphi I,$$

where $\varphi$ is a spatial cut-off function in $C_0^\infty(\mathbf{R})$, $\varepsilon$ is defined as in Theorem 2.1 and $I$ is the identity operator. Then, $\Phi_1$ and $\Phi_2$ satisfy assumption (HS) in Theorem 2.1.

It is instructive to recall known results on global solutions of (1)-(3) without additive noises, that is, in the deterministic case. Suppose that $\Phi_j = 0$ $(j = 1, 2)$. In [5], Bourgain and Colliander proved that when the space dimensions are less than or equal to three, the global existence of solutions in the energy space for (1)-(3). In [7], Ginibre, Tsutsumi and Velo improved the results on the time local well-posedness of (1)-(3) given by [5]. Especially, they showed that the Cauchy problem (1)-(3) is locally well-posed in $L^2 \times H^{-1/2} \times H^{-3/2}$ for the one dimensional case. In [6], Colliander, Holmer and Tzirakis showed that the Cauchy problem (1)-(3) is globally well-posed in $L^2 \times H^{-1/2} \times H^{-3/2}$ for the one dimensional case. We extend the argument by Colliander, Holmer and Tzirakis [6] to the stochastic case.

Few is known about the stochastic case. In [9], B.-L.Guo, Y.Lv and X.-P. Yang study (1)-(3) on a bounded interval with zero Dirichlet boundary condition. In [9], they show that if $\Phi_j dW_j = q_j(x)d\beta^{(j)}(t)$ $(j = 1, 2)$, $q_1 \in H_0^1$, $q_2 \in H^2 \cap H_0^1$ and $(u_0, n_0, n_1) \in (H^2 \cap H_0^1) \times H_0^1 \times L^2$, then there exist the global solutions and the invariant measure. The assumptions in [9] seem to be too restrictive. Their proof is based on the Galerkin method.

# 3 Sketch of Proof for Theorem 2.1

We first considr the deterministic equations with external forces $f$ and $g$ in one dimension.

$$i\partial_t u + \partial_x^2 u = nu + f, \quad t > 0,\ x \in \mathbf{R}, \tag{6}$$

$$\partial_t^2 n - \partial_x^2 n = \partial_x^2(|u|^2) + g, \quad t > 0,\ x \in \mathbf{R}, \tag{7}$$

$$(u, n, \partial_t n)(0) = (u_0(x), n_0(x), n_1(x)) \in L^2 \times H^{-1/2} \times H^{-3/2}. \tag{8}$$

Let $\psi \in C^\infty(\mathbf{R}\backslash\{0\})$ be a time cut-off function such that $\psi(t) = 1$ $(0 < t \le 1)$ and $\psi(t) = 0$ $(t < 0, t \ge 2)$. We put $\psi_T(t) = \psi(t/T)$ for $T > 0$.

We now change the dynamical variables of (6)-(7) into the new ones. We put

$$n_\pm = n \pm i\omega^{-1}\partial_t n, \quad \omega = (1 - \partial_x^2)^{1/2},$$

$$U(t) = e^{it\partial_x^2}, \quad V_\pm(t) = e^{\mp it\omega}.$$

We define the convolution $U *_R h$ and $V_\pm *_R h$ as follows.

$$U *_R h = -i\int_0^t U(t-\tau)h(\tau)\,d\tau,$$
$$V_\pm *_R h = -i\int_0^t V_\pm(t-\tau)h(\tau)\,d\tau$$

We define the new dynamical variables $v$ and $m_\pm$ as follows.

$$v = u - w, \quad m_\pm = n_\pm - w_\pm,$$

where

$$w = \psi_T U *_R f, \quad w_\pm = \psi_T V_\pm *_R \left(\omega^{-1} g\right).$$

Thus, we obtain the following new systems with respect to $(v, m_\pm)$.

$$v(t) = \psi_T U(t)u_0 + \psi_T U *_R \left[\frac{1}{2}(n_+ + n_-)u\right], \tag{9}$$
$$m_\pm(t) = \psi_T V_\pm(t)n_{\pm 0} \mp \psi_T V *_R \left[\omega^{-1}\{\Delta|u|^2 + \frac{1}{2}(n_+ + n_-)\}\right]. \tag{10}$$

We note that we keep the notation $u$ and $n_\pm$ on the right hand sides of (9)-(10), because the complete use of $v$ and $m_\pm$ makes the equations much more lengthy.

Let us next recall the argument by Colliander, Holmer and Tzirakis [6]. The proof by Colliander, Holmer and Tzirakis [6] may be thought of as a generalization of the Gronwall inequality in terms of Fourier restriction norms. If we try to apply their proof to the stochastic case, we have a serious problem with the regularity in time of paths for cylindrical Wiener processes $\Phi_j W_j$, which are slightly less than 1/2-Hölder continuous. While the Fourier restriction method converts time regularity to spatial regularity, paths of cylindrical Wiener processes $\Phi_j W_j$ barely fail to have regularity in time, which the proof in [6] requires. This is one of the difficulties to apply the Fourier restiriction method to stochastic nonlinear dispersive equations.

The following lemma about the bilinear estimates is used in [6].

**Lemma 3.1** *(bilinear estimates)*
*(i) Assume that* $1/4 < b_1, c_1, b < 1/2$ *and* $b + b_1 + c_1 \geq 1$. *Then, we have*

$$\left\|n_\pm u\right\|_{X^{0,-c_1}} \lesssim \|n_\pm\|_{Y_\pm^{-1/2,b}} \|u\|_{X^{0,b_1}}.$$

*(ii) Assume that* $1/4 < b_1, c < 1/2$ *and* $2b_1 + c \geq 1$. *Then, we have*

$$\left\|\partial_x\left(u_1 \bar{u}_2\right)\right\|_{Y_\pm^{-1/2,-c}} \lesssim \|u_1\|_{X^{0,b_1}} \|u_2\|_{X^{0,b_1}}.$$

**Remark 3.1** *Lemma 3.1 can be proved by the argument of [7] if $b+b_1+c_1>1$ and $2b_1+c_1>1$ and by the refined argument of [6] if $b+b_1+c_1=2b_1+c_1=1$. Lemma 3.1 for the latter case plays a crucial role in the proof of global a priori estimate corresponding to the solution $n(t)$ of the ion-acoustic part. We can choose $b=b_1=c=c_1=1/3$, which satisfy all the assumptions in Lemma 3.1 including $b+b_1+c_1=2b_1+c_1=1$.*

We now explain what the argument in [6] is like. We assume

$$\exists L>0;\ \left\|\partial_x\left(|w|^2\right)\right\|_{Y_{\pm}^{-1/2,-1/3}}\leq L^2T^{1/3}, \tag{11}$$

$$T^{1/2}\left(\|u_0\|_{L^2}+L\right)^2\lesssim\|n_{\pm 0}\|_{H^{-1/2}}. \tag{12}$$

Unless (12) holds, $\|n_{\pm 0}\|_{H^{-1/2}}$ can be controlled by $\|u_0\|_{L^2}$. Obviously, in this case, the solutions $(v,m_{\pm})$ can be extended. Therefore, we have only to show that as long as (12) holds, the solutions $(v,m_{\pm})$ can be extended. Suppose that one-time application of the contraction argument extends the solutions by the length $T$ and

$$T\sim\|n_{\pm 0}\|_{H^{-1/2}}^{-2}.$$

Here, we note that the influence by $w_{\pm}$ on $T$ is negligible, which follows from the contraction argument for the local well-posedness.

Then, Lemma 3.1 and the linear estimates yield

$$\|m_{\pm}(T)\|_{H^{-1/2}}\leq\|n_{\pm 0}\|_{H^{-1/2}}+CT^{1/2}\left(\|u_0\|_{L^2}+L\right)^2.$$

This inequality implies that every time we extend the solutions by $T$, $\|m_{\pm}\|_{H^{-1/2}}$ grows at most by $CT^{1/2}\left(\|u_0\|_{L^2}+L\right)^2$. Denote by $m$ the number of repetition of the contraction argument until $\|m_{\pm}(t)\|_{H^{-1/2}}$ becomes twice as large as $\|n_{\pm 0}\|_{H^{-1/2}}$. Then, we have

$$m\sim\frac{\|n_{\pm 0}\|_{H^{-1/2}}}{T^{1/2}\left(\|u_0\|_{L^2}+L\right)^2}.$$

Accordingly, the $m$-time repetition of the contraction argument enables us to extend the solutions by the following time length:

$$\begin{aligned} mT &\sim\frac{T^{1/2}\|n_{\pm 0}\|_{H^{-1/2}}}{\left(\|u_0\|_{L^2}+L\right)^2}\\ &\sim\left(\|u_0\|_{L^2}+L\right)^{-2}, \end{aligned}$$

which shows that $mT$ is independent of $\|n_{\pm 0}\|_{H^{-1/2}}$. Thus, we have

$$\begin{gathered}\forall T_0>0,\ \exists C>0;\ \|m_{\pm}(t)\|_{H^{-1/2}}\leq C,\\ T_0\geq t>0,\end{gathered}$$

where $C$ depends only on $\|u_0\|_{L^2}$, $\|n_{\pm 0}\|_{H^{-1/2}}$, $w$, $w_\pm$ and $T_0$. This yields the global existence of solutions $(v, m_\pm)$ in $L^2 \times H^{-1/2}$.

The argument by [6] would still work if the following inequality held:

$$\|\psi_T f\|_{X^{0,b_1}} \lesssim T^{\frac{1}{2}-b_1} \log(1/T) \|f\|_{X^{0,1/2}},$$
$$1/2 > b_1 \geq 1/3.$$

But even if this were true, we can *NOT* choose

$$f = \psi_T \int_0^t U(t-\tau)\Phi_1 dW_1 \notin X^{0,a} \quad (a \geq 1/2).$$

To overcome this difficulty, instead of Lemma 3.1, we use the following lemma.

**Lemma 3.2** *(i) Assume that $c_1, b \geq 1/3$ and $1 \gg \varepsilon > 0$. Then, we have*

$$\|n_\pm w\|_{X^{0,-c_1}} \lesssim \|n_\pm\|_{Y_\pm^{-1/2,b}} \|w\|_{X^{2\varepsilon,(1-\varepsilon)/3}}.$$

*(ii) Assume that $c, b_1 \geq 1/3$ and $1 \gg \varepsilon > 0$. Then, we have*

$$\|\partial_x(u\bar{w})\|_{Y_\pm^{-1/2,-c}} \lesssim \|u\|_{X^{0,b_1}} \|w\|_{X^{2\varepsilon,(1-\varepsilon)/3}}.$$

*(iii) Assume that $c \geq 1/3$ and $1 \gg \varepsilon > 0$. Then, we have*

$$\|\partial_x(|w|^2)\|_{Y_\pm^{-1/2,-c}} \lesssim \|w\|^2_{X^{2\varepsilon,(1-\varepsilon)/3}}.$$

**Remark 3.2** *Lemma 3.2 trades off the spatial regularity for the time regularity of stochastic convolution term $w$. In fact, Lemma 2 limits the lower bound of the regularity for the covariance operator $\Phi_1$ of the Schrödinger part. Estimate (iii) in Lemma 2 ensures assumption (11), which runs the algorithm by [6].*

### Sketch of Proof of Lemma 3.2

Estamate (i) is almost equivalent to (ii) by duality. We now prove (ii) and (iii). We consider the product of two waves in the Fourier space.

$$\hat{u}(\tau_1, \xi_1), \quad \hat{w}(\tau_2, \xi_2),$$
$$\tau = \tau_1 + \tau_2, \quad \xi = \xi_1 + \xi_2.$$

The interaction of two waves $\hat{u}(\tau_1, \xi_1)$ and $\widehat{\bar{w}}(\tau_2, \xi_2)$ in the acoustic wave sector is represented as follows:

$$\begin{aligned} &\tau \pm |\xi| - (\tau_1 - \xi_1^2) - (\tau_2 + \xi_2^2) \\ &= |\xi| \left[ \frac{\xi}{|\xi|}(\xi_1 - \xi_2) \pm 1 \right]. \end{aligned} \qquad (13)$$

The problem is how to recover $1/2$ derivative. We first prove the following estimate.

$$\left\|\partial_x\left(u\bar{w}\right)\right\|_{Y_{\pm}^{-1/2,-1/3}} \lesssim \|u\|_{X^{0,1/3}}\|w\|_{X^{1/2,1/4}}. \tag{14}$$

In either case of $|\xi_1| \ll |\xi_2|$ or $|\xi_1| \gg |\xi_2|$, one can pick out the factor $\left(|\xi||\xi_1-\xi_2|\right)$ from the modulus of (13), which yields the gain of extra derivative. Otherwise, in the case of $|\xi_1| \sim |\xi_2|$, one can let $1/2$ derivative act on $\hat{w}(\tau_2, \xi_2)$. Thus, estimate (14) is proved.

Let $u \in X^{0,1/3}$ be fixed. We first consider the linear operator: $\bar{w} \mapsto \partial_x\left(u\bar{w}\right)$. We interpolate between Lemma 3.1 (ii) and (14) to obtain Lemma 3.2 (ii). We next consider the bilinear operator: $(u, \bar{w}) \mapsto \partial_x\left(u\bar{w}\right)$. We use the symmetry of the above mapping with respect to $u$ and $w$ to obtain Lemma 3.2 (iii) by the bilinear interpolation (see Theorem 4.1 in Appendix below for the bilinear interpolation).

# 4 Appendix

We have the following theorem concerning the bilinear interpolation (see Exercise 5(b) in Section 3.13 in [3]).

**Theorem 4.1** *(Bilinear Interpolation) $T$ is a bounded bilinear operator such that*

$$\begin{aligned} T &: A_0 \times B_0 \to C_0, \\ T &: A_0 \times B_1 \to C_1, \\ T &: A_1 \times B_0 \to C_1. \end{aligned}$$

*Assume $0 < \theta_0, \theta_1 < \theta < 1$, $1 \le p, q, r \le \infty$, $1 \le 1/p + 1/q$ and $\theta = \theta_0 + \theta_1$. Then,*

$$T : (A_0, A_1)_{\theta_0, pr} \times (B_0, B_1)_{\theta_1, qr} \to (C_0, C_1)_{\theta, r}.$$

In order to obtain Lemma 3.2 (iii), we apply Theorem 4.1 with $A_0 = B_0 = X^{0,1/3}$, $A_1 = B_1 = X^{1/2,1/4}$, $C_0 = C_1 = Y_{\pm}^{-1/2,1/3}$, $p = q = 1$, $r = 2$ and $\theta_0 = \theta_1 = \frac{1}{2}\theta = \eta$, $0 < \eta \ll 1$.

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