数理解析研究所講究録
第 1973 巻 2015 年 1-9

GENERALIZED MAASS RELATIONS AND LIFTS

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1. Introduction

Let $S^{+(1)}_{k-\frac{1}{2}}$ be the space of the cusp forms in the Kohnen plus space of weight $k-\frac{1}{2}$. The purpose of this exposition is to explain the lifting map:

$$\begin{array}{ccccc} S^{+(1)}_{k-n+\frac{1}{2}} & \times & S^{+(1)}_{k-\frac{1}{2}} & \to & S^{+(2n-2)}_{k-\frac{1}{2}} \\ \Psi & & \Psi & & \Psi \\ (\ f & , & g\ ) & \mapsto & \mathcal{F}_{f,g} \end{array} \tag{1}$$

Namely, it is a lifting map from pairs of elliptic modular forms of half-integral weight to Siegel modular forms of half-integral weight of even degree. Here $S^{+(2n-2)}_{k-\frac{1}{2}}$ is the space of the cusp forms of the generalized plus space of weight $k-\frac{1}{2}$ of degree $2n-2$, which is a certain subspace of Siegel cusp forms of weight $k-\frac{1}{2}$ of degree $2n$. The case $2n-2=2$ had been announced in [Ha 11b], and in this exposition we will explain the above lifts for arbitral even positive integers $2n-2$.

In the above lifts we need the assumption that the constructed $\mathcal{F}_{f,g}$ is not identically zero. We checked by numerical calculations that $\mathcal{F}_{f,g}$ is not identically zero for (n,k) which satisfy $n \leq 6$, $k \leq 18$ and $\dim S^{+(1)}_{k-n+\frac{1}{2}} \times \dim S^{+(1)}_{k-\frac{1}{2}} \neq 0$. Therefore, we expect that any $\mathcal{F}_{f,g}$ is not identically zero. The main theorem is

Theorem 1 ([Ha 15]). *Let k be an even integer and n be a natural number. Let $f \in S^{+(1)}_{k-n+\frac{1}{2}}$ and $g \in S^{+(1)}_{k-\frac{1}{2}}$ be Hecke eigenforms. Then there exists $\mathcal{F}_{f,g} \in S^{+(2n-2)}_{k-\frac{1}{2}}$. Under the assumption that $\mathcal{F}_{f,g}$ is not identically zero, the form $\mathcal{F}_{f,g}$ is a Hecke eigenform and the Zhuravlev L-function $L(s,\mathcal{F}_{f,g})$ of $\mathcal{F}_{f,g}$ satisfies*

$$L(s,\mathcal{F}_{f,g}) \;=\; L(s,g) \prod_{i=1}^{2n-3} L(s-i,f).$$

Here the Zhuravlev L-function is a generalization of the Shimura L-function which is a L-function of modular forms of half-integral weight. $L(s,g)$ and $L(s,f)$ are the Shimura L-function of g and f, respectively. The above identity involves also the Euler 2-factors.

The above lift was first conjectured by Ibukiyama and the author [H-I 05] for the case $2n-2=2$. The construction of $\mathcal{F}_{f,g}$ was suggested by T. Ikeda to the author. To show the fact that $\mathcal{F}_{f,g}$ is a Hecke eigenform, we will use a generalized Maass relation.

We remark that $\mathcal{F}_{f,g}$ satisfies the generalized Ramanujan conjecture if $2n-2=2$. It means that the absolute value of the p-th parameters of $\mathcal{F}_{f,g}$ are all the same for each fixed prime p. And if $2n-2 \geq 4$, then the constructed lift $\mathcal{F}_{f,g}$ does not satisfy the generalized Ramanujan conjecture.

We now explain the generalized Maass relations. The usual Maass relation is a certain relation among Fourier coefficients of certain Siegel modular forms of integral weight of degree 2. In particular, the Siegel-Eisenstein series of degree 2 and the Saito-Kurokawa lifts satisfy the Maass relation. Several generalizations of the Maass relation for higher degree or for half-integral weight have been known for the following cases:

- Siegel-Eisenstein series of integral weight of higher degree (cf. [Yk 86, Yk 89, Ha 13])
- Siegel-Eisenstein series of half-integral weight of degree 2 (cf. [Ta 86])
- The Ikeda lifts (cf. [Ko 02, K-K 05, Yn 10, Ha 13, G-H 15]).

In this exposition we also explain a new generalization of the Maass relation for Siegel modular forms of *half-integral weight* of *general degrees.* For example, some Siegel modular forms which satisfy the generalized Maass relation of half-integral weight are constructed by the composition of the three linear maps: the Ikeda lifts, the 1st Fourier-Jacobi map and the Eichler-Zagier-Ibukiyama correspondence. Here the 1st Fourier-Jacobi map is the map from Siegel modular form to Jacobi forms of index 1 by the Fourier-Jacobi expansion. We remark that the Ikeda lift is a linear map, if we regard it as a lifting map from the Kohnen plus-space.

To construct $\mathcal{F}_{f,g} \in S_{k-\frac{1}{2}}^{+(2n-2)}$ we first construct a Siegel modular form of half-integral weight $F \in S_{k-\frac{1}{2}}^{+(2n-1)}$ from $f \in S_{k-n+\frac{1}{2}}^{+(1)}$ in the above manner. Then we will show that F satisfies the generalized Maass relation of half-integral weight, if f is a Hecke eigenform. The form $\mathcal{F}_{f,g}$ is constructed from the pair (F, g). Once we show the fact that F satisfies the generalized Maass relation, it is not difficult to show that $\mathcal{F}_{f,g}$ is a Hecke eigenform and the Hecke eigen values of $\mathcal{F}_{f,g}$ are also calculated by the formula of the generalized Maass relation. The difficult part of the proof of Theorem 1 is to show that F satisfies the generalized Maass relation. It is shown from the fact that the form F is constructed through the Ikeda lift and that F satisfies a similar formula which Siegel-Eisenstein series satisfies. Therefore, we need to investigate the Siegel-Eisenstein series. In particular, we need to investigate a certain Siegel modular form of half-integral weight which is obtained by the Eichler-Zagier-Ibukiyama correspondence of the 1-st Fourier Jacobi coefficient of Siegel-Eisenstein series. Such a Siegel modular form of half-integral weight can be regarded as a generalization of the Cohen-Eisenstein series for general degree.

In Section 2 we will explain a generalized Maass relation for Siegel modular forms of *integral weight.* In Section 3 we will explain a generalized Maass relation for Siegel modular forms of *half-integral weight* and give briefly the proof of Theorem 1.

In this exposition we use the following notation: We denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n and denote by $\mathrm{Sp}(n,K)$ the symplectic group of size $2n$ with entries in a commutative ring K. We set $\Gamma_n := \mathrm{Sp}(n,\mathbb{Z})$. We put $e(A) := \exp(2\pi i\,\mathrm{tr}(A))$ for any symmetric matrix A. The symbol M_k^n denotes the vector space of all Siegel modular forms of weight k of degree n. We write S_k^n for the vector space of all Siegel cusp forms in M_k^n.

2. Generalized Maass relations of integral weight

We start with the following question. Let $F \in M_k^{n+r}$ be a Siegel modular form of degree $n+r$. We consider the pullback of F with respect to the map $\mathfrak{H}_n \times \mathfrak{H}_r \to \mathfrak{H}_{n+r}$:

$$F\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right)\right) = \sum_{g_i\,:\mathit{Hecke\ eigen\ form}} f_i(\tau) g_i(\omega). \quad (\tau \in \mathfrak{H}_n,\ \omega \in \mathfrak{H}_r.) \tag{2}$$

Here g_i runs over all forms in a Hecke eigenbasis of M_k^r. The question is

(Q1) Is f_i a Hecke eigenform?

For usual Siegel modular form F we can not expect that the answer for (Q1) is true. However, if F is a Siegel-Eisenstein series then it is known that f_i is essentially a Klingen type Eisenstein series and the answer for (Q1) is true.

T. Ikeda showed that the answer for (Q1) is true for Ikeda lifts F.

Theorem 2 ([Ik 06]). *Let $k \in 2\mathbb{Z}$ and $n,\ r \in \mathbb{N}$ $(r \leq n)$. Let $F \in S_k^{2n}$ be a Ikeda lift of a Hecke eigenform $h \in S_{2k-2n}^1$. Let f_i be as in (2). If $f_i \not\equiv 0$, then $f_i \in S_k^{2n-r}$ is a Hecke eigenform which satisfies*

$$L(s,f_i,st) = L(s,g_i,st) \prod_{j=1}^{2n-2r} L(s+k-r-j,h).$$

Here $L(s,f_i,st)$ is the standard L-function of f_i. Namely, it gives a lifting map for even integer k:

$$\begin{array}{ccccc} S_{2k-2n}^1 & \times & S_k^r & \to & S_k^{2n-r} \\ \Psi & & \Psi & & \Psi \\ (h & , & g_i) & \mapsto & f_i \end{array} \tag{3}$$

In this exposition we call the lifts in Theorem 2 *Miyawaki lifts.*

The Miyawaki lifts were first conjectured by Miyawaki [Mi 92] in the case $(n,r) = (2,1)$. He calculated some Euler factors of the spinor L-functions of Siegel modular forms of degree 3 of weight 12 and also of weight 14 and he obtained two kinds of conjectures. The above Miyawaki lifts are generalizations of one of the Miyawaki's conjectures. Here the other Miyawaki's conjecture is a conjecture about the existence of the lifting map:

$$S_{2k-2}^1 \quad \times \quad S_{k-2}^1 \quad \to \quad S_k^3$$

and is still an open problem (see also [Ik 06, Remark 2.1]).

As for the expression of the spinor L-function of Miyawaki lifts in the case of $r = 1$ of the map (3), we have

Theorem 3 ([He 12] ($n = 2$), [Ha 14] ($n \geq 3$)). *Assume $r = 1$. Let $f_i \in S_k^{2n-1}$ be as in Theorem 2. If $f_i \not\equiv 0$, then we have*

$$L(s, f_i, spin) = \prod_{m=1}^{n} \prod_{j} L(s - (n-m)(k-n) + j, g_i \otimes sym^{m-1} h)^{R_{n-1,m-1}(j)}.$$

Here $h \in S^1_{2k-2n}$ is the preimage of the Ikeda lift F, and $L(s, f_i, spin)$ is the spinor L-function of f_i. And where $L(s, g_i \otimes sym^m h)$ is a symmetric power L-function of g_i and h. The natural number $R_{n,m}(j)$ is determined by a certain combinatorial way. In the second product j runs over certain integers in a finite set. For the detail of this theorem and of symbols the reader is referred to [He 12, Ha 14].

To show Theorem 3 we used the generalized Maass relations for Siegel modular forms of integral weight of general degrees. We can expect that Theorem 3 for $r > 1$ will be obtained, if we get the corresponding generalized Maass relations of integral weight.

3. Generalized Maass relations of half-integral weight

We consider the same question of (Q1) for the case of the Siegel modular forms of half-integral weight. Let $F \in S_{k-\frac{1}{2}}(\Gamma_0^{(n+r)}(4))$ be a Siegel modular form of weight $k - \frac{1}{2} \in \mathbb{Z} - \frac{1}{2}$ of degree $n + r$. We take an expansion:

$$F\left(\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}\right) = \sum_{g_i : Hecke\ eigen\ form} f_i(\tau) g_i(\omega). \quad (\tau \in \mathfrak{H}_n,\ \omega \in \mathfrak{H}_r.)$$

Here g_i runs over all forms in a Hecke eigenbasis of the vector space $M_{k-\frac{1}{2}}(\Gamma_0^{(r)}(4))$. Here $M_{k-\frac{1}{2}}(\Gamma_0^{(r)}(4))$ denotes the space of Siegel modular forms of weight $k - \frac{1}{2}$ of degree r. The question is

(Q2) Is f_i a Hecke eigenform?

To give a partial answer for the question (Q2) we consider the composition of the following three maps. From now on we assume $k \in 2\mathbb{Z}$.

(4)
$$\begin{tikzcd}
 & I(f) \in S_k^{2n} \arrow[d, "\text{1st F-J}"] & \\
 & \psi_1 \in J_{k,1}^{(2n-1)\,cusp} \arrow[r, "\text{E-Z-I}"'] & F \in S^{+(2n-1)}_{k-\frac{1}{2}} \\
f \in S^{+(1)}_{k-n-\frac{3}{2}} \arrow[uur, "\text{Ikeda lift}"] & &
\end{tikzcd}$$

where $J_{k,1}^{(2n-1)\,cusp}$ is the space of Jacobi cusp forms of weight k of index 1 and of degree $2n-1$, and where $S_{k-\frac{1}{2}}^{+(2n-1)}$ consists of cusp forms in the generalized plus space of weight $k-\frac{1}{2}$ of degree $2n-1$. Here "1st F-J,, is the map which is obtained by Fourier-Jacobi expansion and "E-Z-I,, is the Eichler-Zagier-Ibukiyama correspondence (cf. [Ib 92]). It is shown in [Ha 11a] that if f is a Hecke eigenform, then F is not identically zero and is a Hecke eigenform, and the Zhuravlev L-function $L(s,F)$ of F satisfies

$$L(s,F) \;=\; \prod_{i=0}^{2n-2} L(s-i,f).$$

Here $L(s,f)$ is the Shimura L-function of f which coincides with a usual L-function of an elliptic modular form of integral weight with respect to $\mathrm{SL}(2,\mathbb{Z})$.

For any Hecke eigenform $G \in S_{k-\frac{1}{2}}^{+(n)}$ the Zhuravlev L-function of G is defined by

$$L(s,G) \;:=\; \prod_{p}\prod_{j=1}^{n}\{(1-\beta_{p,j}\,p^{-s+k-\frac{3}{2}})(1-\beta_{p,j}^{-1}\,p^{-s+k-\frac{3}{2}})\}^{-1}.$$

Here $\{\beta_{p,j}^{\pm}\}_{j=1,n}$ is the p-parameters of G in the sense [Zh 84, §10] for odd prime p. For $p=2$ we define $\{\beta_{2,j}^{\pm}\}_{j=1,n}$ by using the Hecke eigenvalues of a Jacobi form in $J_{k,1}^{(n)}$ which corresponds to G by the Eichler-Zagier-Ibukiyama correspondence.

We now explain the generalized Maass relations for half-integral weight.

Let

$$(5) \qquad F\left(\left(\begin{smallmatrix}\tau & z\\ {}^tz & \omega\end{smallmatrix}\right)\right) = \sum_{m\in\mathbb{Z}} \phi_m(\tau,z)e(m\omega) \quad (\tau\in\mathfrak{H}_n, \omega\in\mathfrak{H}_1)$$

be the Fourier-Jacobi expansion of F which is constructed in (4). We remark that ϕ_m is identically zero unless $m \equiv 0,3 \mod 4$, since F belongs to the generalized plus-space $S_{k-\frac{1}{2}}^{+}(\Gamma_0^{(2n-1)}(4))$ and due to the definition of the generalized plus-space.

Theorem 4 ([Ha 15]). *We have the identity*

$$\begin{aligned}&(\phi_m|\tilde{V}_{0,2n-2}(p^2), \phi_m|\tilde{V}_{1,2n-3}(p^2), \ldots, \phi_m|\tilde{V}_{2n-2,0}(p^2))\\ &= \left(\phi_{\frac{m}{p^2}}|U_{p^2},\ \phi_m|U_p,\ \phi_{mp^2}\right)\begin{pmatrix}0 & p^{2k-3}\\ p^{k-2} & p^{k-2}\left(\frac{-m}{p}\right)\\ 0 & 1\end{pmatrix} A(\alpha_p)\end{aligned}$$

for any prime p. (The both sides are vectors of Jacobi forms of index mp^2). Here $A(\alpha_p)$ is a certain $2\times(2n-1)$ matrix which depends on the choice of $f\in S_{k-n-\frac{3}{2}}^{+(1)}$ and does not depend on the choice of m, and where $\left\{\alpha_p^{\pm1}\right\}_p$ are the Satake parameters of f in the sense [Ik 01]*, and where the operators $\tilde{V}_{i,2n-2-i}(p^2)$ $(i=0,\ldots,2n-2)$ and U_{p^j} $(j=1,2)$ are generalizations of V-operator and U-operator* *in the book of Eichler-Zagier* [E-Z 85]. *These are maps*

$$\begin{aligned} \tilde{V}_{i,2n-2-i}(p^2) &: J^{(2n-2)}_{k-\frac{1}{2},m} \to J^{(2n-2)}_{k-\frac{1}{2},mp^2}, \\ U_{p^j} &: J^{(2n-2)}_{k-\frac{1}{2},m} \to J^{(2n-2)}_{k-\frac{1}{2},mp^{2j}} \end{aligned}$$

for odd prime p. Here $J^{(2n-2)}_{k-\frac{1}{2},m}$ *denotes the space of Jacobi forms of weight* $k-\frac{1}{2}$ *of index m of degree* $2n-2$. *For* $p=2$ *we can introduce the operators* $\tilde{V}_{i,2n-2-i}(4)$ *and* U_{2^j} *through the relation between Jacobi forms of half-integral weight of integer index and Jacobi forms of integral weight of matrix index.*

We remark that the generalized Maass relations depends on the choice of f, besides the usual Maass relation does not depend on the choice of the preimage of the Saito-Kurokawa (Maass) lift. To obtain Theorem 4 we needed to show similar identities for Siegel Eisenstein series. It means that we take the Siegel Eisenstein series instead of the Ikeda lift $I(f)$ and Theorem 4 holds for Siegel Eisenstein series. The steps for the proof of Theorem 4 are as follows.

(i) By the virtue of the Ikeda lift, it is enough to show Theorem 4 for the case of generalized Cohen-Eisenstein series. Here the generalized Cohen-Eisenstein series are certain Siegel modular forms of half-integral weight, which are not cusp forms. (As for the definition of Cohen-Eisenstein series, see [Co 75] for degree one and [Ar 98] for general degree).

(ii) Show certain linear isomorphisms between the space of certain Jacobi forms of half-integral weight of integer index and the space of Jacobi forms of integral weight of matrix index.

(iii) Show a compatibility between the linear isomorphisms in (ii) and certain operators which shift the indices of Jacobi forms.

(iv) We use relations between Fourier-Jacobi coefficients of Siegel-Eisenstein series and Jacobi-Eisenstein series of matrix index which is obtained by S. Boecherer [Bo 83].

(v) Calculate the action of shift operators on Jacobi-Eisenstein series of matrix index explicitly.

(vi) Obtain Theorem 4 for the case of generalized Cohen-Eisenstein series by using (ii), (iii), (iv) and (v).

For the detail of the proof of Theorem 4, the reader is referred to [Ha 15, Theorem 8.2].

We also remark that $\phi_m(\tau,0)$ belongs to $S^+_{k-\frac{1}{2}}(\Gamma_0^{(2n-2)}(4))$. By the virtue of the definition of $\tilde{V}_{i,2n-2-i}(p^2)$ we have the identity

$$(\phi_m|\tilde{V}_{i,2n-2-i}(p^2))(\tau,0) = \phi_m(\tau,0)|\tilde{T}_{i,2n-2-i}(p^2). \tag{6}$$

Here $\tilde{T}_{i,2n-2-i}(p^2)$ is a Hecke operator acting on $S^+_{k-\frac{1}{2}}(\Gamma_0^{(2n-2)}(4))$, which corresponds to the double coset $\Gamma_{2n-2}\mathrm{diag}(1_i, p1_{2n-2-i}, p^21_i, p1_{2n-2-i})\Gamma_{2n-2}$.

By using Theorem 4 we shall show Theorem 1. Let F be the form as before which is constructed in (4). We write

$$F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right) = \sum_{g:Hecke\ eigen\ form} \mathcal{F}_{f,g}(\tau)g(\omega). \quad (\tau \in \mathfrak{H}_n,\ \omega \in \mathfrak{H}_1.)$$

Here g runs over all modular forms in a Hecke eigenbasis of the Kohnen plus space $S_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4))$. We normalize g such that the eigenvalues of g are all real numbers. Remark that we write $\mathcal{F}_{f,g}$ instead of f_i in the question (Q2). Thus, we constructed $\mathcal{F}_{f,g} \in S^{+(2n-2)}_{k-\frac{1}{2}}$ from $(f,g) \in S^{+(1)}_{k-n+\frac{1}{2}} \times S^{+(1)}_{k-\frac{1}{2}}$ and it gives the map (1).

We write the matrix $A(\alpha_p) = (a_{j,i})_{j,i}$ $(1 \le j \le 3, 0 \le i \le 2n-2)$. We now assume that $\mathcal{F}_{f,g}$ is not identically zero. The Hecke operator $\tilde{T}_{i,2n-2-i}(p^2)$ (resp. $\tilde{T}_{1,0}(p^2)$) acts on $F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)$ as a function of τ (resp. of ω), and we write it $F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|_\tau \tilde{T}_{i,2n-2-i}(p^2)$ (resp. $F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|_\omega \tilde{T}_{1,0}(p^2)$). We remark

$$\begin{aligned}&F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|_\omega \tilde{T}_{1,0}(p^2)\\ &= \sum_m \left(p^{2k-3}\phi_{\frac{m}{p^2}}(\tau,0) + \left(\frac{-m}{p}\right)p^{k-2}\phi_m(\tau,0) + \phi_{mp^2}(\tau,0)\right)e(m\omega).\end{aligned} \tag{7}$$

Due to the identities (5), (6), (7) and Theorem 4, we have

$$\begin{aligned}F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|_\tau \tilde{T}_{i,2n-2-i}(p^2) &= \sum_m \left(\phi_m(\tau,0)|_\tau \tilde{T}_{i,2n-2-i}(p^2)\right)e(m\omega)\\ &= \sum_m \left(\phi_m|\tilde{V}_{i,2n-2-i}(p^2)\right)(\tau,0)\,e(m\omega)\\ &= \sum_m \sum_{j=1}^{3} a_{j,i}\phi_{mp^{2j-4}}(\tau,0)e(m\omega)\\ &= b_{1,i}p^{k-2}F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right) + b_{2,i}F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|_\omega \tilde{T}_{1,0}(p^2)\end{aligned}$$

Here $b_{1,i}$ and $b_{2,j}$ are complex numbers which are determined by

$$A(\alpha_p) = \begin{pmatrix} b_{1,0} & b_{1,1} & \cdots & b_{1,2n-2}\\ b_{2,0} & b_{2,1} & \cdots & b_{2,2n-2}\end{pmatrix}.$$

Hence

$$\begin{aligned}&\left(\mathcal{F}_{f,g}|\tilde{T}_{i,2n-2-i}(p^2)\right)(\tau)\\ &= \frac{1}{6\langle g,g\rangle}\int_{\Gamma_0(4)\backslash\mathfrak{H}_1}\left(F\left(\left(\begin{smallmatrix}\tau & 0\\ 0 & \omega\end{smallmatrix}\right)\right)|_\tau \tilde{T}_{i,2n-2-i}(p^2)\right)\overline{g(\omega)}\mathrm{Im}(\omega)^{k-\frac{5}{2}}\,d\omega\\ &= b_{1,i}p^{k-2}\mathcal{F}_{f,g}(\tau) + b_{2,i}\lambda_g(p^2)\mathcal{F}_{f,g}(\tau),\end{aligned}$$

where $\lambda_g(p^2)$ denotes the Hecke eigenvalue of g for $\tilde{T}_{1,0}(p^2)$. Thus we conclude that $\mathcal{F}_{f,g}$ is a Hecke eigenform and the Hecke eigenvalues are $\{b_{1,i}p^{k-2} + b_{2,i}\lambda_g(p^2)\}$. The explicit formula of $A(\alpha_p)$ is obtained by a reduction with respect to the degree $2n-2$, which is a generalization of the Krieg-Zarikovskaya's Theorem (cf.

[Kr 86, Ha 15]). The Zhuravlev L-function of $\mathcal{F}_{f,g}$ is calculated by using the fact that $A(\alpha_p)$ is obtained through the eigenvalues of Siegel-Eisenstein series. Thus, we conclude Theorem 1.

If we fix k, n and g, then we can check whether $\mathcal{F}_{f,g} \not\equiv 0$ or not by a numerical computation, because the Fourier coefficients of $F\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right)\right)$ and g are computable (if k and n are small). At least if (n, k) satisfies the conditions $n \leq 6$, $k \leq 18$ and $\dim S^{+(1)}_{k-n+\frac{1}{2}} \times \dim S^{+(1)}_{k-\frac{1}{2}} \neq 0$, then all $\mathcal{F}_{f,g} \in S^{+(2n-2)}_{k-\frac{1}{2}}$ satisfy $\mathcal{F}_{f,g} \not\equiv 0$.

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