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Dirichlet series of 3 variables and Koecher-Maass series of non-holomorphic Siegel-Eisenstein series

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Holomorphic Siegel modular case I would like to talk about certain Dirichlet series associated with the non-holomorphic Siegel-Eisenstein series. As an introduction, I start from holomorphic Siegel modular case. Let f be a holomorphic Siegel modular form of degree n and even weight k. It has a Fourier expansion, indexed by positive semi-definite half integral symmetric matrices as ($e(x) := e^{2\pi ix}$, $Z \in H_n := \{Z = {}^tZ \in M_n(\mathbf{C}); \Im Z > O\}$)

$$f(Z) = \sum_{T \geq O} A(T)e(\mathrm{tr}(TZ)).$$

From the Fourier coefficients, indexed by positive definite T, a certain Dirichlet series called by Koecher-Maass series can be associated for $\Re s \gg 0$ by

$$D_n(f, s) = \sum_{T \in L_n^+/GL_n(\mathbf{Z})} \frac{A(T)}{\sharp\mathrm{Aut}(T)(\det T)^s},$$

where the sum is taken modulo the action $T \to T[U] := {}^tUTU$ of $GL_n(\mathbf{Z})$ and each term is weighted by the order of the unit group of T. It is absolutely convergent for $\Re s$ sufficiently large. Then, we multiply a suitable gamma factor of the form

$$D_n^*(f, s) = 2(2\pi)^{-ns} \prod_{r=1}^{n} \pi^{\frac{r-1}{2}} \Gamma\left(s - \frac{r-1}{2}\right) \cdot D_n(f, s).$$

A fundamental results are

(1) The Dirichlet series $D_n(f, s)$ has a meromorphic continuation to all $s \in \mathbf{C}$.

(2) It satisfies the functional equation $D_n^*(f, k-s) = (-1)^{nk/2} D_n^*(f, s)$.

The proof uses the Mellin transform of the non-degenerate part $f^{(n)}$ of the Fourier series f;

$$D_n^*(f, s) = \int_{GL_n(\mathbf{Z})\backslash \mathcal{P}_n} f^{(n)}(iY)(\det Y)^{s-\frac{n+1}{2}} dY \quad \left(f^{(n)}(Z) := \textstyle\sum_{T>O} A(T)e(\mathrm{tr}(TZ)) \right).$$

To get the Dirichlet series expression, we need Siegel's evaluation of the gamma integral

$$\int_{\mathcal{P}_n} e^{-2\pi \mathrm{tr}(TY)}(\det Y)^{s-\frac{n+1}{2}} dY = (2\pi)^{-ns} \prod_{r=1}^{n} \pi^{\frac{r-1}{2}} \Gamma\left(s - \frac{r-1}{2}\right) \cdot (\det T)^{-s}.$$

Also, some analysis about the degenerate parts in the Fourier expansion are required. We cannot take the Mellin transform for the degenerate parts. Maass applys his invariant differential

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operator to kill (delete) the degenerate parts $f - f^{(n)}$. Arakawa studies the non-degenerate part $f^{(n)}$ to establish his residue formula.

Imai's converse theorem In order to motivate the study of this type of Dirichlet series, I recall some of its applications to modular forms. One of the most impressive one is Imai's converse theorem. For simplicity, I assume that degree is 2. Consider a sequence $\{A(T)\}_{T\in L_2^+}$ indexed by positive definite $T \in L_2^+ = \left\{\left(\begin{smallmatrix} a & b/2 \\ b/2 & c\end{smallmatrix}\right) > O \ ; \ a,b,c \in \mathbf{Z}\right\}$. At this moment, $\{A(T)\}_{T\in L_2^+}$ are not necessarily being Fourier coefficients of modular forms. But, assume that $\exists$ a natural number k, $\exists$ a constant $\delta > 0$ satisfying

(a) $A({}^tUTU) = (\det U)^k A(T) \ \ \forall U \in GL_2(\mathbf{Z})$

(b) $A(T) = O((\det T)^\delta) \ \ \forall T \in L_2^+$

For this sequence, we associate the Fourier series on the Siegel half-space of degree 2

$$F(Z) = \sum_{T\in L_2^+} A(T)e(\mathrm{tr}(TZ)) \quad (Z \in H_2).$$

It is absolute convergent on the Siegel-half space by the assumption (b) and it is holomorphic there. By definition, it is translation invariant; $F(Z+S) = F(Z) \ \ \forall S \in \mathrm{Sym}_2(\mathbf{Z})$. By (a), it is unimodular invariant; $F({}^tUZU) = F(Z) \ \ \forall U \in GL_2(\mathbf{Z})$.

To state the converse theorem, we need the twisting by Maass forms. More precisely, we need the spectral eigenfunctions $\mathcal{U}(\tau)$ of the hyperbolic Laplacian $\Delta = v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$ for L^2 space $L^2(SL_2(\mathbf{Z})\setminus H_1)$ of $SL_2(\mathbf{Z})$ invariant functions on the upper-half plane H_1. They are consisting of the constant function $\sqrt{3/\pi}$, a complete orthonormal system of the cusp forms $\mathcal{U}_m(\tau)$, and the unitary Eisenstein series $E(\tau, 1/2+ir)$ $(r \in \mathbf{R})$. We may assume that $\mathcal{U}_m(\tau)$ is real valued, and either even or odd w.r.t. $u = \Re\tau$.

For such a spectral eigenfunction $\mathcal{U}(\tau)$ and the sequence $\{A(T)\}_{T\in L_2^+}$, we associate the Dirichlet series of Koecher-Maass type like

$$\Psi(F,\mathcal{U},s) = \sum_{T\in L_2^+/SL_2(\mathbf{Z})} \frac{A(T)\mathcal{U}(\tau_T)}{\sharp E(T)(\det T)^s} \quad (\Re(s) > \delta + 3/2),$$

where the sum is taken modulo the action $T \to {}^tUTU$ of $SL_2(\mathbf{Z})$, $E(T) = \{U \in SL_2(\mathbf{Z}) \ ; \ {}^tUTU = T\}$ and $\tau_T = \frac{-b+i\sqrt{\det(2T)}}{2a}$ is the CM point corresponding to $T = \left(\begin{smallmatrix} a & b/2 \\ b/2 & c\end{smallmatrix}\right)$. The Dirichlet series converges absolutely for $\Re(s) > \delta + 3/2$. Finally, we multiply some gamma factor like

$$\Psi^*(F,\mathcal{U},s) = 2\pi^{1/2}(2\pi)^{-2s}\Gamma\left(s - \frac{1}{4} + \frac{ir}{2}\right)\Gamma\left(s - \frac{1}{4} - \frac{ir}{2}\right)\Psi(F,\mathcal{U},s).$$

Here, the number $r \in \mathbf{C}$ comes from the eigenvalue $-\left(\frac{1}{4} + r^2\right)$ of Δ (note $r = i/2$ or $r \in \mathbf{R}$).

Now, we can claim Imai's converse theorem.

Fact Under the above setting, the following conditions are equivalent:

(1) The Fourier series is a Siegel cusp forms; $F(Z) \in S_k(Sp_2(\mathbf{Z}))$

(2) the Dirichlet series $\Psi^*(F,\mathcal{U},s)$ can be continued to an entire function of s, it is bounded in every vertical strip and satisfy the functional equation $\Psi^*(F,\mathcal{U},k-s) = (-1)^k\Psi^*(F,\mathcal{U},s)$ for all spectral eigenfunctions $\mathcal{U}$ with the same parity as k. □

Sketch of the proof The Dirichlet serie has the Mellin transform expression like

$$\Psi^*(F,\mathcal{U},s)=\int_{SL_2(\mathbf{Z})\backslash\mathcal{P}_2}F(iY)\mathcal{U}(Y)(\det Y)^{s-3/2}dY\quad(\Re(s)>\delta+3/2).$$

Assuming (1), the claim (2) follows from the modularity of F. Note that, under the present setting, $F(Z)$ is a cusp form if and only if $F(iY)$ ($Y=\Im Z$) is modular w.r.t. the inversion:

$(*)\quad F(Z)\in S_k(Sp_2(\mathbf{Z}))\iff F(iY^{-1})=(-1)^k(\det Y)^kF(iY)$

The converse statement claims that the assumption (2) implies this inversion modularity of F. To proceed the proof, recall that the Dirichlet series $\Psi^*(F,\mathcal{U},s)$ are the spectral coefficients of $\tilde{F}_s(\tau)\in L^2(SL_2(\mathbf{Z})\setminus H_1)$ w.r.t. the $\mathcal{U}(\tau)$. Here $\tilde{F}_s$ is the partial Mellin transform of F w.r.t. the determinant of the imaginary part of Z as

$$\tilde{F}_s(\tau):=\int_0^\infty F(i\underbrace{\sqrt{t}W_\tau}_{Y})\,t^s\frac{dt}{t},\quad W_\tau:=\begin{pmatrix}v^{-1}&-uv^{-1}\\-uv^{-1}&v^{-1}(u^2+v^2)\end{pmatrix}.$$

By the well-known identification of positive Y with its determinant t and τ in the upper half plane ($\tau=u+iv$, $t=\det Y$, $W_{\gamma\tau}=W_\tau[\gamma^{-1}]$, $\Re(s)>\delta+3/2$), the Dirichlet series has the form (the inner product in the Hilbert space, if $\mathcal{U}(\tau)$ is also L^2)

$$\Psi^*(F,\mathcal{U},s)=\int_{SL_2(\mathbf{Z})\backslash H_1}\tilde{F}_s(\tau)\mathcal{U}(\tau)\,\frac{dudv}{v^2}\qquad\left[\frac{dY}{(\det Y)^{3/2}}=\frac{dt}{t}\frac{dudv}{v^2}\right].$$

So, the assumption (2) of $\Psi^*(F,\mathcal{U},s)$ implies the corresponding properties of $\tilde{F}_s$ through the spectral expansion. And it implies the corresponding properties of the original Fourier series F by the Mellin inversion and Hecke's argument.

In order to work out this, note that one can go back to $F(iY)$ from $\tilde{F}_s(\tau)$ by the Mellin inversion. Note also that the spectral expansion holds pointwise for $\Re(s)\gg 0$. Moreover, one can take the Mellin inversion term by term in the spectral expansion in order to apply Hecke's argument to each $\Psi^*(F,\mathcal{U},s)$ (see Ibukiyama's survey paper on Duke-Imamoglu's work).

One more useful fact is that

$(**)\quad \Psi^*(f,\mathcal{U},s)=\Psi^*(g,\mathcal{U},s)\ \forall\mathcal{U}(\tau)\implies f(Z)=g(Z).$

Known applications The above point of view has several applications. The first actual application is due to Duke-Imamoglu about their analytic proof of the Saito-Kurokawa lift. Kohnen-Breulmann characterize a subset of Fourier coefficients that determines a Siegel Hecke eigen cusp form uniquely. Inspired by these prior researches, I could establish an explicit formula for the Fourier coefficients of a Siegel-Eisenstein series of degree 2 with square-free odd levels. More recently, I got a new proof of Kohnen-Martin's characterization of Siegel cusp forms by the growth of their Fourier coefficients. But I must confess that better results are obtained for each by Katsurada, Heim, Scharlau-Walling, Yamana, Saha, Gunji, Takemori, Böcherer-Das, without Koecher-Maass series.

Theory of explicit formulas Another ingredient for actual applications is Theory of explicit formulas, initiated by Böcherer, Ibukiyama and Katsurada. For example, Böcherer, Duke-Imamoglu established the followings. Suppose the Fourier coefficients is the Saito-Kurokawa type, namely, there exists $c:\mathbf{N}\to\mathbf{C}$ (polynomial growth as $n(\in\mathbf{N})\to\infty$) such that

$A(T)=\sum_{d|e(T)} d^{k-1}c\left(\frac{\det 2T}{d^2}\right)$. Here, $e(T)=(n,r,m)$ for $T=\begin{pmatrix} n & r/2 \\ r/2 & m\end{pmatrix}$. Then the associated Koecher-Maass series is a convolution product of two Dirichlet series for $\Re(s)\gg 0$;

$$\Psi(f,\mathcal{U},s):=\sum_{T\in L_2^+/SL_2(\mathbf{Z})}\frac{A(T)\mathcal{U}(\tau_T)}{\sharp E(T)(\det T)^s}=2^{2s}\zeta(2s-k+1)\sum_{n=1}^{\infty}\frac{c(n)b(-n)n^{3/4}}{n^s},$$

where $E(T)=\{U\in SL_2(\mathbf{Z})\ ;T[U]=T\}$, $b(-n)$ are the average of a spectral eigenfunction over the CM points given by

$$b(-n)=n^{-3/4}\sum_{\substack{T\in L_2^+/SL_2(\mathbf{Z})\\ \det 2T=n}}\frac{\mathcal{U}(\tau_T)}{\sharp E(T)}.$$

To understand the average values $b(-n)$, the Katok-Sarnak result is required. Let $\mathcal{U}(\tau)$ be an even spectral eigenfuntion for $L^2(SL_2(\mathbf{Z})\setminus H_1)$. Then $\exists$ a real analytic modular form $g(\tau)\in T_r^+$ w.r.t. $\Gamma_0(4)$ such that $g(\tau)=\sum_{n\equiv 0,1 \pmod 4} B(n,v)e(nu)\quad (\tau=u+iv\in H_1)$,

$$B(-n,v)=\underbrace{n^{-3/4}\left(\sum_{\substack{T\in L_2^+/SL_2(\mathbf{Z})\\ \det 2T=n}}\frac{\mathcal{U}(\tau_T)}{\sharp E(T)}\right)}_{b(-n)}\cdot W_{-1/4,ir/2}(4\pi|n|v)\quad (n>0)$$

Consequently, the convolution product

$$\Psi(f,\mathcal{U},s)=2^{2s}\zeta(2s-k+1)\cdot\sum_{n=1}^{\infty}\frac{c(n)b(-n)n^{3/4}}{n^s}$$

is likely to be a Rankin-Selberg convolution. Duke-Imamoglu could check the assumptions of the converse theorem directly by the Rankin-Selberg method to get the Saito-Kurokawa lifting.

Non-holomorphic Siegel modular case ? Can we generalize these studies on the Koecher-Maass series to non-holomorphic Siegel modular case ? The approaches applied in the holomorphic Siegel modular case due to Maass, Koecher and Arakawa have not been worked out successfully yet. Maybe because the Fourier expansion of non-holomorphic case is difficult. Firstly, there are many terms in the Fourier expansion, not always indexed by semi-positive definite T. How to separate the terms according to their signatures. Secondly, we must understand the special functions in the Fourier expansion. How to kill the degenerate terms. Analogous to Siegel's Gamma integral, we are required to compute the integral transforms of the special functions of matrix argument. Moreover, there is a problem about $\mu(T)$. Here $\mu(T)=\mathrm{vol}(SO(T,\mathbf{R})/SO(T,\mathbf{Z}))$ is a certain volume of the fundamental domain w.r.t. a suitable Haar measure. It is an indefinite analogue of the inverse of the order of the unit groups $\sharp SO(T,\mathbf{Z})$ for positive T. Then, $\mu(T)$ is not finite for indefinite T of size 2 such that $\sqrt{-\det(T)}\in\mathbf{Q}$.

Maass raised the question in his L.N.M 216, p. 308 "whether it is possible to attach Dirichlet series by means of integral transforms to the non analytic Eisenstein series or, more generally, to automorphic forms of the same type,....." "already in the case degree is two difficulties come up which show that one can not proceed in the usual way." The second comment indicates the volume problem. This talk is concerned with the non-holomorphic Siegel-Eisenstein series as a typical example.

Non-holomorphic Siegel-Eisenstein series Let $H_n = \{Z = {}^tZ \in M_n(\mathbf{C}); \Im Z > O\}$ be the Siegel half space. Let k be even, and $\sigma \in \mathbf{C}$ $(2\Re\sigma + k > 3)$. The non-holomorphic Siegel-Eisenstein series is defined by

$$E_{n,k}(Z,\sigma) = \sum_{\{C,D\}} \det(CZ+D)^{-k}|\det(CZ+D)|^{-2\sigma} \quad (Z \in H_n),$$

where the sum is taken over all non-associated coprime symmetric pairs $\{C, D\}$. It has the Fourier expansion w.r.t. the real part of Z, denoted by X $(Z = X + iY)$

$$E_{n,k}(Z,\sigma) = \sum_{T \in L_n} C(T,\sigma,Y)e(\mathrm{tr}(TX)),$$

where L_n is the set of all half-integral symmetric matrices of size n (not necessarily positive semi-definite). An explicit form of the Fourier coefficients is known by Maass, Shimura, Katsurada. It is a product of the singular series $b(T, k+2\sigma)$ and the confluent hypergeometric function $\xi(Y, T, \sigma + k, \sigma)$ as

$$C(T,\sigma,Y) = b(T, k+2\sigma) \cdot \xi(Y,T,\sigma+k,\sigma) \quad (\det T \neq 0).$$

Arakawa, Suzuki Arakawa and Suzuki defined the Koecher-Maass series directly from the arithmetic part of the Fourier coefficients. The Koecher-Maass series for $n \geq 3$ is ($\Re s \gg 0$, σ : real $> \frac{n(n+1)}{2} + 1$)

$$\zeta_i(s,\sigma) := \sum_{T \in L_n^{(i)}/SL_n(\mathbf{Z})} \frac{\mu(T)b(T,\sigma)}{|\det T|^{s-k+\frac{n+1}{2}}},$$

where $\mu(T) = \mathrm{vol}(SO(T,\mathbf{R})/SO(T,\mathbf{Z}))$ is the volume ($\mu(T) = c_n \cdot \sharp E(T)^{-1}$ for $T > O$), and $L_n^{(i)}$ is the set of all half-integral symmetric matrices of size n, signature $(i, n-i)$. The degree n is assumed to be larger than 2, in view of the volume problem. Arakawa, Suzuki established the followings. Put

$$\eta_i(s) := (2\pi)^{-ns} \prod_{r=1}^{n} \Gamma\left(s - \frac{r-1}{2}\right) \cdot \zeta_i(s,\sigma).$$

Fact Each $\eta_i(s)$ has a meromorphic continuation to all $s \in \mathbf{C}$. They satisfy the system of functional equations $\mathbf{D}^*(k-s) = \mathbf{D}^*(s)$,

$$\mathbf{D}^*(s) := e(ns/4)\sigma(s)(\eta_0(s),\ \eta_1(s),\ \cdots\cdots,\ \eta_n(s))U\left(\frac{n+1}{2} - k + s\right),$$

where $\sigma(s)$ is a pruduct of sin functions, $U(s)$ is size $n+1$, whose entries are sum of exponential functions. Remark that Suzuki's Automorphic distribution approach is generalized by Sato-Tamura-Ueno-Sugiyama-Miyazaki.

Ibukiyama-Katsurada On the other hand, Ibukiyama-Katsurada established explicit descriptions of the Koecher-Maass series. Put $L^{(j)}_{n,k}(s,\sigma) := c_{n,k,\sigma} \cdot \zeta_j(s, k+2\sigma)$. The formula depends on the parity of the degree $n \geq 3$. If degree n is odd, we need only the Riemann zeta function;

$$L^{(j)}_{n,k}(s,\sigma) = 2^{(n-1)s} \frac{\prod_{i=1}^{(n-1)/2} \zeta(1-2i)}{\zeta(1-k-2\sigma)\prod_{i=1}^{(n-1)/2}\zeta(1-2k-4\sigma+2i)}$$
$$\Bigg\{ \zeta(s)\zeta(s-k-2\sigma+1)\prod_{i=1}^{(n-1)/2}\zeta(2s-2i)\zeta(2s-2k-4\sigma+2i+1)$$
$$+(-1)^{\frac{n^2-1}{8}}(-1)^{(n-j)(n-j-1)/2+j(n+1)/2}\zeta\left(s-\frac{n-1}{2}\right)\zeta\left(s-k-2\sigma+\frac{n+1}{2}\right)$$
$$\times\prod_{i=1}^{(n-1)/2}\zeta(2s-2i+1)\zeta(2s-2k-4\sigma+2i)\Bigg\}.$$

If degree n is even, the result is similar, but a non-trivial factor appears in their formula;

$$L^{(j)}_{n,k}(s,\sigma) = 2^{ns} \frac{\prod_{i=1}^{n/2-1} \zeta(1-2i)}{\zeta(1-k-2\sigma)\prod_{i=1}^{n/2}\zeta(1-2k-4\sigma+2i)}$$
$$\Bigg\{ D(s,\sigma;(-1)^{n/2+j})\prod_{i=1}^{n/2-1}\zeta(2s-2i)\zeta(2s-2k-4\sigma+2i+1)$$
$$+\frac{1+(-1)^{n/2+j}}{2}(-1)^{(n-j)(n-j-1)/2+\frac{n(n+2)}{8}}\zeta\left(1-\frac{n}{2}\right)\zeta\left(1-k-2\sigma+\frac{n}{2}\right)$$
$$\times\prod_{i=1}^{n/2}\zeta(2s-2i+1)\zeta(2s-2k-4\sigma+2i)\Bigg\}.$$

Here, the non-trivial factor $D(s,\sigma;\pm1)$ are the following type of the Dirichlet series; it is obtained from the 3 variable Dirichlet series by a specialization w.r.t. the 3 parameters $(s,\sigma,\eta) \in \mathbf{C}^3$ with $\Re s \gg 0$ like

$$\sum_{\substack{d>0 \\ -d\equiv 0,1 \pmod 4}} \frac{L_{-d}(\sigma-1)L_{-d}(\eta-1)}{|d|^{s-\frac{\sigma}{2}+1}}, \qquad \sum_{\substack{d<0 \\ -d\equiv 0,1 \pmod 4}} \frac{L_{-d}(\sigma-1)L_{-d}(\eta-1)}{|d|^{s-\frac{\sigma}{2}+1}},$$

Here for $\forall\ D \neq 0, D \equiv 0,1 \pmod 4$, the quadratic L-function is

$$L_D(s) := L(s,\chi_K)\sum_{a|f}\mu(a)\chi_K(a)a^{-s}\sigma_{1-2s}(f/a),$$

where the natural number f is defined by $D = d_K f^2$ with the discriminant d_K of $K = \mathbf{Q}(\sqrt{D})$, χ_K is the Kronecker symbol, μ is the Möbius function and $\sigma_s(n) = \sum_{d|n} d^s$. The main aim of this talk is analysis about these Dirichlet series of 3 variables. By Ibukiyama-Saito, these can be regarded as the Rankin-Selberg convolution of real analytic Eisenstein series of half-integral weight. We shall start from recalling some setting and definition.

Maass forms Let $H_1 := \{\tau = u+iv; v>0\}$ be the upper-half plane. We always take the principal branch of log. For odd k and a complex number λ, we define the space of Maass forms of weight $-k/2$ by the usual manner. Namely, a Maass form f is a smooth function on H_1, satisfying the modularity for $\Gamma_0(4)$ of weight $-k/2$, having the eigenvalue λ of the Laplacian, and satisfies the cusp condition, that is,

$$\mathcal{F}(\Gamma_0(4), \chi, \lambda, -k/2) := \{f : H_1 \to \mathbf{C} \text{ ; smooth and (1), (2), (3)}\}$$

$$(1) \quad \frac{\left(\frac{\theta(\gamma\tau)}{\theta(\tau)}\right)^k}{|c\tau + d|^{k/2}} f(\gamma\tau) = f(\tau) \quad \forall\, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4),$$

$$(2) \quad [v^2(\partial_u^2 + \partial_v^2) + i(k/2)v\partial_u]\, f = -\lambda f,$$

(3) $f(\tau)$ has polynomial growth at every cusps of $\Gamma_0(4)$.

Any $f(\tau) \in \mathcal{F}(\Gamma_0(4), \chi, \lambda, -k/2)$ has the Fourier expansion

$$f(\tau) = A_0(v) + \sum_{d \neq 0} a_d\, W_{-\mathrm{sgn}(d)k/4, \rho}(4\pi|d|v)\; e(du).$$

Here W is the Whittaker function, ρ comes from the eigenvalue λ of the Laplacian by $\lambda = 1/4 - \rho^2$.

Plus space There is a nice subspace, called the Plus space. It is defined in terms of the Fourier expansion at the cusp infinity

$$\mathcal{F}^+(\Gamma_0(4), \chi, \lambda, -k/2) := \{f \in \mathcal{F}(\Gamma_0(4), \chi, \lambda, -k/2); a_d = 0 \text{ if } (-1)^{(k+1)/2}d \equiv 2, 3 \pmod 4\}.$$

The condition means that the d-th Fourier coefficient appears only when $(-1)^{(k+1)/2}d$ is a discriminant.

Let $f(\tau) \in \mathcal{F}^+(\Gamma_0(4), \chi, \lambda, -k/2)$ with

$$f(\tau) = \sum_{(-1)^{(k+1)/2}d \equiv 0,1 \pmod 4} c(d, v) e(du), \quad c(d, v) = a_d W_{-\mathrm{sgn}(d)k/4, \rho}(4\pi|d|v).$$

For $\mu = 0$ or 1, we define the partial Fourier series $f^{(\mu)}$

$$f^{(\mu)}(\tau) = \sum_{(-1)^{(k+1)/2}d \equiv \mu \pmod 4} c\left(d, \frac{v}{4}\right) e\left(d\frac{u}{4}\right).$$

These are obtained by picking up the half of the Fourier series, then changing the variable τ to its quarter $\tau/4$. Then these behave like a vector valued modular form on $SL_2(\mathbf{Z})$ as

$$\begin{pmatrix} f^{(0)}(\gamma\tau) \\ f^{(1)}(\gamma\tau) \end{pmatrix} = U(\gamma) \begin{pmatrix} f^{(0)}(\tau) \\ f^{(1)}(\tau) \end{pmatrix} \quad \forall\gamma \in SL_2(\mathbf{Z}), \forall\tau \in H_1.$$

Here $U(\gamma)$ is a certain unitary matrix independent of f. This property is useful, when we discuss the Rankin-Selberg convolution of Maass forms belonging to the plus space.

Differential operator When we apply the Rankin-Selberg method involving two Maass forms, we only get the Dirichlet series $\sum_{d \neq 0} \frac{L_{-d}(\sigma-1)L_{-d}(\eta-1)}{|d|^{s-\frac{a}{2}+1}}$ indexed by all non-zero integers. Later, we required to study its subseries indexed by positive d and negative d separately. To do so, we use a differential operator, following Maass and Muller's idea. The differential operator is defined by $E_{-k/2} := v(i\partial_u - \partial_v) - k/4$.

Fact

$$(1)\ \ E_{-k/2}\left(W_{-k/4,\rho}(4\pi dv)e(du)\right)=\begin{cases}\gamma(-k/4,\rho)W_{-k/4-1,\rho}(4\pi dv)e(du), & d>0,\\ W_{1+k/4,\rho}(4\pi|d|v)e(du), & d<0.\end{cases}$$

where $\gamma(\alpha,\rho):=\rho^2-(\alpha-1/2)^2$. Hence, the plus condition is stable under $E_{-k/2}$.
(2) $f(\tau)\in\mathcal{F}^+(\Gamma_0(4),\chi,\lambda,-k/2)\implies(E_{-k/2}f)(\tau)\in\mathcal{F}^+(\Gamma_0(4),\chi,\lambda,-k/2-2)$ □

Accordingly, $\begin{pmatrix}(E_{-k/2}f)^{(0)}(\gamma\tau)\\(E_{-k/2}f)^{(1)}(\gamma\tau)\end{pmatrix}=U(\gamma)\begin{pmatrix}(E_{-k/2}f)^{(0)}(\tau)\\(E_{-k/2}f)^{(1)}(\tau)\end{pmatrix}\quad\forall\gamma\in SL_2(\mathbf{Z}).$

Cohen-Ibukiyama-Saito's Eisenstein series There exists a Maass form of half-integral weight, whose Fourier coefficients are the quadratic L-functions $L_d(s)$. It belongs to the plus space. Let k be odd, $\sigma\in\mathbf{C}$ with$-k+2\Re\sigma-4>0$, and $\tau\in H_1$. The nice Eisenstein series is defined by Ibukiyama and Saito as the sum of two real analytic Eisenstein series for $\Gamma_0(4)$ like

$$F(k,\sigma,\tau):=E(k,\sigma,\tau)+2^{k/2-\sigma}(e^{2\pi i\frac{k}{8}}+e^{-2\pi i\frac{k}{8}})E\left(k,\sigma,-\frac{1}{4\tau}\right)(-2i\tau)^{k/2},$$

$$E(k,\sigma,\tau)=(\Im\tau)^{\sigma/2}\sum_{d=1,\mathrm{odd}}^{\infty}\sum_{c=-\infty}^{\infty}\left(\frac{\theta(\gamma\tau)}{\theta(\tau)}\right)^k|4c\tau+d|^{-\sigma}.$$

This corresponds to Arakawa's real analytic Jacobi Eisenstein series

$$E_{k,1}(\tau,z,s)=\frac{(\Im\tau)^s}{2}\sum_{\substack{c,d\in\mathbf{Z}\\(c,d)=1}}\sum_{\lambda\in\mathbf{Z}}\frac{e^{2\pi i(\lambda^2\frac{a\tau+b}{c\tau+d}+2\lambda\frac{z}{c\tau+d}-\frac{cz^2}{c\tau+d})}}{(c\tau+d)^k|c\tau+d|^{2s}}\quad(2\Re(s)+k>3).$$

To fit into our setting, we put $f(k,\sigma,\tau):=(\Im\tau)^{-k/4}F(k,\sigma,\tau)$. Then the eigenvalue of the Laplacian is $\lambda:=(\sigma/2-k/4)(1-\sigma/2+k/4)$, and the corresponding ρ is $\rho:=\sigma/2-k/4-1/2$. We know the following facts from the works of Ibukiyama-Saito and Shimura.

Fact (1) $f(k,\sigma,\tau)$ has the Fourier expansion

$$f(k,\sigma,\tau)=A_0(k,\sigma,v)+\sum_{\substack{d\neq0\\(-1)^{(k+1)/2}d\equiv0,\,1\pmod 4}}a_d(k,\sigma)\,W_{-sgn(d)k/4,\rho}(4\pi|d|v)e(du)$$

$$a_d(k,\sigma)=c(d,\sigma,k)\cdot i^{k/2}\pi^{\sigma/2-k/4}|d|^{\sigma/2-k/4-1}\cdot\begin{cases}\Gamma(\sigma/2-k/2)^{-1}, & d>0,\\ \Gamma(\sigma/2)^{-1}, & d<0.\end{cases}$$

$$c(d,\sigma,k)=2^{k+3/2-2\sigma}e^{(-1)^{(k+1)/2}(\pi i/4)}\frac{L_{(-1)^{(k+1)/2}d}(\sigma-\frac{k+1}{2})}{\zeta(2\sigma-k-1)},$$

where, for $\forall$ discriminant $D\neq0, D\equiv0,1\pmod 4$ $(D=d_Kf^2,\ K:=\mathbf{Q}(\sqrt{D}))$,

$$L_D(s)=L(s,\chi_K)\sum_{a|f}\mu(a)\chi_K(a)a^{-s}\sigma_{1-2s}(f/a)$$

(2) $f(k,\sigma,\tau)\in\mathcal{F}^+(\Gamma_0(4),\chi,\lambda,-k/2)$
(3) $\left(\sigma-\dfrac{k+2}{2}\right)\left(\sigma-\dfrac{k+3}{2}\right)\zeta(2\sigma-k-1)f(k,\sigma,\tau)$ is holomorphic on $\sigma\in\mathbf{C}$.

Convolution product For later use, it is sufficient to assume $k \equiv 1 \pmod 4$. We consider the following convolution series S^δ of 3 variables. Here $\delta = +$ or $-$ indicates $\sum_{d>0}$ or $\sum_{d<0}$.

$$S^\delta(s,k,\sigma,\eta) := \sum_{\delta d>0} a_d(k,\sigma)\overline{a_d(k,\overline{\eta})}|d|^{-(s-1)} \qquad (\Re s \gg 0)$$
$$= C_{k,\sigma,\eta} \cdot \sum_{\delta d>0} \frac{L_{-d}(\sigma - \frac{k+1}{2})L_{-d}(\eta - \frac{k+1}{2})}{|d|^{s-\frac{\sigma+\eta-k}{2}+1}} \times \begin{cases} \Gamma(\frac{\sigma-k}{2})^{-1}\Gamma(\frac{\eta-k}{2})^{-1}, & d>0, \\ \Gamma(\frac{\sigma}{2})^{-1}\Gamma(\frac{\eta}{2})^{-1}, & d<0, \end{cases}$$

where $C_{k,\sigma,\eta}$ is some constant. Each of $S^\delta(s,k,\sigma,\eta)$ is a half of the Rankin-Selberg convolution of two real analytic Eisenstein series $f(k,\sigma,\tau)$ and $f(k,\eta,\tau)$. To study these convolution series, one must take into account the followings;

(a) Rankin-Selberg method for two Eisenstein series (both of them are not of rapid decay)

(b) To pick up (or separate) $\sum_{d>0}$ and $\sum_{-d>0}$ from $\sum_{d\neq 0}$

(c) To study the Gamma factor $\int_0^\infty v^{s-2}W_{\alpha,\rho}(v)W_{\alpha,\kappa}(v)\,dv$

(d) To get a simple Gamma matrix in the functional equation

First of all, we record the region of the convergence of the convolution Dirichlet series. This follows from the estimation of $L_D(s)$ given later.

Fact Suppose that $(s,\rho,\kappa) \in \mathbf{C}^3$ satisfy $\Re s > \frac{3}{2} + |\Re\rho| + |\Re\kappa|$.

(1) The following series (defining $S^\delta(s,k,\sigma,\eta)$)

$$\sum_{\delta d>0,\,-d\neq\square} \frac{L_{-d}(2\rho+\frac{1}{2})L_{-d}(2\kappa+\frac{1}{2})}{|d|^{s-\rho-\kappa}}, \qquad (\ \rho := \sigma/2 - k/4 - 1/2, \quad \kappa := \eta/2 - k/4 - 1/2\ )$$

$$\left(2\rho - \frac{1}{2}\right)\left(2\kappa - \frac{1}{2}\right) \sum_{-d=\square} \frac{L_{-d}(2\rho+\frac{1}{2})L_{-d}(2\kappa+\frac{1}{2})}{|d|^{s-\rho-\kappa}}$$

are absolutely convergent for $\Re s > \frac{3}{2} + |\Re\rho| + |\Re\kappa|$.

(2) They are holomorphic for the three variables on $\Re s > \frac{3}{2} + |\Re\rho| + |\Re\kappa|$. □

To allow our manipulation freely, we note the following estimation of $L_D(s)$. The first statement is easy. The second statement follows from the functional equation of $L_D(s)$. The third statement follows from Rademacher's Phragment-Lindelof theorem. Any way, we need only a polynomial growth estimate w.r.t. the discriminant Δ.

Fact Suppose $\Delta \neq 0$ and $s \in \mathbf{C}$.

(1) If $\Re s > 1$, one has $|L_\Delta(s)| \le \zeta(\Re s)^2\zeta(2\Re s - 1)$.

(2) If $\Re s < 0$, one has $|L_\Delta(s)| \le |\Delta|^{\frac{1}{2}-\Re s}\zeta(1-\Re s)^2\zeta(1-2\Re s)\cdot|\gamma_{\mathrm{sgn}(\Delta)}(s)|$ by (1) and

$$L_\Delta(s) = |\Delta|^{\frac{1}{2}-s}\gamma_{\mathrm{sgn}(\Delta)}(s)L_\Delta(1-s),\ \gamma_{\mathrm{sgn}(\Delta)}(s) = \begin{cases} \pi^{-\frac{1}{2}+s}\dfrac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}, & \Delta>0, \\ \pi^{-\frac{1}{2}+s}\dfrac{\Gamma(\frac{2-s}{2})}{\Gamma(\frac{s+1}{2})}, & \Delta<0. \end{cases}$$

(3) On the strip $S(-\xi, 1+\xi) = \{s \in \mathbf{C} : -\xi \le \Re s \le 1+\xi\}$ $(0 < \xi \le \frac{1}{2}$: fixed),

(3 − 1) If $\Delta \neq \square$, one has $|L_\Delta(s)| \le \left(\frac{|\Delta|}{2\pi}\right)^{\frac{1+\xi-\Re s}{2}} |1+s|^{\frac{1+\xi-\Re s}{2}}\zeta(1+\xi)^2\zeta(1+2\xi)$.

(3 − 2) If $\Delta = \square$, one has $L_\Delta(0) = -|\Delta|^{\frac{1}{2}}/2$ and

$$|s(1-s)L_\Delta(s)| \le \left(\frac{|\Delta|}{2\pi}\right)^{\frac{1+\xi-\Re s}{2}} \left(\frac{1+\xi}{1-\xi}\right)^{\frac{1+\xi-\Re s}{1+2\xi}} |1+s|^{2+\frac{1+\xi-\Re s}{2}} \zeta(1+\xi)^2\zeta(1+2\xi).$$

Whittaker function Next, we recall some basic properties of $W_{\alpha,\mu}(v)$ $(v>0)$. It can be continued to a holomorphic function for all $(\alpha,\mu) \in \mathbf{C}^2$. It satisfies the relation $W_{\alpha,-\mu}(v) = W_{\alpha,\mu}(v)$, and the differential equations ($' = d/dv$)

$$v^2 W''_{\alpha,\mu}(v) = \left(\frac{1}{4}v^2 - \alpha v + \mu^2 - \frac{1}{4}\right) W_{\alpha,\mu}(v),\ \ vW'_{\alpha,\mu}(v) = -\left(\alpha v - \frac{1}{2}v\right) W_{\alpha,\mu}(v) - W_{\alpha+1,\mu}(v).$$

Its asymptotic behaviour are well known,

$$W_{\alpha,\mu}(v) \sim v^\alpha e^{-\frac{v}{2}}\ as\ v\to\infty,\ \ W_{\alpha,\mu}(v) = \begin{cases} O(v^{\frac{1}{2}-|\Re\mu|}), & \mu\neq 0 \\ O(v^{\frac{1}{2}}|\log v|), & \mu = 0 \end{cases}\ \ as\ v\to 0.$$

Finally, a uniform estimation is known by Shimura; For $\forall$ compact set K of $\mathbf{C}^2$, $\exists$ positive constants $A,\ B>0$ such that

$$|W_{\alpha,\mu}(v)| \le A v^{\Re\alpha} e^{-\frac{v}{2}}(1+v^{-B})\ \ \forall v>0, \forall(\alpha,\mu)\in K.$$

Gamma factor The Mellin transforms of the product of two Whittaker functions arise naturally, when we treat the Rankin-Selberg convolution of two Maass forms. We follow the Muller's treatment. For $\forall\alpha\in\mathbf{R}$ and $\forall s,\rho,\kappa\in\mathbf{C}$, we define

$$G_{\alpha,\rho,\kappa}(s) := \int_0^\infty v^{s-2} W_{\alpha,\rho}(v) W_{\alpha,\kappa}(v)\ dv \qquad (\Re s > |\Re\rho| + |\Re\kappa|).$$

Fact Put $t_1 = \rho+\kappa$ and $t_2 = \rho-\kappa$.
(1) The integral defining $G_{\alpha,\rho,\kappa}(s)$ is absolutely convergent and holomorphic for $(s,\rho,\kappa)\in\mathbf{C}^3$ on the region $\Re s > |\Re\rho| + |\Re\kappa|$. It satisfies the reccurence

$$s(s+1)G_{\alpha,\rho,\kappa}(s+2) = 2\alpha s(2s+1)G_{\alpha,\rho,\kappa}(s+1) + (s^2-t_1^2)(s^2-t_2^2)G_{\alpha,\rho,\kappa}(s).$$

(2) For $\forall\ M\in\mathbf{N}$, $\exists$ polynomials $p_M(s)$ and $q_M(s)\in\mathbf{R}[s]$ satisfying

$$G_{\alpha,\rho,\kappa}(s)\prod_{j=0}^{M}\prod_{l=1}^{2}\{(s+j)^2-t_l^2\} = p_M(s)G_{\alpha,\rho,\kappa}(s+M+1) + q_M(s)G_{\alpha,\rho,\kappa}(s+M+2).$$

This gives a meromorphic continuation of $G_{\alpha,\rho,\kappa}(s)$ to all $(s,\rho,\kappa)\in\mathbf{C}^3$. □

In fact, we may take M sufficiently large such that the two integrals on the r.h.s. are absolutely convergent and holomorphic as a function of the 3 complex variables in wider region. The possible polar divisors arise from the product of linear forms $\prod_{j=0}^{M}\prod_{l=1}^{2}\{(s+j)^2-t_l^2\}$ of s,t_1,t_1, in other words, s,σ,ρ. The following evaluation formulas are required to get a simple functional equation. Consider the following 3 functions;

$$D_{\rho,\kappa}(s) := G_{\alpha,\rho,\kappa}(s)G_{1-\alpha,\rho,\kappa}(s) - \gamma(\alpha,\rho)\gamma(\alpha,\kappa)G_{\alpha-1,\rho,\kappa}(s)G_{-\alpha,\rho,\kappa}(s),$$

$$\mathcal{V}^-_{\alpha,\rho,\kappa}(s) := G_{1-\alpha,\rho,\kappa}(s)G_{-\alpha,\rho,\kappa}(1-s) - G_{1-\alpha,\rho,\kappa}(1-s)G_{-\alpha,\rho,\kappa}(s),$$

$$\mathcal{V}^+_{\alpha,\rho,\kappa}(s) := G_{1-\alpha,\rho,\kappa}(s)G_{\alpha,\rho,\kappa}(1-s) - \gamma(\alpha,\rho)\gamma(\alpha,\kappa)G_{\alpha-1,\rho,\kappa}(1-s)G_{-\alpha,\rho,\kappa}(s).$$

Here $\gamma(\alpha,\rho) := \rho^2 - (\alpha - 1/2)^2$. These arise naturally as a product of 2 by 2 matrix, whose entries are the Gamma factors G. We can describe these functions in terms of the usual gamma functions and the trigonometric functions.

Fact Let $\alpha \in \mathbf{R}$, $s, \rho, \kappa \in \mathbf{C}$ and $\mathcal{J} = \{\pm t_1, \pm t_2\}$ with $t_1 = \rho + \kappa$, $t_2 = \rho - \kappa$. One has

$$D_{\rho,\kappa}(s) = \frac{\prod_{t\in\mathcal{J}} \Gamma(s+t)}{\Gamma(s)^2}, \qquad \mathcal{V}^-_{\alpha,\rho,\kappa}(s) = E(\alpha,\rho,\kappa)\frac{\sin(2\pi s)}{\prod_{t\in\mathcal{J}} \sin\pi(s+t)},$$

$$\mathcal{V}^+_{\alpha,\rho,\kappa}(s) = \pi\sin(\pi s)\frac{\cos(\pi s)\cos\pi(s+2\alpha) + \cos(\pi t_1)\cos(\pi t_2)}{\prod_{t\in\mathcal{J}} \sin\pi(s+t)}.$$

Here $E(\alpha,\rho,\kappa) = \frac{-\pi^3}{\Gamma(\frac{1}{2}+\alpha+\rho)\Gamma(\frac{1}{2}+\alpha-\rho)\Gamma(\frac{1}{2}+\alpha+\kappa)\Gamma(\frac{1}{2}+\alpha-\kappa)}$.

Rankin-Selberg method for two Eisenstein series From Cohen-Ibukiyama-Saito's Eisenstein series, we define $\mathcal{H}_{k,\sigma,\eta}(\tau) = \mathcal{F}_{k,\sigma,\eta}(\tau)$ or $\mathcal{G}_{k,\sigma,\eta}(\tau)$, where

$$\mathcal{F}_{k,\sigma,\eta}(\tau) := \sum_{\mu=0,1} f^{(\mu)}(k,\sigma,\tau)\ \overline{f^{(\mu)}(k,\overline{\eta},\tau)},$$

$$\mathcal{G}_{k,\sigma,\eta}(\tau) := \sum_{\mu=0,1} (E_{-k/2}f)^{(\mu)}(k,\sigma,\tau)\ \overline{(E_{-k/2}f)^{(\mu)}(k,\overline{\eta},\tau)}.$$

Recall that $f^{(\mu)}(k,\sigma,\tau)$ behave like a vector valued modular form on $SL_2(\mathbf{Z})$, and it is described by a certain unitary matrix. Hence, these newly defined functions behave like

$$\mathcal{H}_{k,\sigma,\eta}(\gamma\tau) = \mathcal{H}_{k,\sigma,\eta}(\tau) \quad \forall\, \gamma \in SL_2(\mathbf{Z}), \quad \forall\, \tau \in H_1.$$

This observation simplifies the Rankin-Selberg method, since the level 4 decrease to 1. To this newly defined functions, we associate the Rankin-Selberg transform following Zagier

$$R(\mathcal{H}_{k,\sigma,\eta}, s) := \int_0^\infty \int_0^1 [\mathcal{H}_{k,\sigma,\eta}(\tau) - \psi_{\mathcal{H}_{k,\sigma,\eta}}(v/4)]v^{s-2}\, dudv \quad (\Re s \gg 0,\ \tau = u+iv).$$

Here $\psi_{\mathcal{F}_{k,\sigma,\eta}}(v) := A_0(k,\sigma,v)\overline{A_0(k,\overline{\eta},v)}$ and $\psi_{\mathcal{G}_{k,\sigma,\eta}}(v) := (E_{-k/2}A_0)(k,\sigma,v)\overline{(E_{-k/2}A_0)(k,\overline{\eta},v)}$ and A_0 comes from the constant term of Cohen-Ibukiyama-Saito's Eisenstein series. We must subtract ψ for the convergence of the integral. By Zagier's Rankin-Selberg method, we can study this integral transforms.

Fact Put $\rho := \sigma/2 - k/4 - 1/2$, $\kappa := \eta/2 - k/4 - 1/2$.

(1) The integral is absolutely convergent for $\Re s > 2 + |\Re\rho| + |\Re\kappa|$, and has the expression

$$\pi^{s-1}R(\mathcal{F}_{k,\sigma,\eta}, s) = G_{-k/4,\rho,\kappa}(s)S^+(s,k,\sigma,\eta) + G_{k/4,\rho,\kappa}(s)S^-(s,k,\sigma,\eta),$$

$$\pi^{s-1}R(\mathcal{G}_{k,\sigma,\eta}, s) = \gamma(-k/4,\rho)\gamma(-k/4,\kappa)G_{-k/4-1,\rho,\kappa}(s)S^+(s,k,\sigma,\eta) + G_{1+k/4,\rho,\kappa}(s)S^-(s,k,\sigma,\eta).$$

$$\left[\ S^\pm(s,k,\sigma,\eta) = \sum_{\pm d>0} \frac{L_{-d}(\sigma - \frac{k+1}{2})L_{-d}(\eta - \frac{k+1}{2})}{|d|^{s-\frac{\sigma+\eta-k}{2}+1}},\quad G_{\alpha,\rho,\kappa}(s) := \int_0^\infty v^{s-2}W_{\alpha,\rho}(v)W_{\alpha,\kappa}(v)\, dv\ \right]$$

(2) $R^*(\mathcal{H}_{k,\sigma,\eta}, s) := \zeta^*(2s)R(\mathcal{H}_{k,\sigma,\eta}, s)$ can be meromorphically continued to all $(s,\sigma,\eta) \in \mathbf{C}^3$.

(3) They satisfy the functional equation $R^*(\mathcal{H}_{k,\sigma,\eta}, s) = R^*(\mathcal{H}_{k,\sigma,\eta}, 1-s)$. □

In fact, these statements follows from the integral representation

$$R^*(\mathcal{H}_{k,\sigma,\eta},s)=\int\int_{D_T}\mathcal{H}_{k,\sigma,\eta}(\tau)E^*(\tau,s)\,\frac{dudv}{v^2}-\zeta^*(2s)h_{T,\mathcal{H}_{k,\sigma,\eta}}(s)-\zeta^*(2s-1)h_{T,\mathcal{H}_{k,\sigma,\eta}}(1-s)$$
$$+\int\int_{D-D_T}[\mathcal{H}_{k,\sigma,\eta}(\tau)E^*(\tau,s)-\psi_{\mathcal{H}_{k,\sigma,\eta}}(v/4)e(v,s)]\,\frac{dudv}{v^2}.$$

Here $D:=\{\tau=u+iv\in H;|\tau|\geq 1,|u|\leq 1/2\}$ and $D_T:=\{u+iv\in D;v\leq T\}$ $(T\gg 0)$,

$$E^*(\tau,s):=\frac{1}{2}\zeta^*(2s)\cdot\sum_{\substack{c,d\in\mathbf{Z}\\(c,d)=1}}\frac{v^s}{|c\tau+d|^{2s}},\qquad h_{T,\mathcal{H}_{k,\sigma,\eta}}(s):=\sum_{j=1}^{4}c_j\cdot\frac{T^{s+\alpha_j-1}}{s+\alpha_j-1}$$

with explicit c_j,α_j, and $e(v,s)$ is the constant term of $E^*(\tau,s)$.

Separating $S^+(s,k,\sigma,\eta)$ and $S^-(s,k,\sigma,\eta)$ from $R(\mathcal{H}_{k,\sigma,\eta},s)$ Now, write the relation between the Rankin-Selberg transforms and convolution series S^δ in a matrix form like

$$\pi^{s-1}\begin{pmatrix}R(\mathcal{F}_{k,\sigma,\eta},s)\\R(\mathcal{G}_{k,\sigma,\eta},s)\end{pmatrix}=\begin{pmatrix}G_{-k/4,\rho,\kappa}(s)&G_{k/4,\rho,\kappa}(s)\\\gamma(-k/4,\rho,\kappa)G_{-k/4-1,\rho,\kappa}(s)&G_{1+k/4,\rho,\kappa}(s)\end{pmatrix}\begin{pmatrix}S^+(s,k,\sigma,\eta)\\S^-(s,k,\sigma,\eta)\end{pmatrix}.$$

Inverting this, we can separate each one sided convolution series S^δ as desired;

$$\therefore\ \begin{pmatrix}S^+(s,k,\sigma,\eta)\\S^-(s,k,\sigma,\eta)\end{pmatrix}$$
$$=\frac{\pi^{s-1}}{\mathcal{D}_{\rho,\kappa}(s)}\begin{pmatrix}G_{1+k/4,\rho,\kappa}(s)&-G_{k/4,\rho,\kappa}(s)\\-\gamma(-k/4,\rho,\kappa)G_{-k/4-1,\rho,\kappa}(s)&G_{-k/4,\rho,\kappa}(s)\end{pmatrix}\begin{pmatrix}R(\mathcal{F}_{k,\sigma,\eta},s)\\R(\mathcal{G}_{k,\sigma,\eta},s)\end{pmatrix}.$$

Since the desired analytic properties of the Rankin-Selberg transforms R are well understood, we have a meromorphic continuation of each convolution series $S^\pm$. Moreover, possible polar divisors can be given explicitly. That is the 2 functions

$$S^\pm(s,k,\sigma,\eta)\cdot\zeta^*(2s)\Gamma(s)^{-2}\cdot s(s-1)(s-1/2)\prod_{j=1}^{4}\{(s+\alpha_j-1)(s-\alpha_j)\}$$
$$\times(\sigma-(k+2)/2)(\sigma-(k+3)/2)\zeta(2\sigma-k-1)\cdot(\eta-(k+2)/2)(\eta-(k+3)/2)\cdot\zeta(2\eta-k-1)$$

are holomorphic functions for all $(s,\sigma,\eta)\in\mathbf{C}^3$. In summary, we have the main theorem.

Theorem The Dirichlet series $S^\pm(s,k,\sigma,\eta)$ can be meromorphically continued to all $(s,\sigma,\eta)\in\mathbf{C}^3$. They satisfy the vector functional equation

$$\begin{pmatrix}S^+(s,k,\sigma,\eta)\\S^-(s,k,\sigma,\eta)\end{pmatrix}=\frac{\pi^{2s-1}\varphi(s)}{D_{\rho,\kappa}(s)}\begin{pmatrix}\mathcal{V}^+_{-k/4,\rho,\kappa}(s)&\mathcal{V}^-_{-k/4,\rho,\kappa}(s)\\\mathcal{V}^-_{k/4,\rho,\kappa}(s)&\mathcal{V}^+_{k/4,\rho,\kappa}(s)\end{pmatrix}\begin{pmatrix}S^+(1-s,k,\sigma,\eta)\\S^-(1-s,k,\sigma,\eta)\end{pmatrix},$$

$\rho:=\sigma/2-k/4-1/2$, $\kappa:=\eta/2-k/4-1/2$, $\varphi(s)=\dfrac{\zeta^*(2-2s)}{\zeta^*(2s)}$ with $\zeta^*(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$.

The gamma matrix can be described explicitly using the usual gamma functions and the trigonometric functions. For $\alpha\in\mathbf{R}$, $s,\rho,\kappa\in\mathbf{C}$ and $\mathcal{J}=\{\pm t_1,\pm t_2\}$ with $t_1=\rho+\kappa$, $t_2=\rho-\kappa$,

$$D_{\rho,\kappa}(s)=\frac{\prod_{t\in\mathcal{J}}\Gamma(s+t)}{\Gamma(s)^2},\qquad \mathcal{V}^-_{\alpha,\rho,\kappa}(s)=E(\alpha,\rho,\kappa)\frac{\sin(2\pi s)}{\prod_{t\in\mathcal{J}}\sin\pi(s+t)},$$

$$\mathcal{V}^+_{\alpha,\rho,\kappa}(s)=\pi\sin(\pi s)\frac{\cos(\pi s)\cos\pi(s+2\alpha)+\cos(\pi t_1)\cos(\pi t_2)}{\prod_{t\in\mathcal{J}}\sin\pi(s+t)}.$$

Koecher-Maass series for $E_{n,k}(Z,\sigma)$ (even degree $n \geq 4$, even weight k) We apply the above result to the Koecher-Maass series of non-holomorphic Siegel-Eisenstein series. Suppose the degree is even and greater than 2. In Ibukiyama-Katsurada's explicit formula, the non-trivial factor is given by the following Dirichlet series ($\Re s \gg 0$).

$$G_n^+(s,\sigma) = \pi^{-2s}\zeta(2s)\Gamma(s+t_1)\Gamma(s+t_2), \quad G_n^-(s,\sigma) = \pi^{-2s}\zeta(2s)\Gamma(s-t_1)\Gamma(s-t_2),$$

$$t_1 = \sigma + k/2 - 1/2, \quad t_2 = n/2 - k/2 - \sigma,$$

$$\Omega_n^+(s,\sigma) := G_n^+(s,\sigma)\cdot \sum_{(-1)^{\frac{n}{2}+1}d>0} \frac{L_{-d}(\frac{n}{2})L_{-d}(2\sigma+k-\frac{n}{2})}{|d|^{s-\sigma-\frac{k}{2}+\frac{1}{2}}},$$

$$\Lambda_n^-(s,\sigma) := G_n^-(s,\sigma)\cdot \sum_{(-1)^{\frac{n}{2}}d>0} \frac{L_{-d}(\frac{n}{2})L_{-d}(2\sigma+k-\frac{n}{2})}{|d|^{s-\sigma-\frac{k}{2}+\frac{1}{2}}}.$$

By our Theorem, a simple specialization of the parameters implies the following results.

Theorem The Dirichlet series $\Omega_n^+(s,\sigma)$ and $\Lambda_n^-(s,\sigma)$ can be meromorphically continued to all $(s,\sigma)\in \mathbf{C}^2$. They satisfy the functional equations

$$\Omega_n^+(s,\sigma) = \Omega_n^+(1-s,\sigma),$$

$$\Lambda_n^-(s,\sigma) = \Lambda_n^-(1-s,\sigma) - 2(-1)^{\frac{k}{2}} \frac{\cos(\pi\sigma)\cos(\pi s)}{\cos\pi(s-\sigma)\sin\pi(s+\sigma)} \frac{G_n^-(1-s,\sigma)}{G_n^+(1-s,\sigma)}\Omega_n^+(1-s,\sigma). \quad \square$$

These functional equations can be used to simplify Arakawa and Suzuki's functional equation.

Toward to the degree 2 case Note the followings;

- We cannot put $n=2$ in Ibukiyama-Katsurada's explicit formula,

$$\sum_{\mp(-1)^{\frac{n}{2}}d>0} \frac{L_{-d}(\frac{n}{2})L_{-d}(2\sigma+k-\frac{n}{2})}{|d|^{s-\sigma-\frac{k}{2}+\frac{1}{2}}}.$$

- But it is only when $-d = \square = f^2$, in which case $L_{-d}(s) = \zeta(s)\sum_{a|f}\mu(a)a^{-s}\sigma_{1-2s}(f/a)$.
- On the other hand, even when degree is 2, Ibukiyama-Katsurada's explicit formula holds true for almost all terms, if we ignore the terms such that $\mu(T)$ is infinite. More precisely

$$(\sharp) \qquad \sum_{T\in(L_2^-)'/SL_2(\mathbf{Z})} \frac{\mu(T)b(T,\sigma)}{|\det T|^s} \doteqdot \sum_{-d>0,\ -d\neq\square} \frac{L_{-d}(\sigma-1)\cdot|d|^{\frac{1}{2}}L_{-d}(1)}{|d|^s},$$

$$(L_2^-)' := \left\{T = \begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}; \text{indefinite}, a,b,c\in\mathbf{Z}, \sqrt{-\det(T)}\notin\mathbf{Q}\right\}.$$

Hence, one can define the Koecher-Maass series by the Dirichlet series ($\sharp$). While then, its analytic continuation and its functional equation turned out to be non-trivial.

Prof. Ibukiyama's suggestion Prof. Ibukiyama suggested to me the following approach in order to treat the case degree 2. First, prove an analytic continuation and a functional equation of the Dirichlet series with parameter η like $\displaystyle\sum_{\substack{d<0\\ -d\equiv 0,1 \ (\mathrm{mod}\ 4)}} \frac{L_{-d}(\sigma-1)L_{-d}(\eta-1)}{|d|^{s-\frac{\sigma}{2}+1}}$.

Next, consider the Laurent expansion around $\eta = 2$ on the both sides of the functional equation. Then, as the constant term of the Laurent expansion, we should get "the main part ($\sharp$)" and "a natural correction term", in the sense that the Dirichlet series with "a correction term" has an analytic continuation and a functional equation.

In fact, this is Ibukiyama-Saito's approach on Shintani's zeta functions of symmetric matrices of size 2. We have established the analytic continuation and the functional equation of the Dirichlet series of 3 variables. I worked out the computation of the constant term of the Laurent expansion. The results are as follows.

The case of degree 2 For any sign $\delta = +$ or $\delta = -$, put

$$G_2^\delta(s,\sigma) := \pi^{-2s}\zeta(2s)\Gamma\left(s+\delta\cdot\frac{\sigma-1}{2}\right)\Gamma\left(s-\delta\cdot\frac{\sigma-2}{2}\right).$$

For $(s,\sigma)\in\mathbf{C}^2$ with $\Re s \gg 0$, we define

$$\begin{aligned}\Omega^-(s,\sigma) &:= G_2^-(s,\sigma)\cdot\sum_{-d>0,\ -d\neq\square}\frac{L_{-d}(\sigma-1)\cdot|d|^{\frac12}L_{-d}(1)}{|d|^{s-\frac{\sigma}{2}+1}}\\ &+\ \zeta(\sigma-1)\frac{\zeta(2s-\sigma+1)\zeta(2s+\sigma-2)}{\zeta(2s)}G_2^-(s,\sigma)\\ &\cdot\left(\frac{\zeta'}{\zeta}(2s+\sigma-1)+\frac{\zeta'}{\zeta}(2s-\sigma+2)-\frac{\zeta'}{\zeta}(2s+\sigma-2)-\frac{\zeta'}{\zeta}(2s-\sigma+1)+P(s,\sigma)\right),\end{aligned}$$

where $P(s,\sigma) := \sum_p \frac{(p^{-2s-1}-p^{-2s})\log p}{(1-p^{-2s-\sigma+1})(1-p^{-2s+\sigma-2})}$ for $\Re s \gg 0$.

Similarly, we define $\Omega^+(s,\sigma) := G_2^+(s,\sigma)\cdot\frac{1}{2\pi}\sum_{d>0}\frac{L_{-d}(\sigma-1)\cdot d^{\frac12}L_{-d}(1)}{|d|^{s-\frac{\sigma}{2}+1}}$, and

$$\begin{aligned}\mathcal{G}(s,\sigma) &:= \frac{-\pi}{\sin\pi\left(s-\frac{\sigma}{2}\right)\cos\pi\left(s+\frac{\sigma}{2}\right)}\\ &+\ \frac{\Gamma'}{\Gamma}\left(s+\frac{\sigma-1}{2}\right)-\frac{\Gamma'}{\Gamma}\left(s-\frac{\sigma-1}{2}\right)-\frac{\Gamma'}{\Gamma}\left(s+\frac{\sigma-2}{2}\right)+\frac{\Gamma'}{\Gamma}\left(s-\frac{\sigma-2}{2}\right).\end{aligned}$$

Theorem The Dirichlet series $\Omega^\pm(s,\sigma)$ can be meromorphically continued to the whole $(s,\sigma)\in\mathbf{C}^2$, and satisfy the functional equations

$$\begin{aligned}\Omega^-(1-s,\sigma) &= \Omega^-(s,\sigma)-\frac{2^2\pi\cos(\frac{\pi\sigma}{2})\cos(\pi s)}{\sin\pi\left(s-\frac{\sigma}{2}\right)\cos\pi\left(s+\frac{\sigma}{2}\right)}\frac{G_2^-(s,\sigma)}{G_2^+(s,\sigma)}\Omega^+(s,\sigma)\\ &\quad+2^{-1}\zeta(\sigma-1)\frac{\zeta(2s-\sigma+1)\zeta(2s+\sigma-2)}{\zeta(2s)}G_2^-(s,\sigma)\mathcal{G}(s,\sigma),\end{aligned}$$

$$\Omega^+(1-s,\sigma) = \Omega^+(s,\sigma)+\frac{\zeta(\sigma-1)\sin\left(\frac{\pi\sigma}{2}\right)\cos(\pi s)}{2\cos\pi(s-\frac{\sigma}{2})\sin\pi(s+\frac{\sigma}{2})}\frac{G_2^+(s,\sigma)\zeta(2s+\sigma-2)\zeta(2s-\sigma+1)}{\zeta(2s)}.$$

Application to Koecher-Maass series Recall Kaufhold's formula for the singular series;

$$b(T,\sigma) = \frac{1}{\zeta(\sigma)\zeta(2\sigma-2)}\cdot\sum_{d|e(T)} d^{2-\sigma}L_{\frac{-(\det 2T)}{d^2}}(\sigma-1),\quad e(T)=(n,r,m)\ \text{ for }\ T=\begin{pmatrix} n & r/2\\ r/2 & m\end{pmatrix}.$$

The Koecher-Maass series for *positive-definite* Fourier coefficients can be defined for $\Re s \gg 0$ by

$$\xi_2^+(s,\sigma) := (2\pi)^{-2s}\zeta(\sigma)\zeta(2\sigma-2)\Gamma\left(s+\sigma-\frac{3}{2}\right)\Gamma(s)\cdot \sum_{T\in L_2^+/SL_2(\mathbf{Z})} \frac{b(T,\sigma)}{\sharp E(T)(\det T)^s},$$

$$L_2^+ = \left\{T = \begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix} > O \;;\; a,b,c\in\mathbf{Z}\right\},\quad E(T) = \{U\in SL_2(\mathbf{Z})\;;\; T[U]=T\}.$$

By Böcherer, one has

$$\xi_2^+(s,\sigma) = \pi^{\sigma-2}\Omega^+\left(s+\frac{\sigma}{2}-1,\sigma\right) \doteqdot \sum_{d>0} \frac{L_{-d}(\sigma-1)\cdot d^{\frac{1}{2}}L_{-d}(1)}{|d|^{s-\frac{\sigma}{2}+1}}.$$

Theorem The Koecher-Maass series $\xi_2^+(s,\sigma)$ can be meromorphically continued to the whole $(s,\sigma)\in\mathbf{C}^2$. It satisfies a functional equation similar to $\Omega^+\left(s+\frac{\sigma}{2}-1,\sigma\right)$. □

The Koecher-Maass series for *indefinite* Fourier coefficients should be defined for $\Re s \gg 0$ by

$$\xi_2^-(s,\sigma) := (2\pi)^{-2s}\zeta(\sigma)\zeta(2\sigma-2)\Gamma\left(s-\frac{1}{2}\right)\Gamma(s+\sigma-2)\cdot \sum_{T\in(L_2^-)'/SL_2(\mathbf{Z})} \frac{\mu(T)b(T,\sigma)}{|\det T|^s}$$

$$+\quad 2\pi^{-2s}\Gamma\left(s-\frac{1}{2}\right)\Gamma(s+\sigma-2)\,\zeta(2s-1)\zeta(2s+2\sigma-4)$$

$$\cdot\left(\frac{\zeta'}{\zeta}(2s+2\sigma-3)+\frac{\zeta'}{\zeta}(2s)-\frac{\zeta'}{\zeta}(2s+2\sigma-4)-\frac{\zeta'}{\zeta}(2s-1)+P\left(s+\frac{\sigma}{2}-1,\sigma\right)\right).$$

Here $(L_2^-)' = \left\{T=\begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix};\text{indefinite}, a,b,c\in\mathbf{Z}, \sqrt{-\det(T)}\notin\mathbf{Q}\right\}$, and for $T=\begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix}\in(L_2^-)'$, $S_T=\{\tau=u+iv\;;\; v>0,\; a(u^2+v^2)+bu+c=0\}$, $\mu(T)$ is the non-Euclidean length of a fundamental domain on S_T for $E(T)=\{U\in SL_2(\mathbf{Z})\;;\;T[U]=T\}$. Similar to Böcherer, one has

$$\xi_2^-(s,\sigma) = 2\pi^{\sigma-2}\Omega^-\left(s+\frac{\sigma}{2}-1,\sigma\right) \doteqdot \sum_{-d>0,\ -d\neq\square} \frac{L_{-d}(\sigma-1)\cdot|d|^{\frac{1}{2}}L_{-d}(1)}{|d|^{s-\frac{\sigma}{2}+1}}.$$

Theorem The Koecher-Maass series $\xi_2^-(s,\sigma)$ can be meromorphically continued to the whole $(s,\sigma)\in\mathbf{C}^2$. It satisfies a functional equation similar to $\Omega^-\left(s+\frac{\sigma}{2}-1,\sigma\right)$. □

In order to relate the 3 variable Dirichlet series and the Koecher-Maass series, we have applied Böcherer-Duke-Imamoglu type computation combined with Kaufhold's formula and the Class number formulas

$$L_{-d}(1) = \frac{2\pi}{d^{1/2}}\sum_{\substack{T\in L_2^+/SL_2(\mathbf{Z})\\ \det 2T=d}}\frac{1}{\sharp E(T)}\ (d>0),\quad L_d(1)=\frac{1}{2d^{1/2}}\sum_{\substack{T\in(L_2^-)'/SL_2(\mathbf{Z})\\ -\det(2T)=d}}\mu(T)\ \ (d>0; d\neq\square).$$

Average of the Hurwitz class numbers For any negative discriminant $-d$, define $H(d)$ by $H(d) := \sum_{\substack{T\in L_2^+/SL_2(\mathbf{Z})\\ \det 2T=d}} \frac{1}{\sharp E(T)}$. Let α_σ be a certain constant $\alpha_\sigma := \frac{\pi}{12}\frac{\zeta(\sigma+2)\zeta(2\sigma+2)}{\zeta(\sigma+3)}$.

For a fixed $\sigma \geq 0$, one has by Tauberian theorem

$$\sum_{d \leq X} L_{-d}(\sigma+1)H(d) \sim \frac{\alpha_\sigma}{3}X^{3/2}, \qquad \sum_{d \leq X} H(d)^2 \sim \frac{\pi^4}{2^7 \cdot 3^3 \cdot \zeta(3)}X^2,$$

$$\sum_{d \leq X} d^{\frac{\sigma+1}{2}} L_{-d}(\sigma+1)H(d) \sim \frac{\alpha_\sigma}{\sigma+4}X^{2+\sigma/2}.$$

Böcherer obtained the case $\sigma = k-2$ using Arakawa's residue formula. There exists Arakawa's unpublished work about $\sum_{d=1}^{\infty} \frac{H(d)^2}{d^s}$ and its application to the average $\sum_{d \leq X} H(d)^2$.

References

[1] T. Arakawa, Dirichlet series related to the Eisenstein series on the Siegel upper half-plane. Comment. Math. Univ. St. Paul. **27** (1978), no. 1, 29–42.

[2] S. Böcherer, Bemerkungen über die Dirichletreihen von Koecher und Maass, Mathematica Göttingensis, Schriftenreihe des SFB Geometrie und Analysis, Heft **68** (1986).

[3] T. Ibukiyama, H. Katsurada, Koecher-Maass series for real analytic Siegel Eisenstein series, "Automorphic Forms and Zeta Functions, Proceedings of the conference in memory of Tsuneo Arakawa" pp. 170–197, World Scientific 2006.

[4] T. Ibukiyama, H. Saito, On zeta functions associated to symmetric matrices, II: Functional equations and special values. Nagoya Math. J. **208** (2012), 265–316.

[5] H. Maass, Siegel's modular forms and Dirichlet series. Lecture Notes in Mathematics, **216**, Springer-Verlag, Berlin-New York. v+328 pp. (1971)

[6] W. Müller, The Rankin-Selberg method for non-holomorphic automorphic forms. J. Number Theory. **51** (1995), no. 1, 48–86.

[7] A. Pitale, Jacobi Maass forms. Abh. Math. Semin. Univ. Hambg. **79** (2009), no. 1, 87–111.

[8] D. Zagier, The Rankin-Selberg method for automorphic functions which are not of rapid decay, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 415–437.

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