STRONG LOCAL–GLOBAL PHENOMENA FOR GALOIS AND AUTOMORPHIC REPRESENTATIONS

KIMBALL MARTIN

ABSTRACT. Many results are known regarding how much local information is required to determine a global object, such as a modular form, or a Galois or automorphic representation. We begin by discussing some things that are known and expected, and then explain recent joint work with Dinakar Ramakrishnan about comparing degree 2 Artin and automorphic representations which *a priori* may not correspond at certain infinite sets of places.

These notes are based on a talk I gave at the RIMS workshop, "Modular forms and automorphic representations," from Feb 2–6, 2015, which was in turn based on the joint work [MR] with Ramakrishnan. I am grateful to the organizers for the opportunity to present this exposition.

1. INTRODUCTION

A local–global principle, or phenomenon, is a situation where certain local conditions are sufficient to imply a corresponding global condition. Examples both of local–global principles (e.g., zeroes of quadratic forms, norms in cyclic extensions, *Grunwald*–Wang, splitting of central simple algebras) as well as examples of failures of local–global principles (e.g., unique factorization, Grunwald–*Wang*, points on varieties, zeros or poles of L-functions, vanishing of periods) abound in number theory and are of consummate interest. See, for example, Mazur's (8th out of 11 so far!) *Bulletin* article [Maz93] for local–global principles and obstructions for varieties.

On the other hand, for certain objects like idele class characters, modular (new) forms, Galois representations or automorphic representations, we have much more rigid local–global phenomena. Here the usual local–global principle is more-or-less tantamount to the existence of an Euler product for the associated L-function. We will discuss stronger versions of this, where knowing local L-factors at a sufficiently large set of places determines the global L-function (and hence, often, the global object up to isomorphism).

Specifically, consider the following 3 results. Let F be a number field, Σ_F the set of places of F and Γ_F the absolute Galois group of F. Denote by ρ, ρ' irreducible n-dimensional complex representations of Γ_F (i.e., irreducible Artin representations) and by π, π' irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$.

(1) If $L(s, \rho_v) = L(s, \rho'_v)$ for almost all v, then $L(s, \rho) = L(s, \rho')$, and in fact $\rho \simeq \rho'$.
(2) If $L(s, \pi_v) = L(s, \pi'_v)$ for almost all v, then $L(s, \pi) = L(s, \pi')$, and in fact $\pi \simeq \pi'$.
(3) If $L(s, \rho_v) = L(s, \pi_v)$ for almost all v, then $L(s, \rho) = L(s, \pi)$.

To be more precise, by the notation $L(s, \rho)$, $L(s, \pi)$, etc., for global L-functions we will mean the incomplete L-function (the product over all finite places of local factors). When we want to denote completed L-functions, we will write $L^*(s, \rho)$, $L^*(s, \pi)$, etc. By equality of two global L-functions, we mean as Euler products over the base field, i.e., not just equality of meromorphic functions but all local factors are equal as well.

Date: July 31, 2015.

Result (1) is an elementary consequence of Chebotarev density, and (2) is the strong multiplicity one (SMO) theorem for $GL(n)$ due to Jacquet and Shalika [JS81]. Result (3) follows from an argument of Deligne and Serre [DS74] (see Appendix A of my thesis [Mar04]).

While the first two statements are usually stated just with the conclusion of the two representations being isomorphic, stating the conclusion in terms of a global L-function equality puts all three results on the same footing. In addition, if one wants to think about representations of other groups, this seems to be the right point of view. E.g., cuspidal representations of $SO_n(\mathbb{A})$ will not satisfy SMO in the usual sense, but equality at almost all places should give an equality of global L-functions (in fact, L-packets).

Now one can ask a more general type of question. Suppose two global L-functions over F agree at all primes outside of some set $S \subset \Sigma_F$. Under what conditions can we conclude that the L-factors are equal everywhere? The above 3 results are about when S is a finite set, but some results and conjectures exist generalizing (1) and (2) if S is "not too big", or of a certain form. We will discuss each of these situations, and conclude by explaining recent joint work with Ramakrishnan [MR], where we generalized (3) to certain kinds of infinite sets for $n = 2$.

Of course it is interesting to consider when ρ and ρ' are ℓ-adic Galois representations as well. We will make some remarks about ℓ-adic representations, but for simplicity focus on Artin representations.

2. GALOIS REPRESENTATIONS

Suppose ρ and ρ' are irreducible n-dimensional Artin representations of Γ_F. They both factor through Galois groups of finite extensions of F, so we can choose a single finite Galois extension K/F such that ρ and ρ' may be considered as representations of $G = \mathrm{Gal}(K/F)$.

Our basic problem is to determine if knowing $L^S(s, \rho) = L^S(s, \rho')$ for some fixed $S \subset \Sigma_F$ implies $L(s, \rho) = L(s, \rho')$. Since a representation of a finite group is determined by its character, for Artin representations it suffices to consider a weaker hypothesis.

Namely, let $S \subset \Sigma_F$ and suppose $\mathrm{tr}\,\rho(Fr_v) = \mathrm{tr}\,\rho'(Fr_v)$ for $v \notin S$. (We may assume S contains all places where ρ and ρ' are ramified, so that this makes sense.) Note this is weaker than the condition on L-factors because, at unramified places, $L_v(s, \rho) = (\det(I - \rho(\mathrm{Fr}_v)q_v^{-s}))^{-1}$ determines $\mathrm{tr}\,\rho(\mathrm{Fr}_v)$ but not conversely.

Recall we define the (natural) density of S to be

$$\mathrm{den}(S) := \lim_{x \to \infty} \frac{\#\{v \in S : q_v < x\}}{\#\{v \in \Sigma_F : q_v < x\}},$$

if this limit exists. Now Chebotarev density says that if $\mathrm{den}(S) < \frac{1}{|G|}$, then $\{\mathrm{Fr}_v : v \notin S\}$ hits all conjugacy classes in G. So if $\mathrm{den}(S) < \frac{1}{|G|}$, then $\mathrm{tr}\,\rho(g) = \mathrm{tr}\,\rho'(g)$ for all $g \in G$, whence $\rho \simeq \rho'$. Often it is easier to work with Dirichlet density, which is defined by

$$\delta(S) := \lim_{s \to 1^+} \frac{\sum q_v^{-s}}{\log \frac{1}{s-1}}$$

If $\mathrm{den}(S)$ exists, so does $\delta(S)$ and they are equal.

Proposition 1. *Suppose ρ and ρ' are n-dimensional Artin representations of $\mathrm{Gal}(K/F)$. If $\mathrm{tr}\,\rho(\mathrm{Fr}_v) = \mathrm{tr}\,\rho'(\mathrm{Fr}_v)$ for v outside of a set S of places with $\delta(S) < \frac{1}{2n^2}$, then $\rho \simeq \rho'$.*

This follows from combining the above Chebotarev density argument and the following result about finite group characters.

Lemma 2. *If χ and χ' are irreducible characters of degree n of a finite group G and $X = \{g \in G : \chi(g) = \chi'(g)\}$ has size $> |G|(1 - 1/2n^2)$, then $\chi = \chi'$.*

Proof. Put $Y = G - X$. Since χ, χ' have maximum absolute value n, we see

$$\sum_{g \in Y} |\chi(g)\bar{\chi}(g)|, \sum_{g \in Y} |\chi(g)\bar{\chi}'(g)| \leq |Y|n^2 < \frac{|G|}{2}.$$

Since $\sum_{g \in G} \chi(g)\bar{\chi}(g) = |G|$, this means $\sum_{g \in X} \chi(g)\bar{\chi}(g) \geq \frac{G}{2}$. Then

$$\sum_{g \in G} \chi(g)\bar{\chi}(g) = \sum_{g \in X} \chi(g)\bar{\chi}(g) + \sum_{g \in Y} \chi(g)\bar{\chi}'(g) \neq 0,$$

which implies $\chi = \chi'$ by orthogonality relations and irreducibility. $\square$

It is known that one cannot do better than this, cf. [Ram94b]. Namely, if $n = 2^m$ then Buzzard, Edixhoven and Taylor constructed distinct n-dimensional irreducibles ρ and ρ' such that $\text{den}(S) = |G|(1 - 1/2n^2)$ where G is a central quotient of Q_8^m (Q_8 is the quaternion group of order 8) times $\{\pm 1\}$. Serre showed the existence of similar examples for arbitrary n.

We remark that Rajan [Raj98] proved an analogue for (semisimple, finitely ramified) ℓ-adic Galois representations of $\text{Gal}(\bar{F}/F)$.

3. AUTOMORPHIC REPRESENTATIONS

Let π and π' be irreducible automorphic cuspidal unitary representations of $\text{GL}_n(\mathbb{A}_F)$. For a finite place v, we can write

$$L(s, \pi_v) = \prod_{i=1}^{k} (1 - \alpha_{v,i} q_v^{-s})^{-1}$$

for some $0 \leq k \leq n$ and nonzero complex numbers $\alpha_{v,i}$. Note that $L(s, \pi_v)$ is a nowhere vanishing meromorphic function whose set of poles are precisely the values of s such that $q_v^s = \alpha_{v,i}$ for some $1 \leq i \leq k$. The latter condition implies $q_v^{\text{Re}(s)} = |\alpha_{v,i}|$, and conversely for each real x with $q_v^x = |\alpha_{v,i}|$ for some i, there exists an s such that $\text{Re}(s) = x$ and there is a pole at s. (If $k = 0$, then $L(s, \pi_v) = 1$ and there are no poles.)

Similarly, write

$$L(s, \pi_v') = \prod_{i=1}^{k'} (1 - \alpha_{v,i}' q_v^{-s})^{-1}.$$

The following observation, while simple, will be key for us in several places, so I will set it off to highlight it.

Fact. *Fix a finite place v with $k, k' \geq 1$. If the first (rightmost) pole for $L(s, \pi_v)$ occurs on the vertical line $\text{Re}(s) = x_0$, then $x_0 = \max\{\frac{\log |\alpha_{v,i}|}{\log q_v} : 1 \leq i \leq k\}$. Similarly, if the first pole for $L(s, \pi_v \times \bar{\pi}_v')$ occurs on the vertical line $\text{Re}(s) = x_0$, then $x_0 = \max\{\frac{\log |\alpha_{v,i}\alpha_{v,j}'|}{\log q_v} : 1 \leq i \leq k, 1 \leq j \leq k'\}$.*

Theorem 3 (Strong Multiplicity One, [JS81]). *Suppose $L(s, \pi_v) = L(s, \pi_v')$ for all v outside of a finite set S. Then $\pi \simeq \pi'$.*

This generalizes earlier results of Miyake [Miy71] for $n = 2$ and Piatetski–Shapiro [PS79], who needed to also assume the archimedean components match. For $n = 1$, this follows from strong approximation.

Proof. There are two ingredients, both proved in [JS81].

(i) We have $\pi \simeq \pi'$ if and only if $L(s, \pi \times \bar{\pi}')$ has a pole at $s = 1$ (use the integral representation and orthogonality of cusp forms).

(ii) For finite v, we have the bound $|\alpha_{v,i}| < q_v^{1/2}$. (The ramified case reduces to the unramified case.)

Now to prove strong multiplicity one, consider the ratio

$$(1) \qquad \frac{L(s, \pi \times \bar{\pi})}{L(s, \pi \times \bar{\pi}')} = \frac{L_S(s, \pi \times \bar{\pi})}{L_S(s, \pi \times \bar{\pi}')}.$$

By (ii), we see that $L_S(s, \pi \times \bar{\pi})$ and $L_S(s, \pi \times \bar{\pi}')$ both have no poles on $\mathrm{Re}(s) \geq 1$. Since these functions are also never zero, the right hand side has no pole in $\mathrm{Re}(s) \geq 1$. Since $L(s, \pi \times \bar{\pi})$ has a pole at $s = 1$ by (i), it must be canceled out by a pole of $L(s, \pi \times \bar{\pi}')$ at $s = 1$ in order for the ratio on the left to not have a pole there. Thus, again by (i), we get $\pi \simeq \pi'$. $\qquad\square$

We remark that Moreno [Mor85] proved an "analytic" SMO: if π and π' have bounded conductors and archimedean parameters, there is an effective (but exponential) constant X (depending on the bounds on conductors and archimedean parameters, F and n) such that if $\pi_v \simeq \pi'_v$ for all v with $q_v < X$, then $\pi \simeq \pi'$. Note that for $n = 2$, such a result gives you a bound on the number of Fourier coefficients needed to distinguish modular forms of bounded level and weight with the same nebentypus. (Of course there will be some finite bound because the space of such forms is finite dimensional.) Further work has been done along these lines (for $n = 2$ and general n), but this is not our focus now and we will not discuss it further. We are interested in results where one does not impose *a priori* bounds on ramification or infinity types.

Coming back to the usual SMO, note that in the above proof it was crucial S be finite to conclude the RHS of (1) has no pole in $\mathrm{Re}(s) \geq 1$. To refine this, we need a couple more ingredients.

First is an improvement on (ii). Recall the Generalized Ramanujan Conjecture (GRC) asserts that each π_v is tempered, i.e., each $|\alpha_{v,i}| = 1$. For general n, the best that is known is the Luo–Rudnick–Sarnak bound from [LRS99], which says $|\alpha_{v,i}| < q_v^{1/2 - 1/(n^2+1)}$. For $n = 2$ we can do better. Using Sym^2, Gelbart–Jacquet [GJ78] got a bound of $q_v^{1/4}$. With Sym^3 this was improved to exponent $\frac{1}{9}$ by Kim–Shahidi [KS02], then further improved to $\frac{7}{64}$ by Kim–Sarnak [Kim03] and Blomer–Brumley [BB11] using Sym^4. In fact, for us, a bound of the form q_v^δ for some $\delta < \frac{1}{4}$ is sufficient.

The second ingredient we need is Landau's lemma, which we explain now.

Let us say a Dirichlet series $L(s)$ is of **positive type** if it has an Euler product (on some right half plane) and $\log L(s)$ is a Dirichlet series with positive (≥ 0) coefficients. Note

$$\log \frac{1}{1 - \alpha q^{-s}} = \sum_n \frac{\alpha^n/n}{q^{ns}}$$

is a Dirichlet series with positive coefficients if $\alpha \geq 0$. Since the sum of Dirichlet series with positive coefficients is again a Dirichlet series with positive coefficients (admitting convergence) we see an L-series of the form

$$L(s) = \prod_i \frac{1}{1 - \alpha_i q_i^{-s}}$$

with each $\alpha_i \geq 0$ is of positive type (admitting convergence), e.g., a Dedekind zeta function. More important for us will be examples like $L(s, \pi \times \bar{\pi})$, which are also of positive type.

Lemma 4 (Landau). *Suppose $L(s)$ is a Dirichlet series of positive type. Then no zero of $L(s)$ occurs to the right of the first (rightmost) pole, and the first pole occurs on the real axis.*

This will be extremely useful because we can now control the locations of not just poles of L-functions, but also zeroes.

Theorem 5 (Refined SMO, Ramakrishnan [Ram94a]). *Suppose $n = 2$ and $L(s, \pi_v) = L(s, \pi'_v)$ for all v outside a set S with $\delta(S) < \frac{1}{8}$. Then $\pi \simeq \pi'$.*

This was used by Taylor [Tay94] for constructing families of ℓ-adic Galois representations to modular forms over imaginary quadratic field.

Proof (Sketch). Suppose $\pi \neq \pi'$ and assume S contains all places of ramification. Put

$$Z(s) = \frac{L(s, \pi \times \bar{\pi})L(s, \pi' \times \bar{\pi}')}{L(s, \pi \times \bar{\pi}')L(s, \pi' \times \bar{\pi})}.$$

Then, by (i), the numerator has a double pole at $s = 1$ while the denominator has no pole there. Hence $Z(s)$ has a pole of order 2 at $s = 1$. By definition, $Z_v(s) = 1$ for any $v \notin S$, so we also have

$$Z(s) = Z_S(s) = \frac{L_S(s, \pi \times \bar{\pi})L_S(s, \pi' \times \bar{\pi}')}{L_S(s, \pi \times \bar{\pi}')L_S(s, \pi' \times \bar{\pi})}.$$

Put

$$D_S(s) = L_S(s, \pi \times \bar{\pi})L_S(s, \pi' \times \bar{\pi}')L_S(s, \pi \times \bar{\pi}')L_S(s, \pi' \times \bar{\pi})$$

so

$$Z_S(s) = \frac{L_S(s, \pi \times \bar{\pi})^2 L_S(\pi' \times \bar{\pi}')^2}{D_S(s)}.$$

This is convenient because $D_S(s)$ is a Dirichlet series of positive type, and one can check it is nonvanishing for $s \geq 1$, so it has no zero at $s = 1$ by Landau's lemma. We would like to get a contradiction by saying that $L_S(s, \pi \times \bar{\pi})$ and $L_S(\pi' \times \bar{\pi}')$ can't have poles at $s = 1$ for S of sufficiently small density, but there is no reason they even need to be meromorphic at $s = 1$. Instead, we observe that $Z_S(s)$ having a pole of order 2 at $s = 1$ means

$$\lim_{s \to 1^+} \frac{\log Z_S(s)}{\log \frac{1}{s-1}} = 2.$$

Hence to obtain a contradiction, it will suffice to show

(2)
$$\lim_{s \to 1^+} \frac{\log L_S(s, \pi \times \bar{\pi})}{\log \frac{1}{s-1}} < \frac{1}{2},$$

as the same argument will apply to $L_S(s, \pi \times \bar{\pi}')$. For simplicity, assume $F = \mathbb{Q}$. Say π_p has Satake parameters $\{\alpha_{1,p}, \alpha_{2,p}\}$. Then

$$\log L(s, \pi_p \times \bar{\pi}_p) = \sum_{1 \leq i,j \leq 2} \log \frac{1}{1 - \alpha_{i,p}\alpha_{j,p}p^s} = \sum_{1 \leq i,j \leq 2} \sum_{n \geq 1} \frac{(\alpha_{i,p}\alpha_{j,p})^n}{np^{ns}}$$

$$= \frac{c_p}{p^s} + O(p^{-2s}),$$

where

$$c_p = \sum_{1 \leq i,j \leq 2} \alpha_{i,p}\alpha_{j,p}.$$

It is well known that the "prime zeta function" satisfies

$$\sum_p \frac{1}{p^s} = \log \frac{1}{s-1} + O(1), \quad s \to 1^+.$$

So if π is tempered at p, and then $|c_p| \leq 4$ and one deduces

(3)
$$\log L_S(s, \pi \times \bar{\pi}) \leq 4\delta(S) \log \frac{1}{s-1} + o(\log \frac{1}{s-1}),$$

and we are done as $\delta(S) < \frac{1}{8}$.

So the difficulty is when π is not tempered. Here one needs the above-mentioned bound towards GRC: $|\alpha_{i,v}| \leq q_v^\delta$ with $\delta < \frac{1}{4}$. Then Ramakrishnan does a careful analysis involving $L(s, \mathrm{Ad}(\pi))$ and $L(s, \mathrm{Ad}(\pi) \times \mathrm{Ad}(\pi))$ to treat the case when $\mathrm{Ad}(\pi)$ is cuspidal. (This is the technical crux of the proof, but it will not come up later for us, so we will not explain this analysis.) If $\mathrm{Ad}(\pi)$ is not cuspidal, then π is induced from a character of a quadratic extension, and therefore tempered everywhere. $\square$

In fact, Rajan [Raj03] observed this is also true if one just assumes equality of coefficients of Dirichlet series (i.e., sums of Satake parameters—or, for modular forms, Fourier coefficients) at primes $v \notin S$. This is analogous to only requiring equalities of traces $\operatorname{tr} \rho(\operatorname{Fr}_v) = \operatorname{tr} \rho'(\operatorname{Fr}_v)$ for Galois representations.

Note that Ramakrishnan's result is sharp, which one can deduce from $n = 2$ examples which show Proposition 1 is sharp. Nevertheless, Walji [Wal14] was able to prove some refinements, such as the following: if $n = 2$ and π and π' are not dihedral (induced from quadratic extensions), then a refined SMO is true with the stronger bound $\delta(S) < \frac{1}{4}$.

Now let's go back to considering arbitrary n.

Conjecture 6 (Ramakrishnan [Ram94b]). *A refined SMO is true with $\delta(S) < \frac{1}{2n^2}$.*

For $n = 1$ this is true by class field theory and Proposition 1. For $n = 2$, this is precisely the content of Theorem 5.

Let's think back to the proofs of Theorems 3 and 5 to see what is needed to prove a refined SMO result. In the proof of the usual SMO (Theorem 3) we wanted to show $L_S(s, \pi \times \bar{\pi})$ has no pole at $s = 1$ and $L_S(s, \pi \times \bar{\pi}')$ has no zero at $s = 1$. To prove refined SMO for GL(2) (Theorem 5), Ramakrishnan considered a ratio $Z(s)$ and used Landau's lemma to essentially translate the problem into showing both $L_S(s, \pi \times \bar{\pi})$ and $L_S(s, \pi \times \bar{\pi}')$ have no poles in $\operatorname{Re}(s) \geq 1$. Let's just consider $L_S(s, \pi \times \bar{\pi})$ since the idea for $L_S(s, \pi \times \bar{\pi}')$ is similar.

Suppose we have a bound towards GRC which says each $L(s, \pi_v \times \bar{\pi}_v)$ has no pole in $\operatorname{Re}(s) > 2\delta < 1$. Then, morally, if S is not too dense the bound for the first pole of $L_S(s, \pi \times \bar{\pi})$ should not be pushed too far to the right of 2δ. (If S has density 1, then $L_S(s, \pi \times \bar{\pi})$ can have a pole up to 1 unit to the right of 2δ.) The actual argument is more subtle than this, but we will return to this moral shortly.

Looking at the argument for the tempered case of Theorem 5, we see the fact that $n = 2$ was not really crucial. For general n, the $4\delta(S)$ in (3) becomes $n^2\delta(S)$, and this is less than the $\frac{1}{2}$ required in (2) precisely when $\delta(S) < \frac{1}{2n^2}$. In other words, this conjecture should follow from GRC and. Moreover, the bound $\delta(S) < \frac{1}{2n^2}$ must be sharp by the existence of examples of Galois representations showing Proposition 1 is sharp—here one can take these examples to be of finite nilpotent Galois groups, where one knows modularity by Arthur–Clozel [AC89].

Unfortunately, the Luo–Rudnick–Sarnak bounds toward GRC only tell us each $L(s, \pi_v \times \bar{\pi}_v)$ has no pole in $\operatorname{Re}(s) \geq 1 - \frac{2}{n^2+1}$, which does not seem to be enough to force $L_S(s, \pi \times \bar{\pi})$ to have no pole in $\operatorname{Re}(s) \geq 1$ for any S of positive density. So for not-necessarily tempered representations of GL(n) we don't know any refined SMO for $n > 2$ and S of positive density at present, but we can treat certain infinite sets S of density 0. (In fact, at the time of his conjecture, Ramakrishnan announced he had a weak result for $n > 2$ ([Ram94a], [Ram94b]), but did not publish a result of this type until recently—see below.)

We remark that there have been spectacular results on proving GRC for certain classes of representations for GL(n) to which one often knows how to associate Galois representations, e.g., cohomological self-dual representations over a totally real field. For instance, see Clozel's aphoristically titled article [Clo13].

In [Raj03], Rajan showed that a refined SMO is true for arbitrary n if

$$\sum_{v \in S} q_v^{-\frac{2}{n^2+1}} < \infty.$$

This is not difficult—this condition implies that the first pole of $L_S(s, \pi \times \bar{\pi})$ is not more than $\frac{2}{n^2+1}$ to the right of the first pole of a local factor (the argument is the same as for the Key Observation below). So by the Luo–Rudnick–Sarnak bound, this is precisely what one needs to conclude $L_S(s, \pi \times \bar{\pi})$ has no pole in $\operatorname{Re}(s) \geq 1$. However this condition only holds for *very* sparse sets of primes. It is much stronger than $\sum q_v^{-1} < \infty$, which is in turn stronger than

requiring $\delta(S) = 0$, so one cannot handle S of positive density. An example of where this applies is: let $F/\mathbb{Q}$ be cyclic of prime degree $p > \frac{n^2+1}{2}$ and let $S \subset \Sigma_F$ consist of inert primes in $F/\mathbb{Q}$. (In [Raj03], Rajan says S has positive density in this example, but presumably he means the corresponding primes of $\mathbb{Q}$, rather than F, have positive density: by Chebotarev, the density of the underlying primes of S in $\Sigma_{\mathbb{Q}}$ has density $\frac{p-1}{p}$ in $\Sigma_{\mathbb{Q}}$.)

Recently, Ramakrishnan proved the following result.

Theorem 7 (Ramakrishnan [Ram15]). *Suppose F is a cyclic extension of prime degree p of some number field k. A refined SMO is true when $S \subset \Sigma_F$ contains only finitely many primes which are split over k.*

This is still density 0, and satisfies Rajan's criterion when p is large, so the main content is for p small. In fact, $p = 2$ is the hardest case, and this case was used in a crucial way in trace formula comparisons of Wei Zhang [Zha14] and Feigon–Martin–Whitehouse [FMW]. More recently, Ramakrishnan [Ram] has extended this to the arbitrary Galois case, where a quite different approach was required.

Proof (sketch). As explained above, the key point is to show that $L_S(s, \pi \times \bar{\pi})$ has no pole in $\mathrm{Re}(s) \geq 1$. There are two ingredients to the proof. First is the following elementary but key fact, which we want to highlight because we will use it again in the next section.

Key Observation. *Let S_j be the set of primes of degree j. Suppose for each $v \in S_j$ we have numbers α_v such that $|\alpha_v| < q_v^\delta$. Then $\prod_{S_j} \frac{1}{1 - \alpha_v q_v^{-s}}$ converges absolutely in $\Re(s) > \delta + \frac{1}{j}$.*

This is a special case where we can make our above-mentioned "moral" precise. It says that a product of local factors over primes of degree j will not have in a pole which is more than $\frac{1}{j}$ to the right of a pole of any local factor.

Proof. Note

$$\log \prod_{v \in S_j} \frac{1}{1 - \alpha_v q_v^{-s}} = \sum_v \sum_m \frac{\alpha_v^m}{m q_v^{s m v}} \leq \sum_v \sum_m \frac{1}{q_v^{(s-\delta)m}}$$

If we denote by p_v the rational prime below q_v, then $q_v \geq p_v^j$ so the above is bounded (absolutely) by

$$\sum_v \sum_m \frac{1}{p_v^{(s-\delta)jm}} \leq \sum_m \frac{1}{m^{(s-\delta)j}},$$

which converges if $(\mathrm{Re}(s) - \delta)j > 1$, i.e., if $\mathrm{Re}(s) > \delta + \frac{1}{j}$. $\qquad\square$

Now by Luo–Rudnick–Sarnak, we know each local factor of $L(s, \pi \times \bar{\pi})$ has no pole in $\mathrm{Re}(s) \geq 1 - \frac{2}{n^2+1}$, so the above observation tells us $L_S(s, \pi \times \bar{\pi})$ has no pole in $\mathrm{Re}(s) > 1 - \frac{2}{n^2+1} + \frac{1}{p}$, since all but a finite number (which do not matter) of primes in S have degree p. Hence if $p \geq \frac{n^2+1}{2}$ we are done.

To deal with smaller p, Ramakrishnan uses Kummer theory to prove the following.

Lemma 8. *Suppose K/F is a degree p^{m-1} extension such that K/k is a nested chain of cyclic p^2-extensions. If v is a prime of F of degree p over k, and w is an unramified prime of K over v, then w has degree p^m over k.*

Consequently, given such an extension K/F, the Key Observation tells us that $L_S(s, \pi_K \times \bar{\pi}_K)$ has no poles in $\mathrm{Re}(s) > 1 - \frac{2}{n^2+1} + \frac{1}{p^m}$. Taking m large enough we can conclude $L_S(s, \pi_K \times \bar{\pi}_K)$ has no poles in $\mathrm{Re}(s) \geq 1$, and thus that $\pi_K \simeq \pi_K'$.

To finish the proof, one must carefully vary the field K to get $\pi_K \simeq \pi_K'$ over sufficiently many extension K/F to deduce the isomorphism $\pi \simeq \pi'$ over F. $\qquad\square$

We will a use similar idea for our result in the next section.

4. MODULARITY

Let ρ be an irreducible n-dimensional Artin representation of $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$. Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Here we are interested in comparing ρ and π.

Recall we say ρ is modular if $L(s, \rho)$ agrees with $L(s, \pi(\rho))$ at almost all places, for some cuspidal automorphic representation $\pi(\rho)$ of $\mathrm{GL}_n(\mathbb{A}_F)$. (By SMO, $\pi(\rho)$ is unique up to isomorphism.) The strong Artin, or modularity, conjecture asserts that every ρ is modular. The following well-known result tells us an equivalent definition of modularity is $L(s, \rho) = L(s, \pi)$ (in the sense of equality of Euler products).

Proposition 9. *If $L(s, \rho_v) = L(s, \pi_v)$ for almost all v, then $L(s, \rho) = L(s, \pi)$. Further we have an identity of total archimedean factors $L_\infty(s, \rho) = L_\infty(s, \pi)$.*

The proof follows from an argument due to Deligne and Serre [DS74], and the details are given in [Mar04, Appendix A]. The idea is to twist by a highly ramified character χ at bad places, which makes the L-factors 1 at these places so we get a global equality $L(s, \rho \otimes \chi) = L(s, \pi \otimes \chi)$. We may take χ to be trivial at each archimedean place, and then comparing poles in functional equations allows us to deduce $L_\infty(s, \rho) = L_\infty(s, \pi)$. Now we repeat the argument with χ which is highly ramified at all but one bad place v, where χ is trivial. This gives $L(s, \rho_v) = L(s, \pi_v)$.

In fact, in [MR], when $n = 2$ we show the stronger statement that ρ and π correspond via local Langlands at all (finite and infinite) places. However, this argument relies on the fact that the local Langlands correspondence is characterized by twists of L- and ϵ- factors by characters, which is not true for $n \geq 4$ (see [JPSS79, Remark 7.5.4] for an example with $n = 4$), so this argument does not generalize to arbitrary n.

Now we can ask, to compare ρ and π, how large of a set of places do we need to deduce $L(s, \rho) = L(s, \pi)$? The above proposition says it suffices to compare them at almost all places. But since ρ and π should be determined by their local L-factors outside any set S of places of density less then $\frac{1}{2n^2}$, to show they correspond it should suffice to check matching of local L-factors outside such a set S. Precisely, we have

Conjecture 10. *If $L(s, \rho_v) = L(s, \pi_v)$ for all v outside of some set S of places of density less than $\frac{1}{2n^2}$, then $L(s, \rho) = L(s, \pi)$.*

This is true for $n = 1$ by class field theory. In general, this is a consequence of the strong Artin conjecture together with the refined SMO Conjecture (Conjecture 6). Namely, if ρ is modular and its L-function agrees with that of π outside of S, then $L(s, \pi_v) = L(s, \pi(\rho)_v)$ for $v \notin S$. By Conjecture 6, if $\delta(S) < \frac{1}{n^2}$, then $\pi \simeq \pi(\rho)$.

Consequently, by Theorem 5, we know this conjecture is true whenever $n = 2$ and ρ is modular. This is the case if ρ has solvable image by the Langlands–Tunnell theorem, or if ρ is odd and $F = \mathbb{Q}$ by Khare and Winterberger's work on Serre's conjecture (see [Kha10]). Arguing similarly in the reverse direction, Conjecture 10 is true whenever π corresponds to some Artin representation $\rho(\pi)$ by Proposition 1. This is known if π corresponds to a weight 1 Hilbert modular form by Wiles [Wil88]. However, even for $n = 2$, this is not solved completely, and the case where ρ is even (so π should correspond to a Maass form, say, if $F = \mathbb{Q}$) with nonsolvable image seems particularly difficult.

In any case, one might hope that if one could prove this conjecture independent of modularity, then this may help establish new cases of modularity. Recently, Ramakrishnan and I proved the following mild result towards this conjecture.

Theorem 11 ([MR]). *Suppose $n = 2$ and F is cyclic extension of prime degree p of some number field k. Let $S \subset \Sigma_F$ be a set of primes such that almost all $v \in S$ are inert over k. Then $L(s, \rho_v) = L(s, \pi_v)$ for $v \notin S$ implies $L(s, \rho) = L(s, \pi)$, and in fact $\rho_v \leftrightarrow \pi_v$ in the sense of local Langlands at all v.*

Proof. By the generalization of the Deligne–Serre argument we mentioned above, it suffices to show $L(s, \rho_v) = L(s, \pi_v)$ for almost all v. We show this in 4 steps.

Step 1: Show π is tempered (at each place).

It is immediate from the equality $L(s, \rho_v) = L(s, \pi_v)$ that π_v is tempered for any $v \notin S$, so we just need to show temperedness at $v \in S$. Note, from the Fact before Theorem 3, π_v is tempered if and only if $L(s, \pi_v \times \bar{\pi}_v)$ has no pole in $\mathrm{Re}(s) > 0$. Now consider the ratio

$$\Lambda(s) = \Lambda_F(s) = \frac{L^*(s, \pi \times \bar{\pi})}{L^*(s, \rho \times \bar{\rho})} = \frac{L_S(s, \pi \times \bar{\pi})}{L_S(s, \rho \times \bar{\rho})}.$$

Then $\Lambda(s)$ satisfies a functional equation with $\Lambda(1 - s)$, and if we can show Λ has no poles in $\mathrm{Re}(s) \geq \frac{1}{2}$, this will mean it is entire by the functional equation.

Take $\delta < \frac{1}{4}$ such that the bound of q_v^δ towards GRC is satisfied. The Key Observation implies that $L_S(\pi \times \bar{\pi})$ has no poles in $\mathrm{Re}(s) > 2\delta + \frac{1}{p}$ and $L_S(\rho \times \bar{\rho})$ has no poles (and thus no zeroes by Landau's lemma) in $\mathrm{Re}(s) > \frac{1}{p}$. For p sufficiently large, this means Λ has no poles in $\mathrm{Re}(s) \geq \frac{1}{2}$, so Λ is entire. For small p, we use Lemma 8 to pass to an extension K to push our bounds on the poles of the numerator and denominator to the left of $\mathrm{Re}(s) = \frac{1}{2}$, and get that an analogous ratio Λ_K is entire. In either case, write the ratio as Λ_K, where we take $K = F$ if p is sufficiently large.

If π_v is not tempered for some $v \in S$, then Landau's lemma implies $L_T(s, \pi_K \times \bar{\pi}_K)$ has a pole at some $s_0 > 0$, where T is the set of primes of K above S. But for Λ_K to be entire, we also need a pole at s_0 for $L(s, \rho_K \times \bar{\rho}_K)$. By taking K larger if needed, we can make it so that $L(s, \rho_K \times \bar{\rho}_K)$ has no poles in $\mathrm{Re}(s) > \frac{1}{p^m} < s_0$, a contradiction.

Step 2: Show $L(s, \rho_K)$, for some finite solvable extension K/F, is entire.

Here we consider the ratio

$$\Lambda_K(s) = \frac{L^*(s, \rho_K)}{L^*(s, \pi_K)} = \frac{L_T(s, \rho_K)}{L_T(s, \pi_K)},$$

where as before T is the set of places of K above S. Again, by looking at a functional equation, it suffices to show $L_T(s, \rho_K)$ has no pole in $\mathrm{Re}(s) \geq \frac{1}{2}$. The bound from the Key Observation is that $L_S(s, \rho)$ has no pole in $\mathrm{Re}(s) > \frac{1}{p}$, so can take $K = F$ unless $p = 2$. In this case we need to pass to an extension K, so the bound becomes $\mathrm{Re}(s) > \frac{1}{p^2}$. By a refinement of Lemma 8, we can do this with K/F quadratic or biquadratic, according to whether $\sqrt{-1} \in F$ or not.

Step 3: Deduce $L(s, \rho_K) = L(s, \pi_K)$.

The point is, up until now, everything we did is valid for twists, and the choice of K in the previous step only depends on F/k. So we get that $L(s, \rho_K \otimes \chi)$ is entire for any finite order idele class character of K. For Artin representations, it is known that this is sufficient to use the GL(2) converse theorem, namely one gets boundedness in vertical strips for free. Thus ρ_K corresponds to an automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_K)$, which must have the same L-factors as π_K at all places not above a place in S, i.e., at all places outside a density 0 set. By refined SMO (Theorem 5), this means $\pi_K \simeq \Pi$, so $L(s, \rho_K) = L(s, \pi_K)$.

Step 4: Descend the previous step to F, i.e., show $L(s, \rho) = L(s, \pi)$.

If $p > 2$, then $K = F$ so there is nothing to do. Assume $p = 2$. I will just discuss the proof in the simpler case that K/F is quadratic (i.e., when $\sqrt{-1} \in F$), and refer to [MR] for the biquadratic case. Remember, it suffices to show for almost all $v \in S$ that $L(s, \rho_v) = L(s, \pi_v)$. Fix any place $v \in S$ such that ρ_v and π_v are unramified, and let w be a place above v, which will be inert. By the previous step, we know $\rho_{K,w} \leftrightarrow \pi_{K,w}$ (in the sense of local Langlands, since unramified representations are determined by their local L-factors). This means that ρ_v must correspond to either π_v or $\pi_v \otimes \mu$, where μ is the quadratic character associated K_w/F_v. Now the

point is that we have sufficient flexibility in our choice of K so that we can get the correspondence $\rho_{K,w} \leftrightarrow \pi_{K,w}$ both when K_w/F_v is ramified. But it is impossible for an unramified ρ_v to correspond to a ramified twist of the unramified π_v, so we must have $\rho_v \leftrightarrow \pi_v$, and we are done. $\qquad \square$

Finally we remark that a similar conjecture should be true for (families of compatible) ℓ-adic Galois representations. In order for the proof of the above theorem to go through for ℓ-adic Galois representations, first we would need to know a purity result (which is known in some cases), such as $L(s, \rho_v)$ has no poles in $\operatorname{Re}(s) > 0$, to conclude temperedness of π. Then we would need to know that entirety of the twists $L(s, \rho \otimes \chi)$ also implies boundedness in vertical strips, so we can use the converse theorem in Step 3. (For Artin representations, this follows from a theorem of Brauer which tells us Artin L-functions are quotients of products of degree 1 L-functions.)

References

[AC89] James Arthur and Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989. MR1007299 (90m:22041)

[BB11] Valentin Blomer and Farrell Brumley, *On the Ramanujan conjecture over number fields*, Ann. of Math. (2) **174** (2011), no. 1, 581–605, DOI 10.4007/annals.2011.174.1.18. MR2811610

[Clo13] Laurent Clozel, *Purity reigns supreme*, Int. Math. Res. Not. IMRN **2** (2013), 328–346. MR3010691

[DS74] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids* 1, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975) (French). MR0379379 (52 #284)

[FMW] Brooke Feigon, Kimball Martin, and David Whitehouse, *Periods and nonvanishing of central L-values for* GL(2n) (2014), preprint, available at arXiv:1308.2253.

[GJ78] Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542. MR533066 (81e:10025)

[JL70] H Jacquet and R. P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Mathematics, vol. 114, Springer-Verlag, Berlin-New York, 1970. MR0401654 (70)

[JPSS79] Hervé Jacquet, Ilja Iosifovitch Piatetski-Shapiro, and Joseph Shalika, *Automorphic forms on* GL(3). I, Ann. of Math. (2) **109** (1979), no. 1, 169–212, DOI 10.2307/1971270. MR519356 (80i:10034a)

[JS81] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), no. 3, 499–558, DOI 10.2307/2374103. MR618323 (82m:10050a)

[Kha10] Chandrashekhar Khare, *Serre's conjecture and its consequences*, Jpn. J. Math. **5** (2010), no. 1, 103–125, DOI 10.1007/s11537-010-0946-5. MR2609324 (2011d:11121)

[Kim03] Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, DOI 10.1090/S0894-0347-02-00410-1. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR1937203 (2003k:11083)

[KS02] Henry H. Kim and Freydoon Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J. **112** (2002), no. 1, 177–197, DOI 10.1215/S0012-9074-02-11215-0. MR1890650 (2003a:11057)

[LRS99] Wenzhi Luo, Zeév Rudnick, and Peter Sarnak, *On the generalized Ramanujan conjecture for* GL(n), Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), Proc. Sympos. Pure Math., vol. 66, Amer. Math. Soc., Providence, RI, 1999, pp. 301–310. MR1703764 (2000e:11072)

[Mar04] Kimball Martin, *Four-dimensional Galois representations of solvable type and automorphic forms*, ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–California Institute of Technology. MR2706615

[MR] Kimball Martin and Dinakar Ramakrishnan, *A comparison of automorphic and Artin L-series of GL(2)-type agreeing at degree one primes* (2015), Contemp. Math., to appear.

[Maz93] B. Mazur, *On the passage from local to global in number theory*, Bull. Amer. Math. Soc. (N.S.) **29** (1993), no. 1, 14–50, DOI 10.1090/S0273-0979-1993-00414-2. MR1202293 (93m:11052)

[Miy71] Toshitsune Miyake, *On automorphic forms on* GL_2 *and Hecke operators*, Ann. of Math. (2) **94** (1971), 174–189. MR0299559 (45 #8607)

[Mor85] Carlos J. Moreno, *Analytic proof of the strong multiplicity one theorem*, Amer. J. Math. **107** (1985), no. 1, 163–206, DOI 10.2307/2374461. MR778093 (86m:22027)

[PS79] I. I. Piatetski-Shapiro, *Multiplicity one theorems*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 209–212. MR546599 (81m:22027)

[Raj98] C. S. Rajan, *On strong multiplicity one for l-adic representations*, Internat. Math. Res. Notices **3** (1998), 161–172, DOI 10.1155/S1073792898000142. MR1606395 (99c:11064)

[Raj03] _____, *On strong multiplicity one for automorphic representations*, J. Number Theory **102** (2003), no. 1, 183–190, DOI 10.1016/S0022-314X(03)00066-0. MR1994478 (2004f:11050)

[Ram94a] Dinakar Ramakrishnan, *A refinement of the strong multiplicity one theorem for GL(2). Appendix to: "l-adic representations associated to modular forms over imaginary quadratic fields. II" [Invent. Math.* **116** *(1994), no. 1-3, 619–643; MR1253207 (95h:11050a)] by R. Taylor*, Invent. Math. **116** (1994), no. 1-3, 645–649, DOI 10.1007/BF01231576. MR1253208 (95h:11050b)

[Ram94b] _____, *Pure motives and automorphic forms*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 411–446. MR1265561 (94m:11134)

[Ram15] _____, *A mild Tchebotarev theorem for* GL(n), J. Number Theory **146** (2015), 519–533, DOI 10.1016/j.jnt.2014.08.002. MR3267122

[Ram] _____, *A theorem à là Tchebotarev for* GL(n) (2012), preprint.

[Tay94] Richard Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields. II*, Invent. Math. **116** (1994), no. 1-3, 619–643, DOI 10.1007/BF01231575. MR1253207 (95h:11050a)

[Wal14] Nahid Walji, *Further refinement of strong multiplicity one for* GL(2), Trans. Amer. Math. Soc. **366** (2014), no. 9, 4987–5007, DOI 10.1090/S0002-9947-2014-06103-5. MR3217707

[Wil88] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. **94** (1988), no. 3, 529–573, DOI 10.1007/BF01394275. MR969243 (89j:11051)

[Zha14] Wei Zhang, *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, Ann. of Math. (2) **180** (2014), no. 3, 971–1049. MR3245011

(Permanent address) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OK 73019 USA

(Current address) DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA CITY UNIVERSITY, SUGIMOTO 3-3-138, SUMIYOSHI, OSAKA 558-8585, JAPAN