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IKEDA TYPE CONSTRUCTION OF CUSP FORMS

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ABSTRACT. This is a survey of results on the construction of holomorphic cusp forms on tube domains originally initiated by Ikeda [9]. Besides a survey it includes conjectures and possible applications of our work [19].

1. INTRODUCTION

There are five simple tube domains (cf. [6]). They are of the form $\mathfrak{D}=\{Z=X+iY|X\in \mathbb{R}^n, Y\in C\}$, where C is a self-adjoint homogeneous cone in $\mathbb{R}^n$. Let G be (the real points of) the simply connected, simple real algebraic group which acts transitively on $\mathfrak{D}$. We list the group G and the cone C:

(1) Sp_{2n} (rank n); $n\times n$ positive definite matrices over $\mathbb{R}$;
(2) $SU(n,n)$; $n\times n$ positive definite hermitian matrices over $\mathbb{C}$;
(3) $SU(2n,H)=Spin^*(4n)$; $n\times n$ positive definite hermitian matrices over H (quaternions);
(4) $SO(2,n)^0$; the cone in $\mathbb{R}^{n+1}$ of $(x_0,\ldots,x_n)$ with $x_0>(x_1^2+\cdots+x_n^2)^{\frac{1}{2}}$;
(5) $E_{7,3}$; 3×3 positive definite hermitian matrices over $\mathfrak{C}$ (Cayley numbers).

It is an important problem to explicitly construct holomorphic cusp forms on $\mathfrak{D}$ with respect to $G(\mathbb{Z})$ (we will call such a modular form on $\mathfrak{D}$ "a level one form"). In particular, we focus on the lifting from normalized Hecke cusp eigenforms on the complex upper half-plane $\mathbb{H}$ with respect to $SL_2(\mathbb{Z})$ to holomorphic cusp forms on $\mathfrak{D}$.

Ikeda [9] (see also [8]) gave a (functorial) construction of Siegel cusp forms of weight $n+k$, $n\equiv k \mod 2$ (so that $n+k$ is even) for Sp_{4n} from normalized Hecke eigenforms in $S_{2k}(SL_2(\mathbb{Z}))$ which has been conjectured by Duke and Imamoglu (Independently Ibukiyama formulated a conjecture in terms of Koecher-Maass series). He made use of the uniform property of the Fourier coefficients of Siegel Eisenstein series for Sp_{4n} and together with various deep facts established in [9] to prove Duke-Imamoglu conjecture. When $n=1$, it is nothing but a Saito-Kurokawa lift. Since then, his construction was generalized to unitary

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groups $U(n,n)(K/\mathbb{Q})$ or $SU(n,n)$ for an imaginary quadratic field $K/\mathbb{Q}$ ([10]), quaternion unitary groups $SU(2n,H)$ for a definite quaternion algebra H over $\mathbb{Q}$ ([25]), symplectic groups Sp_{2n} over totally real fields ([11],[12] including some levels), and the exceptional group of type $E_{7,3}$ with $\mathbb{Q}$-rank 3 [19].

In this note we explain main ideas of Ikeda and how they generalize to above cases. We do not discuss a further development by Ikeda [11] though it is important because his new ideas will work beyond "level one" case. We can give a uniform treatment except the case (4), which we will omit since it has been studied thoroughly by Oda [21] and Sugano [22].

Let G be Sp_{4n}, $SU_{2n+1} := SU(2n+1, 2n+1)(K/\mathbb{Q})$ (to ease the notation, we restrict ourselves to this case), $SU(2n,H)$, or $E_{7,3}$, and $P = MN$ the Siegel parabolic subgroup of G with the Levi subgroup M and the abelian unipotent radical N. For any ring R, let $\mathrm{Tr}_G : N(R) \longrightarrow R$ be the trace on N, which is defined as:

$$\mathrm{Tr}_G(n(B)) := \begin{cases} \mathrm{Tr}(B) & \text{if } G = Sp_{4n}, N = \left\{ n(B) = \begin{pmatrix} 1_{2n} & B \\ 0_{2n} & 1_{2n} \end{pmatrix} \,\middle|\, {}^tB = B \right\} \\ \frac{1}{2}\mathrm{Tr}(B+\overline{B}) & \text{if } G = SU_{2n+1}, N = \left\{ n(B) := \begin{pmatrix} 1_{2n+1} & B \\ 0_{2n+1} & 1_{2n+1} \end{pmatrix} \,\middle|\, {}^t\bar{B} = B \right\} \\ \frac{1}{2}\mathrm{Tr}(B+\tau(B)) & \text{if } G = SU(2n,H), N = \left\{ n(B) := \begin{pmatrix} 1_n & B \\ 0_n & 1_n \end{pmatrix} \,\middle|\, {}^t({}^\iota B) = B \right\}, \end{cases}$$

where ${}^\iota x = x_0 - ix_1 - jx_2 - kx_3$ for $x = x_0 + ix_1 + jx_2 + kx_3 \in H$, and $\tau(x) = x + {}^\iota x$. For $E_{7,3}$, see [19].

Set $K = \mathbb{Q}$ if $G = Sp_{4n}$ or $E_{7,3}$, and $K = \mathbb{H}$ if $G = SU(2n,H)$. Let $\mathcal{O}$ be the ring of integers of K if $G \neq SU(2n,H)$, and a maximal order of H if $G = SU(2n,H)$. An element T of $N(K)$ is semi-integral if $\mathrm{Tr}_G(TX) \in \mathbb{Z}$ for any $N(\mathcal{O})$. We denote by L the set of all semi-integral elements in $N(K)$ and denote by L^+ the subset of L consisting of positive definite elements. Here the positivity has the usual meaning as matrices for $G \neq E_{7,3}$, and see [19] for $E_{7,3}$. For instance, if $G = Sp_{4n}$, L consists of matrices $(x_{ij})_{1\le i,j\le 2n}$ so that $x_{ii} \in \mathbb{Z}$ and $x_{ij} = x_{ji} \in \frac{1}{2}\mathbb{Z}$ for $i \neq j$.

For the integers k and d, we denote by $\mathfrak{d}_d$ the discriminant of $\mathbb{Q}\left(\sqrt{(-1)^k d}\right)/\mathbb{Q}$ and χ_d the Dirichlet character associated to $\mathbb{Q}\left(\sqrt{(-1)^k d}\right)/\mathbb{Q}$. Let $\mathfrak{f}_d$ be the positive rational number so that $d = \mathfrak{d}_d \mathfrak{f}_d^2$. Let $L(s,\chi_d)$ be the Dirichlet L-function of χ_d. For $T \in L^+$, put $D_T = \det(2T)$ (resp. $\gamma(T) = (-D_K)^n \det(T)$ where $-D_K$ stands for the fundamental discriminant of $K/\mathbb{Q}$) if $G = Sp_{4n}$ (resp. if $G = SU_{2n+1}$). For $G = SU(2n,H)$, put $D_T = (D_H)^{\frac{n}{2}} \mathrm{Paf}(T)$ where D_H is the product of rational primes p so that $H \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a skew field and Paf is defined in

Section 1 of [25]. When $G = E_{7,3}$, $\det(T)$ is as in [19]. Set

$$\ell(k) := \begin{cases} k+n & \text{if } G = Sp_{4n}, \\ 2k+2n & \text{if } G = SU_{2n+1}, \\ 2k+2n-2 & \text{if } G = SU(2n,H), \\ 2k+8 & \text{if } G = E_{7,3}. \end{cases}$$

For each $\gamma \in G(\mathbb{R})$ and $Z \in \mathfrak{D}$, one can associate the automorphic factor $j(\gamma, Z) \in \mathbb{C}$ so that $j(\gamma, Z)^k$ is used to define modular forms on $\mathfrak{D}$ of weight k for any integer $k \geq 0$. For example, if $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R})$, then $j(\gamma, Z) = \det(CZ + D)$. Put $\Gamma := G(\mathbb{Z})$ and $\Gamma_\infty = \Gamma \cap N(\mathbb{Q})$. Let us consider the Siegel Eisenstein series of weight $\ell(k)$:

$$E_{\ell(k)}(Z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, Z)^{-\ell(k)}.$$

Then we have the Fourier expansion

$$\mathcal{E}_{\ell(k)}(Z) = \frac{1}{C(\ell(k))} E_{\ell(k)}(Z) = \sum_{T \in L} A(T) \exp(2\pi\sqrt{-1} \cdot \mathrm{Tr}_G(TZ)),$$

for a constant $C(\ell(k))$, and for $T \in L^+$, $A(T)$ is given as follows:

$$A(T) = \begin{cases} L(1-k, \chi_{D_T}) \mathfrak{f}_T^{k-\frac{1}{2}} \prod_{p \mid \mathfrak{d}_T} \widetilde{F}_p(T; p^{k-\frac{1}{2}}) & \text{if } G = Sp_{4n} \\ |\gamma(T)|^{k-\frac{1}{2}} \prod_{p \mid \gamma(T)} \widetilde{F}_p(T; p^{k-\frac{1}{2}}) & \text{if } G = SU_{2n+1} \\ {D_T}^{k-\frac{1}{2}} \prod_{p \mid D_T} \widetilde{f}_{p,T}(p^{k-\frac{1}{2}}) & \text{if } G = SU(2n,H) \\ \det(T)^{k-\frac{1}{2}} \prod_{p \mid \det(T)} \widetilde{f}_T^p(p^{k-\frac{1}{2}}) & \text{if } G = E_{7,3}, \end{cases}$$

where $\widetilde{F}_p(T;X), \widetilde{f}_{p,T}(X)$ and $\widetilde{f}_T^p(X)$ are Laurent polynomials over $\mathbb{Q}$ with one variable X which are depending only on T, p and both are identically 1 for all but finitely many p. Introducing multi-variables $\{X_p\}_p$ indexed by rational primes p, we may consider

$$A(\{X_p\}_p) := \begin{cases} L(1-k, \chi_{D_T}) \mathfrak{f}_T^{k-\frac{1}{2}} \prod_{p \mid \mathfrak{d}_T} \widetilde{F}_p(T; X_p) & \text{if } G = Sp_{4n} \\ |\gamma(T)|^{k-\frac{1}{2}} \prod_{p \mid \gamma(T)} \widetilde{F}_p(T; X_p) & \text{if } G = SU_{2n+1} \\ {D_T}^{k-\frac{1}{2}} \prod_{p \mid D_T} \widetilde{f}_{p,T}(X_p) & \text{if } G = SU(2n,H) \\ \det(T)^{k-\frac{1}{2}} \prod_{p \mid \det(T)} \widetilde{f}_T^p(X_p) & \text{if } G = E_{7,3}. \end{cases}$$

Then $A(\{X_p\}_p)$ can be regarded as an element of $\otimes'_p\mathbb{C}[X_p, X_p^{-1}]$. For each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} a(n)q^n$, $q = \exp(2\pi\sqrt{-1}\tau)$, $\tau \in \mathbb{H}$ in $S_{2k}(SL_2(\mathbb{Z}))$ and each rational prime p, we define the Satake p-parameter α_p by $a(p) = p^{k-\frac{1}{2}}(\alpha_p + \alpha_p^{-1})$. For such f, consider the following formal series on $\mathfrak{D}$:

$$F_f(Z) := \sum_{T \in L^+} A_{F_f}(T)\exp(2\pi\sqrt{-1}\mathrm{Tr}_G(TZ)),\ Z \in \mathfrak{D},\quad A_{F_f}(T) = A(\{\alpha_p\}_p).$$

Then

Theorem 1.1. *Assume that H is the Hurwitz quaternion when $G = SU(2n, H)$. Then F_f is a non-zero Hecke eigen cusp form on $\mathfrak{D}$ of weight $\ell(k)$ with respect to $G(\mathbb{Z})$.*

Of course, we have to specify what kind of Hecke theory we use for each case. At any late, the issue is only on the normalization factor of a Hecke action and it does not matter as long as we deal with the adelic form attached to F_f on $G(\mathbb{A}_{\mathbb{Q}})$ because since G is semi-simple, it does not contain the central torus. By virtue of Theorem 1.1, F_f gives rise to a cuspidal automorphic representation $\pi_F = \pi_\infty \otimes \otimes'_p\pi_p$ of $G(\mathbb{A}_{\mathbb{Q}})$. Here π_∞ is a holomorphic discrete series of $G(\mathbb{R})$ of the lowest weight $\ell(k)$, and for each prime p, π_p is unramified at every finite place (but a few exception when $G = SU(2n, H)$), since F_f is of "level one". In fact, π_p turns out to be a degenerate principal series $\pi_p \simeq I(s_p)$, where $s_p \in \mathbb{C}$ so that $p^{s_p} = \alpha_p$ and

$$I(s) = \begin{cases} \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} |\nu(g)|_p^s & \text{if } G = Sp_{4n} \\ \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} |\nu(g)|_p^s & \text{if } G = SU_{2n+1}, \\ \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} |\nu(g)|_p^s & \text{if } G = SU(2n, H) \text{ and } p \nmid D_H, \\ \mathrm{Ind}_{P(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)} |\nu(g)|_p^{2s} & \text{if } G = E_{7,3}, \end{cases}$$

where $\nu : P(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$ is defined as follows:

$$\nu(g) := \begin{cases} det(A) & \text{if } G = Sp_{4n}, P = \left\{ g = \begin{pmatrix} A & B \\ 0_{2n} & {}^tA^{-1} \end{pmatrix} \middle| {}^tB = B \right\} \\ |det(A)|^2 & \text{if } G = SU_{2n+1}, P = \left\{ g = \begin{pmatrix} A & B \\ 0_{2n+1} & {}^t\bar{A}^{-1} \end{pmatrix} \middle| {}^t\bar{B} = B \right\} \end{cases}$$

For $SU(2n, H)$ and $E_{7,3}$, see [25] and [19], resp. The relationship between $I(s)$ and the Eisenstein series is explained in [18]: Let $\Phi(g, s) = \Phi_\infty(g, s) \otimes \otimes_p\Phi_p(g, s)$ be a standard section in $I(s)$ such that $\Phi_\infty(k, s) = \nu(\mathbf{k})^{\ell(k)}$, and $\Phi_p(g, s) = \Phi_p^0(g, s)$ is the normalized spherical section for all p. Then one can define the adelic and classical Eisenstein series

$$E(g, s, \Phi) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash \mathbf{G}(\mathbb{Q})} \Phi(\gamma g, s),\quad E_{\ell(k),s}(Z) = det(Y)^{\frac{s}{2}} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, Z)^{-\ell(k)}|j(\gamma, Z)|^{-s}.$$

Then we have

$$E(g,s,\Phi)=\begin{cases}\det(Y)^{\ell(k)}E_{\ell(k),s+\frac{1}{2}-k}(Z), & \text{if } G\neq E_{7,3},\\ \det(Y)^{\frac{s+9}{2}+\ell(k)}E_{\ell(k),s+1-2k}(Z), & \text{if } G=E_{7,3},\end{cases}$$

Hence the degenerate principal series $I(k-\frac{1}{2})$ corresponds to $E_{\ell(k)}(Z)$ if $G\neq E_{7,3}$, and $I(2k-1)$ corresponds to $E_{\ell(k)}(Z)$ if $G=E_{7,3}$.

In terms of representation theory, Theorem 1.1 can be reformulated as follows: Let π_∞ be the holomorphic discrete series of $G(\mathbb{R})$ of the lowest weight $\ell(k)$, and let π_p be the above degenerate principal series which is irreducible. Then we can form an irreducible admissible representation of $G(\mathbb{A}_\mathbb{Q})$: $\pi=\pi_\infty\otimes\otimes_p'\pi_p$. Then Theorem 1.1 is equivalent to the fact that π is a cuspidal automorphic representation of $G(\mathbb{A})$. In this formulation, at least for Sp_{4n}, Arthur's trace formula [1] may give a more general result as follows: By Adams-Johnson's result on A-packets, π_∞ belongs to a packet with the local character $(-1)^n$. Since π is unramified at every finite place, by the multiplicity formula, π is a cuspidal automorphic representation if and only if the global character $(-1)^n$ is equal to the root number of $L(s,f)$ which is $(-1)^k$. Hence we have the parity condition $k\equiv n \pmod 2$. We have similar results for SU_{2n+1} and $SU(2n,H)$. However, the advantage of Theorem 1.1 is that one can write down the modular form explicitly. Let $L(s,\pi_f)=\prod_p((1-\alpha_p p^{-s})(1-\alpha_p^{-1}p^{-s}))^{-1}$ be the (normalized) automorphic L-function of the cuspidal representation π_f attached to f. In the case of SU_{2n+1}, let $\chi(p)=\left(\frac{-D_K}{p}\right)$ be the quadratic character attached to $K/\mathbb{Q}$, and $L(s,f,\chi)=\prod_p((1-\alpha_p\chi(p)p^{-s})(1-\alpha_p^{-1}\chi(p)p^{-s}))^{-1}$. For each local component π_p, one can associate the local L-factor $L(s,\pi_p,St)$ of the standard L-function of π_F. Set $L(s,\pi_F,St)=\prod_p L(s,\pi_p,St)$:

Theorem 1.2.

$$L(s,\pi_F,St)=\begin{cases}\zeta(s)\prod_{i=1}^{2n}L(s+n+\frac{1}{2}-i,f) & \text{if } G=Sp_{4n},\\ \prod_{i=1}^{2n+1}L(s+n+1-i,f)L(s+n+1-i,f,\chi) & \text{if } G=SU_{2n+1}\\ \prod_{i=1}^{2n}L(s+n+\frac{1}{2}-i,f) & \text{if } G=SU(2n,H)\\ L(s,\mathrm{Sym}^3\pi_f)L(s,f)^2\prod_{i=1}^{4}L(s\pm i,f)^2\prod_{i=5}^{8}L(s\pm i,f) & \text{if } G=E_{7,3},\end{cases}$$

where $L(s,\mathrm{Sym}^3\pi_f)$ is the symmetric cube L-function.

Notice that π_p for $G = E_{7,3}$ is slightly different from other cases (Note $2s_p$ rather than s_p) and the third symmetric power L-function appears in the standard L-function. Note also that in the case $G = SU_{2n+1}$, $L(s, f)L(s, f, \chi) = L(s, \pi_K)$, the L-function of the base change π_K of π_f to K.

In Section 2, we review the tube domains. In Section 3, we review the Jacobi group, Jacobi forms, and a key property of the Fourier-Jacobi expansion of Siegel Eisenstein series, namely, the Fourier-Jacobi coefficients of Eisenstein series are a sum of products of theta functions and Eisenstein series. In Section 4, we will give a sketch of proof of the main theorem. Except for $G = Sp_{4n}$, the situations are similar, in that we do not need to consider half-integral modular forms. Finally in Section 5, we discuss conjectures and problems related to the results in [19].

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2. Description of tube domains

2.1. Sp_{2n}. The tube domain is given by

$$\mathbb{H}_n := \{Z \in M_n(\mathbb{C}) \mid {}^tZ = Z,\ \mathrm{Im}(Z) > 0\} \subset \mathbb{C}^{\frac{n(n+1)}{2}}$$

and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{R})$ acts on $\mathbb{H}_n$ as $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$. Put $j(\gamma, Z) = \det(CZ + D)$.

2.2. SU_{2n+1}. The tube domain is given by

$$\mathcal{H}_{2n+1} := \{Z \in M_{2n+1}(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0\} \subset \mathbb{C}^{(2n+1)^2}$$

and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU_{2n+1}(\mathbb{R})$ acts on $\mathcal{H}_{2n+1}$ as $\gamma(Z) = (AZ + B)(CZ + D)^{-1}$. Put $j(\gamma, Z) = \det(CZ + D)$.

2.3. $SU(2n, H)$. Let H be a definite quaternion algebra with basis $1, i, j, k = ij$ over $\mathbb{Q}$. By Lemma 1.1 of [25], there exists a unique polynomial map (with $4n$ variables) $P : M_n(H) \longrightarrow \mathbb{Q}$ such that $\nu(X) = P(X)^2$ and $P(I_n) = 1$. Put $\mathrm{Paf}(X) = P(X)$ for any $X \in M_n(H)$. The tube domain is given by

$$\mathfrak{H}_n := \{Z \in M_n(H \otimes_{\mathbb{Q}} \mathbb{C}) \mid {}^*Z = Z,\ \mathrm{Im}(Z) > 0\} \subset \mathbb{C}^{2n(n+1)}$$

and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2n,H)(\mathbb{R})$ acts on $\mathfrak{H}_n$ as $\gamma(Z) = (AZ+B)(CZ+D)^{-1}$. Put $j(\gamma, Z) = \nu(CZ+D)^{\frac{1}{2}}$.

2.4. $E_{7,3}$. This group is defined by using Cayley numbers and the structure is rather complicated than previous cases. We refer [2],[4], [15], and Section 2 of [19]. For any field K whose characteristic is different from 2 and 3, the Cayley numbers $\mathfrak{C}_K$ over K is an eight-dimensional vector space over K with basis $\{e_0 = 1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ satisfying certain rules of multiplication. Let $\mathfrak{J}_K$ be the exceptional Jordan algebra consisting of the element:

$$X = (x_{ij})_{1\le i,j\le 3} = \begin{pmatrix} a & x & y \\ \bar{x} & b & z \\ \bar{y} & \bar{z} & c \end{pmatrix},$$

where $a, b, c \in Ke_0 = K$ and $x, y, z \in \mathfrak{C}_K$. We also define

$$\mathfrak{J}_2 = \left\{ \begin{pmatrix} a & x \\ \bar{x} & b \end{pmatrix} \middle| \, a, b \in K, \ x \in \mathfrak{C}_K \right\}.$$

Then the exceptional domain is

$$\mathfrak{D} := \{Z = X + Y\sqrt{-1} \in \mathfrak{J}_{\mathbb{C}} \mid X, Y \in \mathfrak{J}_{\mathbb{R}}, \ Y > 0\}$$

which is a complex analytic subspace of $\mathbb{C}^{27}$. We also define

$$\mathfrak{D}_2 := \{X + Y\sqrt{-1} \in \mathfrak{J}_2(\mathbb{C}) \mid X, Y \in \mathfrak{J}_2(\mathbb{R}), \ Y > 0\}$$

which is the tube domain of $Spin(2,10)$, i.e., $Spin(2,10)$ acts on $Z \in \mathfrak{D}_2$.

3. Jacobi groups and Jacobi forms

In this section we review Jacobi groups and Jacobi forms with a matrix index.

3.1. **Jacobi groups.** We are concerned with the Jacobi group J realized in G, which is a semi-direct product $J \simeq V \rtimes H$ of a semisimple group H and a Heisenberg group V with a 2 step unipotency which has a form $V = X \cdot Y \cdot Z$, where each factor is an additive group (scheme), $\dim(X) = \dim(Y)$, and the center of V is Z. We further require that the action of H on Z is trivial.

In our case, $H = SL_2$ if $G \neq SU_{2n+1}$, and $H = SU_1$ if $G = SU_{2n+1}$. If we write an element as $v = v(x, y, z)$, then by definition, an alternating form $\langle *, * \rangle$ is furnished on $X \oplus Y$ such that the multiplication of two elements in V is given by

$$v(x,y,z) \cdot v(x',y',z') = v(x+x', y+y', z+z' + \frac{1}{2}\langle (x,y),(x',y')\rangle)$$

and further SL_2 or U_1 acts on V as

$$\gamma \cdot v(x,y,z) = v(ax+cy, bx+dy, z),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 \text{ or } U_1.$$

We shall give a table of the dimension $\dim(X)$ of X as a vector scheme over $\mathbb{Z}$ which will be related to the difference of the weights between the original form and the lift.

G	Sp_{4n}	SU_{2n+1}	$SU(2n,H)$	$E_{7,3}$
$\dim(X)$	$2n-1$	$4n$	$4(n-1)$	16

TABLE 1

The difference between $\ell(k)$ and $2k$ is given by $\frac{1}{2}\dim(X)$ except for Sp_{4n}. For Sp_{4n}, we first obtain a cusp form of the half-integral weight $k+\frac{1}{2}$ via Shimura correspondence $S_{2k}(SL_2(\mathbb{Z})) \simeq S_{k+\frac{1}{2}}(\Gamma_0(4))^+$ from the cusp form $f \in S_{2k}(SL_2(\mathbb{Z}))$. Then the difference should be understood as $\ell(k) - (k+\frac{1}{2}) = n - \frac{1}{2}$, which is nothing but $\frac{1}{2}\dim(X)$ for Sp_{4n}.

For Sp_{4n},

$$V = \left\{ v(x,y,z) = \begin{pmatrix} 1_{2n-1} & x & z & y \\ 0 & 1 & {}^t y & 0 \\ & & 1_{2n-1} & 0 \\ & & -{}^t x & 1 \end{pmatrix} \in Sp_{4n} \,\middle|\, {}^t z - y({}^t x) = z - x({}^t y) \right\} = X \cdot Y \cdot Z,$$

where $X = \{v(x,0,0) \in V\}$, $Y = \{v(x,0,0) \in V\}$, and $Z = \{v(0,0,z) \in V\}$, and

$$SL_2 \simeq H := \left\{ \begin{pmatrix} 1_{2n-1} & 0 & 0_{2n-1} & 0 \\ 0 & a & 0 & b \\ 0_{2n-1} & 0 & 1_{2n-1} & 0 \\ 0 & c & 0 & d \end{pmatrix} \in Sp_{4n} \right\}. \tag{3.1}$$

For SU_{2n+1},

$$V = \left\{ v(x,y,z) = \begin{pmatrix} 1_{2n} & x & z & y \\ 0 & 1 & {}^t \bar{y} & 0 \\ & & 1_{2n} & 0 \\ & & -{}^t \bar{x} & 1 \end{pmatrix} \in SU_{2n+1} \,\middle|\, {}^t \bar{z} - y({}^t \bar{x}) = z - x({}^t \bar{y}) \right\} = X \cdot Y \cdot Z,$$

where $X = \{v(x,0,0) \in V\}$, $Y = \{v(x,0,0) \in V\}$, and $Z = \{v(0,0,z) \in V\}$, and

$$U_1 \simeq H := \left\{ \begin{pmatrix} 1_{2n} & 0 & 0_{2n} & 0 \\ 0 & a & 0 & b \\ 0_{2n} & 0 & 1_{2n} & 0 \\ 0 & c & 0 & d \end{pmatrix} \in SU_{2n+1} \right\}. \tag{3.2}$$

We omit details for $SU(2n, H)$ or $E_{7,3}$. Instead we refer Section 5 of [25], and Section 3 and 4 of [19].

Recall L in the introduction. This is the parameter space of Fourier expansion of a modular form on $\mathfrak{D}$. Let Z' be a subgroup of the unipotent radical N of the Siegel parabolic subgroup consisting of matrices whose last low and last column are zero. Then Z' is naturally identified with Z. We denote by L' (resp. L'_+) the subset of $Z'(\mathbb{Q})$ consisting of semi-positive (resp. positive), semi-integral matrices. For any $T \in L^+$, there exists $S \in L'_+$ such that $T = \begin{pmatrix} S & \alpha \\ \beta & x \end{pmatrix}$ with $x \in \mathbb{Z}_+$ and $\beta = \begin{cases} {}^t\alpha, & \text{if } G = Sp_{2n} \\ {}^t\bar{\alpha}, & \text{if } G = SU_{2n+1} \text{ or } SU(2n,H) \\ {}^t\bar{\alpha}, & \text{if } G = E_{7,3}. \end{cases}$

Henceforth we fix $S \in L'_+$. We define the map λ_S on Z by $z \mapsto \frac{1}{2}\mathrm{Tr}_G(Sz)$ if $G \neq E_{7,3}$ and $z \mapsto \frac{1}{2}(S,z)$ for $E_{7,3}$. Then for any domain ring R with characteristic zero, the map $V(R) \longrightarrow X \oplus Y \oplus R$, $v(x,y,z) \mapsto (x,y,\lambda_S(z))$ gives rise to the Heisenberg structure on $X \oplus Y \oplus R$. Hence for any two elements $(x,y,a),(x',y',b) \in X \oplus Y \oplus R$, the multiplication is given by

$$(x,y,a) * (x',y',b) = (x+x', y+y', a+b+\frac{1}{2}\langle (x,y),(x',y')\rangle_S)$$

where $\langle (x,y),(x',y')\rangle_S = \sigma_S(x,y') - \sigma_S(x',y)$. Here $\sigma_S(*,*)$ on $X \oplus Y$ is given by

$$\sigma_S(x,y) = \begin{cases} {}^t xSy & \text{if } G = Sp_{4n}, \\ {}^t\bar{x}Sy & \text{if } G = SU_{2n+1} \text{ or } SU(2n,H) \\ (S, x({}^t\overline{y}) + y({}^t\overline{x})) & \text{if } G = E_{7,3} \end{cases}$$

Put $\mathfrak{X} := X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{D}_1 := \mathbb{H} \times \mathfrak{X}$. The group $J(\mathbb{R})$ acts on $\mathfrak{D}_1$ by

$$\beta(\tau,u) := \left(\gamma\tau, \frac{u}{c\tau+d} + x(\gamma\tau) + y\right), \ \beta = v(x,y,z)h, \ v(x,y,z) \in V(\mathbb{R}), \ h = h(\gamma) \in H(\mathbb{R})$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. Here $\gamma\tau = \dfrac{a\tau+b}{c\tau+d}$ and put $j(\gamma,\tau) := c\tau+d$ for simplicity. For each positive half integer k, the automorphy factor on $J(\mathbb{R}) \times \mathfrak{D}_1$ is defined by

$$j_{k,S}(\beta,(\tau,u)) := j(\gamma,\tau)^k \mathbf{e}(-2\lambda_S(z) + \frac{c}{j(\gamma,\tau)}\sigma_S(u,u) - \frac{2\sigma_S(x,u)}{j(\gamma,u)} - \sigma_S(x,x)(\gamma\tau) - \sigma_S(x,y)),$$

where $\mathbf{e}(*) = \exp(2\pi\sqrt{-1}*)$. When k is not an integer, $j(\gamma, z)^k = (j(\gamma, z)^{\frac{1}{2}})^{2k}$ is defined by the automorphy factor $j(\gamma, z)^{\frac{1}{2}}$ of the metaplectic covering $\widetilde{SL}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$.

For each function $f : \mathfrak{D}_1 \longrightarrow \mathbb{C}$ and $\beta \in V(\mathbb{R})$, we define the "slash" operator $f|_{k,S}[\beta] : \mathfrak{D}_1 \longrightarrow \mathbb{C}$ by

$$f|_{k,S}[\beta](\tau, u) := j_{k,S}(\beta; (\tau, u))^{-1} f(\beta(\tau, u)).$$

3.2. **Jacobi forms with a matrix index.** We define and study Jacobi forms of matrix index on $\mathfrak{D}_1 = \mathbb{H} \times \mathfrak{X}$ in the classical setting. Set

$$\Gamma_J := J(\mathbb{Q}) \cap G(\mathbb{Z}).$$

Definition 3.1. Let k be a positive even integer if $G \neq Sp_{4n}$, a positive half-integral integer if $G = Sp_{4n}$, and S be an element of L'_+. We say a holomorphic function $\phi : \mathfrak{D}_1 \longrightarrow \mathbb{C}$ is a Jacobi form (resp. Jacobi cusp form) of weight k and index S if ϕ satisfies the following conditions:

(1) $\phi|_{k,S}[\beta] = \phi$ for any $\beta \in \Gamma_J$

(2) ϕ has a Fourier expansion of the form

$$\phi(\tau, u) = \sum_{\xi \in X(\mathbb{Q}),\ N \in \mathbb{Z}} c(N, \xi)\mathbf{e}(N\tau + \sigma_S(\xi, u)),$$

where $c(N, \xi) = 0$ unless $S_{\xi,N} := \begin{pmatrix} S & S\xi \\ {}^*\xi S & N \end{pmatrix}$ belongs to L' (resp. L'_+) where ${}^*\xi$ stands for ${}^t\xi$ if $G = Sp_{4n}$, ${}^t\bar{\xi}$ if $G = SU_{2n+1}$ or $SU(2n, H)$, and ${}^t\bar{\xi}$ if $G = E_{7,3}$.

We denote by $J_{k,S}(\Gamma_J)$ (resp. $J^{\rm cusp}_{k,S}(\Gamma_J)$) the space of Jacobi forms (resp. Jacobi cusp forms) of weight k and index S.

Let us extend the quadratic form σ_S linearly to that on $\mathfrak{X}$. We denote by $\mathcal{S}(X(\mathbb{A}_f))$ the space of Schwartz functions on $X(\mathbb{A}_f)$. For each $\varphi \in \mathcal{S}(X(\mathbb{A}_f))$, the classical theta function on $\mathfrak{D}_1 := \mathbb{H} \times \mathfrak{X}$ is given by

$$\theta^S_\varphi(\tau, u) := \sum_{\xi \in X(\mathbb{Q})} \varphi(\xi)\mathbf{e}(\sigma_S(\xi, \xi)\tau + 2\sigma_S(\xi, u)).$$

Define the dual of the lattice $\Lambda := X(\mathbb{Z})$ with respect to the quadratic form σ_S by

$$\widetilde{\Lambda}(S) = \{x \in X(\mathbb{Q}) \mid \sigma_S(x, y) \in \mathbb{Z} \text{ for all } y \in \Lambda\}.$$

If $S \in L'_+$, then the quotient $\widetilde{\Lambda}(S)/\Lambda$ is a finite group. Fix a complete representative $\Xi(S)$ of $\widetilde{\Lambda}(S)/\Lambda$ and denote by φ_ξ the characteristic function $\xi + \prod_{p<\infty} X(\mathbb{Z}_p) \in \mathcal{S}(X(\mathbb{A}_f))$. Any

Jacobi form turns to be the linear combination of products of elliptic modular forms and theta functions by the following lemma.

Lemma 3.2. *Assume $S \in L'_+$. Let $\Xi(S)$ be a complete representative of $\widetilde{\Lambda}(S)/\Lambda$. Then any Jacobi form $\phi \in J_{k,S}(\Gamma_J)$ can be written as*

$$\phi(\tau,u) = \sum_{\xi \in \Xi(S)} \phi_{S,\xi}(\tau)\theta^S_{\varphi_\xi}(\tau,u), \quad \phi_{S,\xi}(\tau) = \sum_{\substack{N\in\mathbf{Z} \\ N-\sigma_S(\xi,\xi)\geq 0}} c(N,\xi)\mathbf{e}((N-\sigma_S(\xi,\xi))\tau).$$

Furthermore, for each $\xi \in \Xi(S)$, $\phi_{S,\xi}(\tau)$ is an elliptic modular form of weight $k - \frac{1}{2}\dim(X)$. .

Proof. See example (iv) at Section 2 of [17] and also the argument at p.656 of [9]. □

Let k be a positive integer and F be a modular form of weight k on $\mathfrak{D}$. We rewrite the variable Z on $\mathfrak{D}$ as $\begin{pmatrix} W & u \\ v & \tau \end{pmatrix}$ where $\tau \in \mathbb{H}$, $u \in X(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, and $W \in \mathbb{H}_{2n-1}, \mathcal{H}_{2n}, \mathfrak{H}_{n-1}$,or $\mathfrak{D}_2$. Note that v is determined by u. Then we have the Fourier-Jacobi expansion

$$F\begin{pmatrix} W & u \\ v & \tau \end{pmatrix} = \sum_{S\in L'} F_S(\tau,u)\mathbf{e}((S,W)). \tag{3.3}$$

Lemma 3.3. *Keep the notation as above. Assume $S \in L'_+$. Then $F_S(\tau,u) \in J_{k,S}(\Gamma_J)$.*

Remark 3.4. *Consider any holomorphic function $F(Z)$ with $Z = \begin{pmatrix} W & u \\ v & \tau \end{pmatrix}$ on $\mathfrak{D}$ which is invariant under $P(\mathbb{Z})$. Then one has the Fourier and the Fourier-Jacobi expansion*

$$F(Z) = \sum_{T\in L} A_F(T)\mathbf{e}((T,Z)) = \sum_{S\in L'} F_S(\tau,u)\mathbf{e}((S,W)),$$

as in (3.3). By Lemma 3.2,

$$F_S(\tau,u) = \sum_{\xi\in\Xi(S)} F_{S,\xi}(\tau)\theta^S_{\varphi_\xi}(\tau,u), \quad F_{S,\xi}(\tau) = \sum_{\substack{N\in\mathbf{Z} \\ N-\sigma_S(\xi,\xi)\geq 0}} A_F(S_{\xi,N})\mathbf{e}((N-\sigma_S(\xi,\xi))\tau),$$

where $S_{\xi,N} = \begin{pmatrix} S & S\xi \\ {}^\star\xi S & N \end{pmatrix}$. The function $F_{S,\xi}$ will be called (S,ξ)-component of F.

Recall the following definition from [9, 10] .

Definition 3.5. For a sufficiently large k_0, a compatible family of Eisenstein series is a family of elliptic modular forms, for even integer $k' \geq k_0$,

$$g_{k'}(\tau) = b_{k'}(0) + \sum_{N\in\mathbb{Q}_{>0}} N^{\frac{k'-1}{2}} b_{k'}(N)q^N, \ q = \mathbf{e}(\tau),$$

satisfying the following three conditions:

(1) $g_{k'} \in \mathcal{V}(E^1_{k'})$ for all $k' \geq k_0$
(2) for each $N \in \mathbb{Q}^\times_+$, there exists $\Phi_N \in \mathcal{R}$ such that $b_{k'}(N) = \Phi_N(\{p^{\frac{k'-1}{2}}\}_p)$.
(3) there exists a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ such that $g_{k'} \in M_{k'}(\Gamma)$ for all $k' \geq k_0$. Here $M_{k'}(\Gamma)$ stands for the space of elliptic modular forms of weight k with respect to Γ.

The following theorem plays a key role in the proof of Theorem 1.1:

Theorem 3.6. *Keep the notations above. Let $E_{\ell(k)}$ be the Siegel Eisenstein series in Section 1. Assume $S \in L'_+$. Then any (S,ξ)-component of $E_{\ell(k)}$ is an Eisenstein series of weight $k - \frac{1}{2}\dim(X)$.*

This theorem was first proved by Böcherer [3] for $G = Sp_{4n}$ in the classical language. However the proof there involves many complicated terms and seems difficult to read off what we need. More sophisticated proof was given by Ikeda [7]. He made a good use of Weil representation and worked over the adelic language. In [25], [19], the authors followed his method. However in case $E_{7,3}$, the group structure is much more complicated than others. So the proof is not a routine at all.

The following Lemma 10.2 of [10] is a crucial ingredient.

Theorem 3.7. *Let $f(\tau) = \sum_{n=1}^{\infty} c(n)q^n$ be a Hecke eigenform of weight k with respect to $SL_2(\mathbb{Z})$ with $c(p) = p^{\frac{k-1}{2}}(\alpha_p + \alpha_p^{-1})$. Assume that there is a finite dimensional representation $(u, \mathbb{C}^d)$ of $SL_2(\mathbb{Z})$ and*

$$\vec{\Phi}_N := {}^t(\Phi_{1,N}, \ldots, \Phi_{d,N}) \in \mathcal{R}^d, \ N \in \mathbb{Q}_{>0}$$

satisfying the following two conditions:

(1) there exists a vector valued modular form $\vec{g}_{k'} = {}^t(g_{1,k'}, \ldots, g_{d,k'})$ which has

$$\vec{g}_{k'}(\tau) = \vec{b}_{k'}(0) + \sum_{N \in \mathbb{Q}_{>0}} N^{\frac{k'-1}{2}} \vec{b}_{k'}(N) q^n, \ (\vec{b}_{k'}(N) = {}^t(b_{1,k'}(N), \ldots, b_{d,k'}(N)), \ N \in \mathbb{Q}_{\geq 0})$$

of weight k' with type u for each sufficiently large even integers k', hence this means that

$$\vec{g}_{k'}(\tau)|_{k'}[\gamma] := {}^t(g_{1,k'}|_{k'}[\gamma], \ldots, g_{d,k'}|_{k'}[\gamma]) = u(\gamma)\vec{g}_k(\tau) \text{ for any } \gamma \in SL_2(\mathbb{Z}),$$

(2) each component $g_{i,k'}$, $(1 \leq i \leq d)$ of $\vec{g}_{k'}(\tau)$ is a compatible family of Eisenstein series such that

$$b_{i,k'}(N) = \Phi_{i,N}(\{p^{\frac{k'-1}{2}}\}_p).$$

Then $\vec{h}(\tau) := \sum_{N\in\mathbb{Q}_{>0}} N^{\frac{k-1}{2}}\vec{\Phi}_N(\{\alpha_p\}_p)q^N$ *is a vector valued modular form of weight* k *with type* u*, hence it satisfies*

$$\vec{h}(\tau)|_k[\gamma] = u(\gamma)\vec{h} \text{ for any } \gamma \in SL_2(\mathbb{Z}).$$

4. Proof of Theorem 1.1 and 1.2

Recall that for each normalized Hecke eigenform $f = \sum_{n=1}^{\infty} a(n)q^n \in S_{2k}(SL_2(\mathbb{Z}))$, we have considered the following formal series on $\mathfrak{D}$:

$$F_f(Z) := \sum_{T\in L^+} A_{F_f}(T)\exp(2\pi\sqrt{-1}\mathrm{Tr}_G(TZ)),\ Z\in\mathfrak{D},\quad A_{F_f}(T) = A(\{\alpha_p\}_p).$$

The first task is to check the absolute convergence for F_f: This is done by using explicit formula of Fourier coefficients of Siegel Eisenstein series and Ramanujan bound $|a(p)| \le 2p^{k-\frac{1}{2}}$.

Next, we use the fact that $\Gamma = G(\mathbb{Z})$ is generated by $P(\mathbb{Z})$ and $H(\mathbb{Z})$, where H is in (3.1) or (3.2). We can easily check, by property of Fourier coefficients of Siegel Eisenstein series, that F_f is invariant under the action of $P(\mathbb{Z})$. Therefore to prove the automorphy of F_f, we have to check only the invariance of F_f under the action of $H(\mathbb{Z})$. For this, we need to use the Fourier-Jacobi expansion.

To unify notation we write the Fourier coefficient of F_f as $A_{F_f}(T) = C_1(T)C_2(T)^{k-\frac{1}{2}}\prod_p \widetilde{F}_p(T;\alpha_p)$ for $T\in L^+$ where $C_1(T) = L(1-k,\chi_T)$ if $G = Sp_{4n}$, $C_1(T) = 1$ otherwise, and other terms should be clear from the definition as in the introduction. Since $F(Z) := F_f(Z)$ is invariant under $P(\mathbb{Z})$, by Remark 3.4, one has the Fourier-Jacobi expansion:

$$F\begin{pmatrix} W & u \\ v & \tau \end{pmatrix} = \sum_{S\in L'_+} F_S(\tau,u)\mathbf{e}(\mathrm{Tr}_G(SW)),\quad F_S(\tau,u) = \sum_{\xi\in\Xi(S)} F_{S,\xi}(\tau)\theta^S_{\varphi_\xi}(\tau,u),$$

and

$$\begin{aligned} F_{S,\xi}(\tau) &= \sum_{\substack{N\in\mathbf{Z}\\ N-\sigma_S(\xi,\xi)\ge 0}} A_F(S_{\xi,N})\mathbf{e}((N-\sigma_S(\xi,\xi))\tau),\ S_{\xi,N} := \begin{pmatrix} S & S\xi \\ \eta S & N \end{pmatrix} \\ &= \sum_{\substack{N\in\mathbf{Z}\\ N-\sigma_S(\xi,\xi)\ge 0}} C_1(S_{\xi,N})C_2(S_{\xi,N})^{k-\frac{1}{2}}\prod_p \widetilde{F}_p(S_{\xi,N};\alpha_p)\mathbf{e}((N-\sigma_S(\xi,\xi))\tau) \\ &= D(S)^{k-\frac{1}{2}} \sum_{\substack{N\in\mathbf{Z}\\ N-\sigma_S(\xi,\xi)\ge 0}} C_1(S_{\xi,N})(N-\sigma_S(\xi,\xi))^{k-\frac{1}{2}}\prod_p \widetilde{F}_p(S_{\xi,N};\alpha_p)\mathbf{e}((N-\sigma_S(\xi,\xi))\tau) \end{aligned}$$

where there exists the constant $D(S)$ depending only on S such that $C_2(S_{\xi,N}) = D(S)(N-\sigma_S(\xi,\xi))$. The invariance under $H(\mathbb{Z})$ is equivalent to claiming that $F_S(\tau,u)\in J_{k,S}(\Gamma_J)$ for any $S\in L'_+$.

By (2,1), p.124 of [23], for each $\gamma \in SL_2(\mathbb{Z})$, there exists a unitary matrix $u_S(\gamma) = (u_S(\gamma)_{\xi\eta})_{\xi,\eta\in\Xi(S)}$ such that

$$\theta^S_{\varphi_\xi}|_{k,S}[\gamma](\tau,u) = \sum_{\eta\in\Xi(S)} u_S(\gamma)_{\xi\eta}\theta^S_{\varphi_\eta}(\tau,u).$$

Further there exists a positive integer Δ_S depending on S such that u_S is trivial on $\Gamma(\Delta_S) \subset SL_2(\mathbb{Z})$. Since $\{\theta^S_{\varphi_\xi} \mid \xi \in \Xi(S)\}$ are linearly independent over $\mathbb{C}$, it suffices to prove that $\{F_{S,\xi}\}_{\xi\in\Xi(S)}$ is a vector valued modular form with type u_S.

For a sufficiently large positive integer k', we now turn to consider (S,ξ)-component $(\mathcal{E}_{\ell(k')})_{S,\xi}$ of the classical Eisenstein series

$$\mathcal{E}_{\ell(k')}(Z) = \sum_{T\in L} A(T)\exp(2\pi\sqrt{-1}\cdot \mathrm{Tr}_G(TZ)),\ A(T) = C_1(T)C_2(T)^{k'-\frac12}\prod_p \widetilde{F}_p(T;p^{k'-\frac12}),$$

Then one has

$$\begin{aligned}&D(S)^{-k'+\frac12}(\mathcal{E}_{\ell(k')})_{S,\xi}(\tau)\\ =&\sum_{\substack{N\in\mathbb{Z}\\ N-\sigma_S(\xi,\xi)\geq 0}} C_1(S_{\xi,N})(N-\sigma_S(\xi,\xi))^{-k'+\frac12}\prod_{p|\det(S_{\xi,N})}\widetilde{F}_p(S_{\xi,N};p^{k'-\frac12})\mathrm{e}((N-\sigma_S(\xi,\xi))\tau)\end{aligned}$$

Then by Theorem 3.6, $\{D(S)^{-k'+\frac12}(\mathcal{E}_{\ell(k')})_{S,\xi}\}_{k'\gg 0}$ makes up a compatible family of Eisenstein series in the sense of Ikeda (see Section 10 of [9] for $G = Sp_{4n}$ and Section 7 of [10] for other cases). Applying Lemma 3.7, one can conclude that

$$F_{S,\xi} = D(S)^{k-\frac12}\sum_{\substack{n\in\mathbb{Z}_{>0}\\ n=N-\sigma_S(\xi,\xi),\ N\in\mathbb{Z}}} C_1(S_{\xi,N})n^{k-\frac12}\prod_{p|\det(S_{\xi,N})}\widetilde{F}_p(S_{\xi,N};p^{k'-\frac12})q^n,$$

is a vector valued modular form with type u_S. The non-vanishing is easy to check except for Sp_{4n}. In this case, a bit of careful study was needed (see p.651 of [9]). At any late one can prove the non-vanishing of F_f.

Since we know Satake parameters of π_F, it is easy to compute $L(s,\pi_F,St)$. For $G = E_{7,3}$, we can use the Langlands-Shahidi method for the case $GE_7 \subset E_8$ (cf. [16], section 2.7.8).

5. Some conjectures and problems

In this section we are concerning with some conjectures and problems related to the results in [19].

5.1. **Conjectural Arthur parameter.** It is worth considering the compatibility with Arthur conjecture in the case $E_{7,3}$: We write the degree 56 standard L-function of $F := F_f$ as

$$L(s,\pi_F,St)=L(s,Sym^3\pi_f)\prod_{i=-4}^{4}L(s+i,\pi_f)\prod_{i=-8}^{8}L(s+i,\pi_f).$$

This suggests the following parametrization of π_F:

Let $\mathcal{L}$ be the (hypothetical) Langlands group over $\mathbb{Q}$, and let $\rho_f:\mathcal{L}\longrightarrow SL_2(\mathbb{C})$ be the 2-dimensional irreducible representation of $\mathcal{L}$ corresponding to π_f. Let Sym^n be the irreducible $(n+1)$-dimensional representation of $SL_2(\mathbb{C})$. Note that if $n=2m-1$, $Im(Sym^n)\subset Sp_{2m}(\mathbb{C})$, and if $n=2m$, $Im(Sym^n)\subset SO_{2m+1}(\mathbb{C})$. We have the tensor product maps $SL_2(\mathbb{C})\times Sp_{2m}(\mathbb{C})\longrightarrow Sp_{4m}(\mathbb{C})$ and $SL_2(\mathbb{C})\times SO_{2m+1}(\mathbb{C})\longrightarrow Sp_{4m+2}(\mathbb{C})$. Hence

$\rho_f\otimes Sym^{16}:\mathcal{L}\times SL_2(\mathbb{C})\longrightarrow Sp_{34}(\mathbb{C})$, and $\rho_f\otimes Sym^8:\mathcal{L}\times SL_2(\mathbb{C})\longrightarrow Sp_{18}(\mathbb{C})$.

Let $Sym^3\rho_f:\mathcal{L}\times SL_2(\mathbb{C})\longrightarrow Sp_4(\mathbb{C})$ be the parameter of $Sym^3\pi_f$, where it is trivial on $SL_2(\mathbb{C})$. Consider the parameter

$$\rho=Sym^3\rho_f\oplus(\rho_f\otimes Sym^{16})\oplus(\rho_f\otimes Sym^8):\mathcal{L}\times SL_2(\mathbb{C})\longrightarrow Sp_4(\mathbb{C})\times Sp_{34}(\mathbb{C})\times Sp_{18}(\mathbb{C})\subset Sp_{56}(\mathbb{C})$$

Note that $E_7(\mathbb{C})\subset Sp_{56}(\mathbb{C})$. We expect that ρ will factor through $E_7(\mathbb{C})$, and give rise to a parameter $\rho:\mathcal{L}\times SL_2(\mathbb{C})\longrightarrow E_7(\mathbb{C})$, which parametrizes π_F.

5.2. **Ikeda lift as CAP form.** If $G=Sp_{4n}$, the Ikeda lift F_f is a CAP form. Namely, π_F is nearly equivalent to the quotient of the induced representation

$$Ind_{P_{2,..,2}}^{Sp_{4n}}\ \pi_f|det|^{n-\frac{1}{2}}\otimes\pi_f|det|^{n-\frac{3}{2}}\otimes\cdots\otimes\pi_f|det|^{\frac{1}{2}},$$

where $P_{2,\dots,2}$ is the standard parabolic subgroup of Sp_{4n} with the Levi subgroup $GL_2\times\cdots\times GL_2$ (n factors) (see also p.114 of [8]).

If $G=E_{7,3}$, π_F cannot be a CAP form in a usual sense since there are not many $\mathbb{Q}$-parabolic subgroups of $E_{7,3}$. We expect that π_F will be a CAP form in a more general sense: Namely, there exists a parabolic subgroup $Q=M'N'$ of the split E_7, and a cuspidal representation $\tau=\otimes_p'\tau_p$ of M', and a parameter Λ_0 such that for all finite prime p, π_p is a quotient of $Ind_{Q(\mathbb{Q}_p)}^{E_7(\mathbb{Q}_p)}\ \tau_p\otimes exp(\Lambda_0,H_Q(\ ))$.

5.3. **Miyawaki type lift to $GSpin(2,10)$.** This work is in progress [20]. For $Z\in\mathfrak{D}_2$, let $\begin{pmatrix}Z&0\\0&\tau\end{pmatrix}\in\mathfrak{D}$. For $f\in S_{2k}(SL_2(\mathbb{Z}))$, let F be the Ikeda lift of f, which is a cusp form of weight $2k+8$ on $\mathfrak{D}$. For $h\in S_{2k+8}(SL_2(\mathbb{Z}))$, consider the integral

$$\mathcal{F}_{f,h}(Z) = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} F\begin{pmatrix} Z & 0 \\ 0 & \tau \end{pmatrix} \overline{h(\tau)}(Im\tau)^{2k+6}\, d\tau.$$

When $\mathcal{F}_{f,h}$ is not zero, it is a cusp form of weight $2k+8$ on $\mathfrak{D}_2$. It is expected that $\mathcal{F}_{f,h}$ is a Hecke eigen form, and it would give rise to a cuspidal representation $\pi_{\mathcal{F}_{f,h}}$ on $GSpin(2,10)$: Let $\pi_{\mathcal{F}_{f,h}} = \pi_\infty \otimes \otimes'_p \pi_p$. Let $\{\alpha_p, \alpha_p^{-1}\}$ and $\{\beta_p, \beta_p^{-1}\}$ be the Satake parameter of f and h at the prime p, resp. Then for each prime p, it is expected that the Satake parameter of π_p is

$$\{(\beta_p\alpha_p)^{\pm 1}, (\beta_p\alpha_p^{-1})^{\pm 1}, 1, 1, p^{\pm 1}, p^{\pm 2}, p^{\pm 3}\}.$$

Then the standard L-function of $\pi_{\mathcal{F}_{f,h}}$ is

$$L(s, \pi_{\mathcal{F}_{f,h}}, St) = L(s, h\times f)\zeta(s)^2\zeta(s\pm 1)\zeta(s\pm 2)\zeta(s\pm 3),$$

where the first factor is the Rankin-Selberg L-function. This can be explained by Arthur parameter as follows: Let $\phi_f, \phi_h : \mathcal{L} \longrightarrow SL_2(\mathbb{C})$ be the hypothetical Langlands parameter. Then due to the tensor product map $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow SO_4(\mathbb{C})$, we have $\phi_f \otimes \phi_h : \mathcal{L} \longrightarrow SO_4(\mathbb{C})$. The distinguished unipotent orbit $(7,1)$ of $SO_8(\mathbb{C})$ gives rise to the map $SL_2(\mathbb{C}) \longrightarrow SO_8(\mathbb{C})$. It defines the map $\phi_u : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow SO(8,\mathbb{C})$. Then consider

$$\phi = (\phi_h \otimes \phi_f) \oplus \phi_u : \mathcal{L} \times SL_2(\mathbb{C}) \longrightarrow SO_4(\mathbb{C}) \times SO_8(\mathbb{C}) \subset GSO_{12}(\mathbb{C}).$$

We expect that ϕ parametrizes $\pi_{\mathcal{F}_{f,h}}$.

5.4. **Pertersson formula and its possible application.** In case $E_{7,3}$, it may be interesting to give an explicit formula of the Petersson inner product formula for F_f. (See [5] for its importance.) Since $L(s, \pi_F, St)$ involves the third symmetric power L-function $L(s, Sym^3\pi_f)$, we expect to somehow figure out an "algebraic part" of $L(s, Sym^3\pi_f)$.

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