Local existence of solutions for the heat equation with a nonlinear boundary condition

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1　Introduction

This paper is concerned with the heat equation with a nonlinear boundary condition,

$$\begin{cases} \partial_t u = \Delta u, & x \in \Omega,\ t > 0, \\ \nabla u \cdot \nu(x) = |u|^{p-1} u, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = \varphi(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $N \geq 1$, $p > 1$, Ω is a smooth domain in $\mathbf{R}^N$, $\partial_t = \partial/\partial t$ and $\nu = \nu(x)$ is the outer unit normal vector to $\partial\Omega$. For any $\varphi \in BUC(\Omega)$, problem (1.1) has a unique solution

$$u \in C^{2,1}(\Omega \times (0, T]) \cap C^{1,0}(\overline{\Omega} \times (0, T]) \cap BUC(\overline{\Omega} \times [0, T])$$

for some $T > 0$ and the maximal existence time $T(\varphi)$ of the solution can be defined. If $T(\varphi) < \infty$, then

$$\limsup_{t \to T(\varphi)} \|u(t)\|_{L^\infty(\Omega)} = \infty$$

and we call $T(\varphi)$ the blow-up time of the solution u.

Problem (1.1) has been studied in many papers from various points of view (see e.g. [4]–[6], [8]–[12], [14]–[18], [20]–[25], [27], [28], [33], [35] and references therein). In particular, the local well-posedness of the solutions of (1.1) in $L^r(\Omega)$ ($1 \leq r \leq \infty$) was studied in [4]. See also [6]. However, for problem (1.1), there are few results related to the dependence of the blow-up time on the initial function.

Let $L^r_{uloc,\rho}(\Omega)$ be the uniformly local L^r space in Ω equipped with the norm

$$\|f\|_{r,\rho} := \sup_{x \in \overline{\Omega}} \left(\int_{\Omega \cap B(x,\rho)} |f(y)|^r dy \right)^{1/r},$$

where $1 \leq r < \infty$ and $\rho > 0$. Let $\mathcal{L}^r_{uloc,\rho}(\Omega)$ be the completion of bounded uniformly continuous functions in Ω with respect to the norm $\| \cdot \|_{r,\rho}$, that is,

$$\mathcal{L}^r_{uloc,\rho}(\Omega) := \overline{BUC(\Omega)}^{\| \cdot \|_{r,\rho}}.$$

We set $L^\infty_{uloc,\rho}(\Omega) = L^\infty(\Omega)$ and $\mathcal{L}^\infty_{uloc,\rho}(\Omega) = BUC(\Omega)$. The spaces $L^r_{uloc,\rho}(\Omega)$ and $\mathcal{L}^r_{uloc,\rho}(\Omega)$ are useful for the study of the solutions of parabolic equations in unbounded domains with non-decaying initial functions (see e.g., [7], [31] and references therein).

In this paper we prove the local existence and the uniqueness of the solutions of problem (1.1) with initial functions in $\mathcal{L}^r_{uloc,\rho}(\Omega)$ and obtain the estimates of the blow-up time of the solutions by using the scaling parameter ρ of $\|\varphi\|_{r,\rho}$. The blow-up time of the solution is involved with the degree of the concentration of the initial function, which can be estimated by the scaling parameter ρ of the norm $\|\varphi\|_{r,\rho}$. We give the estimates of the blow-up time by the norm $\|\varphi\|_{r,\rho}$ with a suitable choice of ρ. This also gives a sufficient condition for the existence of global-in-time solutions for problem (1.1) (see Corollary 1.1 and Remark 1.1).

Throughout this paper, following [34, Section 1], we assume that $\Omega \subset \mathbf{R}^N$ is a uniformly regular domain of class C^1. For any $x \in \mathbf{R}^N$ and $\rho > 0$, define

$$B(x,\rho) := \{y \in \mathbf{R}^N : |x - y| < \rho\}, \quad \Omega(x,\rho) := \Omega \cap B(x,\rho), \quad \partial\Omega(x,\rho) := \partial\Omega \cap B(x,\rho).$$

By the trace inequality for $W^{1,1}(\Omega)$-functions and the Gagliardo–Nirenberg inequality we can find $\rho_* \in (0,\infty]$ with the following properties (see Lemma 2.2).

- There exists a positive constant c_1 such that

$$\int_{\partial\Omega(x,\rho)} |v|\, d\sigma \leq c_1 \int_{\Omega(x,\rho)} |\nabla v|\, dy \tag{1.2}$$

for all $v \in C_0^1(B(x,\rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$.

- Let $1 \leq \alpha$, $\beta \leq \infty$ and $\sigma \in [0,1]$ be such that

$$\frac{1}{\alpha} = \sigma \left(\frac{1}{2} - \frac{1}{N} \right) + (1 - \sigma)\frac{1}{\beta}. \tag{1.3}$$

Assume, if $N \geq 2$, that $\alpha \neq \infty$ or $N \neq 2$. Then there exists a constant c_2 such that

$$\|v\|_{L^\alpha(\Omega(x,\rho))} \leq c_2 \|v\|_{L^\beta(\Omega(x,\rho))}^{1-\sigma} \|\nabla v\|_{L^2(\Omega(x,\rho))}^{\sigma} \tag{1.4}$$

for all $v \in C_0^1(B(x,\rho))$, $x \in \overline{\Omega}$ and $0 < \rho < \rho_*$.

We remark that, in the case

$$\Omega = \{(x', x_N) \in \mathbf{R}^N : x_N > \Phi(x')\},$$

where $N \geq 2$ and $\Phi \in C^1(\mathbf{R}^{N-1})$ with $\|\nabla\Phi\|_{L^\infty(\mathbf{R}^{N-1})} < \infty$, (1.2) and (1.4) hold with $\rho_* = \infty$ (see Lemma 2.2). Inequalities (1.2) and (1.4) are used to treat the nonlinear boundary condition.

Next we state the definition of the solution of (1.1).

Definition 1.1 *Let $0 < T \leq \infty$ and $1 \leq r < \infty$. Let u be a continuous function in $\overline{\Omega} \times (0, T]$. We say that u is a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, T]$ if*

- *$u \in L^\infty(\tau, T : L^\infty(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0, R)))$ for any $\tau \in (0, T)$ and $R > 0$,*

- *$u \in C([0, T) : L^r_{uloc,\rho}(\Omega))$ with $\lim_{t \to 0} \|u(t) - \varphi\|_{r,\rho} = 0$ for some $\rho > 0$,*

- *u satisfies*

$$\int_0^T \int_\Omega \{-u\partial_t\phi + \nabla u \cdot \nabla\phi\} \, dy\,ds = \int_0^T \int_{\partial\Omega} |u|^{p-1}u\phi \, d\sigma\,ds \qquad (1.5)$$

for all $\phi \in C_0^\infty(\mathbf{R}^N \times (0, T))$.

Here $d\sigma$ is the surface measure on $\partial\Omega$. Furthermore, for any continuous function u in $\overline{\Omega} \times (0, T)$, we say that u is a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, T)$ if u is a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, \eta]$ for any $\eta \in (0, T)$.

We remark the following for any ρ, $\rho' \in (0, \infty)$:

- $f \in L^r_{uloc,\rho}(\Omega)$ is equivalent to $f \in L^r_{uloc,\rho'}(\Omega)$;

- $u \in C([0, T] : L^r_{uloc,\rho}(\Omega))$ is equivalent to $u \in C([0, T] : L^r_{uloc,\rho'}(\Omega))$.

These follow from property (i) in Section 2.

Now we are ready to state the main results of this paper. Let $p_* = 1 + 1/N$.

Theorem 1.1 *Let $N \geq 1$ and $\Omega \subset \mathbf{R}^N$ be a uniformly regular domain of class C^1. Let p_* satisfy (1.2) and (1.4). Then, for any $1 \leq r < \infty$ with*

$$\begin{cases} r \geq N(p-1) & \text{if} \quad p > p_*, \\ r > 1 & \text{if} \quad p = p_*, \\ r \geq 1 & \text{if} \quad 1 < p < p_*, \end{cases} \qquad (1.6)$$

there exists a positive constant γ_1 such that, for any $\varphi \in \mathcal{L}^r_{uloc,\rho}(\Omega)$ with

$$\rho^{\frac{1}{p-1}-\frac{N}{r}} \|\varphi\|_{r,\rho} \leq \gamma_1 \qquad (1.7)$$

for some $\rho \in (0, \rho_*/2)$, problem (1.1) possesses a $L^r_{uloc}(\Omega)$-solution u in $\Omega \times [0, \mu\rho^2]$ satisfying

$$\sup_{0 < t < \mu\rho^2} \|u(t)\|_{r,\rho} \leq C\|\varphi\|_{r,\rho}, \qquad (1.8)$$

$$\sup_{0 < t < \mu\rho^2} t^{\frac{N}{2r}} \|u(t)\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{r,\rho}. \qquad (1.9)$$

Here C and μ are constants depending only on N, Ω, p and r.

Theorem 1.1 implies that $T(\varphi) \geq \mu\rho^2$ under assumption (1.6). Furthermore, we have:

Theorem 1.2 *Assume the same conditions as in Theorem 1.1. Let v and w be $L^r_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0, T)$ such that $v(x, 0) \leq w(x, 0)$ for almost all $x \in \Omega$, where $T > 0$ and r is as in (1.6). Assume, if $r = 1$, that*

$$\limsup_{t \to +0} t^{\frac{1}{2(p-1)}} \left[\|v(t)\|_{L^\infty(\Omega)} + \|w(t)\|_{L^\infty(\Omega)} \right] < \infty. \tag{1.10}$$

Then there exists a positive constant γ_2 such that, if

$$\rho^{\frac{1}{p-1}-\frac{N}{r}} \left[\|v(0)\|_{r,\rho} + \|w(0)\|_{r,\rho} \right] \leq \gamma_2 \tag{1.11}$$

for some $\rho \in (0, \rho_*/2)$, then

$$v(x, t) \leq w(x, t) \quad in \quad \Omega \times (0, T).$$

We give some comments related to Theorems 1.1 and 1.2.

(i) Let u be a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, T)$. It follows from Definition 1.1 that $u \in L^\infty(\tau, \sigma : L^\infty(\Omega))$ for any $0 < \tau < \sigma < T$. This together with Theorem 6.2 of [12] implies that $u(t) \in BUC(\Omega)$ for any $t \in (0, T)$. This means that $u(0) \in \mathcal{L}^r_{uloc,\rho}(\Omega)$ for any $\rho > 0$.

(ii) Let $1 \leq r < \infty$. If, either

(a) $f \in L^r_{uloc,1}(\Omega)$, $r > N(p-1)$ or (b) $f \in L^r(\Omega)$, $r \geq N(p-1)$,

then, for any $\gamma > 0$, we can find a constant $\rho > 0$ such that $\rho^{\frac{1}{p-1}-\frac{N}{r}} \|f\|_{r,\rho} \leq \gamma$.

As a corollary of Theorem 1.1, we have:

Corollary 1.1 *Assume the same conditions as in Theorem 1.1 and $p > p_*$.*

(i) *For any $\varphi \in L^{N(p-1)}(\Omega)$, problem (1.1) has a unique $L^{N(p-1)}_{uloc}(\Omega)$-solution in $\Omega \times [0, T]$ for some $T > 0$.*

(ii) *Assume $\rho_* = \infty$. Then there exists a constant γ such that, if*

$$\|\varphi\|_{L^{N(p-1)}(\Omega)} \leq \gamma, \tag{1.12}$$

then problem (1.1) has a unique $L^{N(p-1)}_{uloc}(\Omega)$-solution u such that

$$\sup_{0 < t < \infty} \|u(t)\|_{L^{N(p-1)}(\Omega)} + \sup_{0 < t < \infty} t^{\frac{1}{2(p-1)}} \|u(t)\|_{L^\infty(\Omega)} < \infty.$$

Remark 1.1 *Let $\Omega = \mathbf{R}^N_+ := \{(x', x_N) \in \mathbf{R}^N : x_N > 0\}$. If $1 < p \leq p_*$, then problem (1.1) possesses no positive global-in-time solutions. See [11] and [18]. For the case $p > p_*$, it is proved in [28] (see also [27]) that, if $\varphi \geq 0$, $\varphi \not\equiv 0$ in Ω and*

$$\|\varphi\|_{L^1(\mathbf{R}^N_+)} \|\varphi\|_{L^\infty(\mathbf{R}^N_+)}^{N(p-1)-1} \quad is \ sufficiently \ small,$$

then there exists a positive global-in-time solution of (1.1). This also immediately follows from assertion (ii) of Corollary 1.1 and the comparison principle.

We explain the idea of the proof of Theorem 1.1. Under the assumptions of Theorem 1.1, there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset BUC(\Omega)$ such that

$$\lim_{n\to\infty} \|\varphi - \varphi_n\|_{r,\rho} = 0, \qquad \sup_n \|\varphi_n\|_{r,\rho} \le 2\|\varphi\|_{r,\rho}. \tag{1.13}$$

For any $n = 1, 2, \ldots$, let u_n satisfy in the classical sense

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \times (0, T_n), \\ \nabla u \cdot \nu(x) = |u|^{p-1}u & \text{on } \partial\Omega \times (0, T_n), \\ u(x,0) = \varphi_n(x) & \text{in } \Omega, \end{cases} \tag{1.14}$$

where T_n is the blow-up time of the solution u_n. By regularity theorems for parabolic equations (see e.g. [12] and [29, Chapters III and IV]) we see that

$$u_n \in BUC(\overline{\Omega} \times [0, T]), \qquad \nabla u_n \in L^\infty(\Omega \times (\tau, T)), \tag{1.15}$$

for any $0 < \tau < T < T_n$, which imply that u_n is a $L^r_{uloc}(\Omega)$-solution in $\Omega \times [0, T_n)$ for any $1 \le r < \infty$. Set

$$\Psi_{r,\rho}[u_n](t) := \sup_{0 \le \tau \le t} \sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} |u_n(y,\tau)|^r \, dy, \qquad 0 \le t < T_n.$$

It follows from (1.7) and (1.13) that

$$\Psi_{r,\rho}[u_n](0)^{\frac{1}{r}} = \|\varphi_n\|_{r,\rho} \le 2\|\varphi\|_{r,\rho} \le 2\gamma_1 \rho^{-\frac{1}{p-1}+\frac{N}{r}}. \tag{1.16}$$

Define

$$T_n^* := \sup\left\{\sigma \in (0, T_n) \,:\, \Psi_{r,\rho}[u_n](t) \le 6M\Psi_{r,\rho}[u_n](0) \quad \text{in} \quad [0,\sigma]\right\},$$
$$T_n^{**} := \sup\left\{\sigma \in (0, T_n) \,:\, \rho^{-1} + \|u_n(t)\|_{L^\infty(\Omega)}^{p-1} \le 2t^{-\frac{1}{2}} \quad \text{in} \quad (0,\sigma]\right\}, \tag{1.17}$$

where M is the integer given in Lemma 2.1. We adapt the arguments in [2], [3] and [26] to obtain uniform estimates of u_n and $u_m - u_n$ with respect to $m, n = 1, 2, \ldots$, and prove that

$$\inf_n T_n^* \ge \mu\rho^2, \qquad \inf_n T_n^{**} \ge \mu\rho^2,$$

for some $\mu > 0$. This enables us to prove Theorem 1.1. Theorem 1.2 follows from a similar argument as in Theorem 1.1.

2 Preliminaries

In this section we recall some properties of uniformly local L^r spaces and prove some lemmas related to ρ_*. Furthermore, we give some inequalities used in Section 3. In what follows, the letter C denotes a generic constant independent of $x \in \overline{\Omega}$, n and ρ.

Let $1 \le r < \infty$. We first recall the following properties of $L^r_{uloc,\rho}(\Omega)$:

(i) if $f \in L^r_{uloc,\rho}(\Omega)$ for some $\rho > 0$, then, for any $\rho' > 0$, $f \in L^r_{uloc,\rho'}(\Omega)$ and

$$\|f\|_{r,\rho'} \leq C_1 \|f\|_{r,\rho}$$

for some constant C_1 depending only on N, ρ and ρ';

(ii) there exists a constant C_2 depending only on N such that

$$\|f\|_{r,\rho} \leq C_2 \rho^{N(\frac{1}{r}-\frac{1}{q})} \|f\|_{q,\rho}, \qquad f \in L^q_{uloc,\rho}(\Omega), \tag{2.1}$$

for any $1 \leq r \leq q < \infty$ and $\rho > 0$;

(iii) if $f \in L^r(\Omega)$, then $f \in L^r_{uloc,\rho}(\Omega)$ for any $\rho > 0$ and

$$\lim_{\rho \to +0} \|f\|_{r,\rho} = 0. \tag{2.2}$$

Properties (ii) and (iii) are proved by the Hölder inequality and the absolute continuity of $|f|^r \, dy$ with respect to dy. Property (i) follows from the following lemma.

Lemma 2.1 *Let $N \geq 1$ and Ω be a domain in $\mathbf{R}^N$. Then there exists $M \in \{1, 2, \dots\}$ depending only on N such that, for any $x \in \overline{\Omega}$ and $\rho > 0$,*

$$\Omega(x, 2\rho) \subset \bigcup_{k=1}^{n} \Omega(x_k, \rho) \tag{2.3}$$

for some $\{x_k\}_{k=1}^{n} \subset \overline{\Omega}$ with $n \leq M$.

We state a lemma on the existence of ρ_* satisfying (1.2) and (1.4).

Lemma 2.2 *Let $N \geq 1$ and Ω be a uniformly regular domain of class C^1. Then there exists $\rho_* > 0$ such that (1.2) and (1.4) hold. In particular, if*

$$\Omega = \{(x', x_N) \in \mathbf{R}^N : x_N > \Phi(x')\}, \tag{2.4}$$

where $N \geq 2$ and $\Phi \in C^1(\mathbf{R}^{N-1})$ with $\|\nabla \Phi\|_{L^\infty(\mathbf{R}^{N-1})} < \infty$, then (1.2) and (1.4) hold with $\rho_* = \infty$.

We obtain the following two lemmas by using (1.2) and (1.4).

Lemma 2.3 *Let $N \geq 1$ and $\Omega \subset \mathbf{R}^N$ be a uniformly regular domain of class C^1. Let ρ_* satisfy (1.2) and (1.4). Then there exists a constant C_1 such that*

$$\int_{\partial\Omega(x,\rho)} \phi^2 \, d\sigma \leq \epsilon \int_{\Omega(x,\rho)} |\nabla \phi|^2 \, dy + \frac{C_1}{\epsilon} \int_{\Omega(x,\rho)} \phi^2 \, dy \tag{2.5}$$

for all $\phi \in C_0^1(B(x,\rho))$, $\epsilon > 0$, $x \in \overline{\Omega}$ and $\rho \in (0, \rho_*)$. Furthermore, for any $p > 1$ and $r > 0$, there exists a constant C_2 such that

$$\int_{\Omega(x,\rho)} f^{2p+r-2} \, dy \leq C_2 \left(\int_{\Omega(x,\rho)} f^{N(p-1)} \, dy \right)^{\frac{2}{N}} \int_{\Omega(x,\rho)} |\nabla f^{\frac{r}{2}}|^2 \, dy \tag{2.6}$$

for all nonnegative functions f satisfying $f^{r/2} \in C^1(\Omega(x,\rho))$ with $f = 0$ near $\Omega \cap \partial B(x,\rho)$, $\rho \in (0, \rho_*)$ and $x \in \overline{\Omega}$.

Proof. It follows from (1.4) that

$$\int_{\partial\Omega(x,\rho)} \phi^2 \, d\sigma \le C \int_{\Omega(x,\rho)} |\nabla\phi^2| \, dy \le 2C \int_{\Omega(x,\rho)} |\phi|\,|\nabla\phi| \, dy$$

$$\le \epsilon \int_{\Omega(x,\rho)} |\nabla\phi|^2 \, dy + \frac{C^2}{\epsilon} \int_{\Omega(x,\rho)} \phi^2 \, dy$$

for all $\phi \in W_0^{1,2}(B(x,\rho))$, $\epsilon > 0$, $x \in \overline{\Omega}$ and $\rho \in (0, \rho_*)$. This implies (2.5).

Let $r > 0$ and $0 < \rho < \rho_*$. If $2N(p-1) \ge r$, then, by (1.4) we have

$$\int_{\Omega(x,\rho)} g^{\frac{4}{r}(p-1)+2} \, dy \le C \left(\int_{\Omega(x,\rho)} g^{\frac{2N(p-1)}{r}} \, dy \right)^{\frac{2}{N}} \int_{\Omega(x,\rho)} |\nabla g|^2 \, dy \qquad (2.7)$$

for all $g \in C_0^1(B(x,\rho))$ and $x \in \overline{\Omega}$. Furthermore, we obtain (2.7) by the Hölder inequality and (1.4) even for the case $2N(p-1) < r$ (see e.g. [32, Lemma 3]). Then, setting $g = f^{r/2}$, we obtain (2.6), and the proof is complete. $\square$

Lemma 2.4 *Assume the same conditions as in Theorem 1.1. Let $r \ge 1$, $T > 0$ and f be a nonnegative function such that*

$$f \in C([0,T] : L_{uloc,\rho}^r(\Omega)) \cap L^2(\tau, T : W^{1,2}(\Omega \cap B(0,R)))$$

for any $\rho \in (0, \rho_*/2)$, $\tau \in (0,T)$ and $R > 0$. Let $x \in \overline{\Omega}$ and ζ be a smooth function in $\mathbf{R}^N$ such that

$$0 \le \zeta \le 1 \quad and \quad |\nabla\zeta| \le 2\rho^{-1} \quad in \quad \mathbf{R}^N,$$
$$\zeta = 1 \quad on \quad B(x,\rho), \quad \zeta = 0 \quad outside \quad B(x,2\rho).$$

Set $f_\epsilon = f + \epsilon$ for $\epsilon > 0$. Then, for any sufficiently large $k \ge 2$, there exists a constant C such that

$$\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\partial\Omega(x,2\rho)} f_\epsilon^{p+r-1} \zeta^k \, d\sigma ds$$
$$\le C \left[\rho^{\frac{r}{p-1}-N} \Psi_{r,\rho}[f_\epsilon](t) \right]^{\frac{p-1}{r}} \left[\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2 \, dy ds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[f_\epsilon](t) \right] \qquad (2.8)$$

for all $0 < \tau < t \le T$, $\rho \in (0, \rho_*/2)$ and $\epsilon > 0$.

Proof. Let $\rho \in (0, \rho_*/2)$. It suffices to consider the case where $\partial\Omega(x,\rho) \ne \emptyset$. Let $k \ge 2$ be such that

$$\frac{k}{2p+r-2} \cdot \frac{r}{2} \ge 1. \qquad (2.9)$$

By (1.2) and Lemma 2.1, for any $\delta > 0$, we have

$$\int_\tau^t \int_{\partial\Omega(x,2\rho)} f_\epsilon^{p+r-1}\zeta^k \, d\sigma ds \leq C \int_\tau^t \int_{\Omega(x,2\rho)} \left|\nabla[f_\epsilon^{p+r-1}\zeta^k]\right| \, dyds$$

$$\leq C \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{p+\frac{r}{2}-1}|\nabla f_\epsilon^{\frac{r}{2}}|\zeta^k \, dyds + C \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{p+r-1}|\nabla\zeta|\zeta^{k-1} \, dyds$$

$$\leq C\delta \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{2p+r-2}\zeta^k \, dyds$$

$$+ C\delta^{-1} \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2\zeta^k \, dyds + C\delta^{-1} \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^r \zeta^{k-2}|\nabla\zeta|^2 \, dy \, ds$$

$$\leq C\delta \int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^{2p+r-2}\zeta^k \, dyds$$

$$+ C\delta^{-1} \sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2 \, dyds + C\delta^{-1}\rho^{-2}(t-\tau)\Psi_{r,\rho}[f_\epsilon](t)$$

(2.10)

for $0 < \tau < t \leq T$, where C is a constant independent of ϵ and δ. Set $g_\epsilon := f_\epsilon\zeta^{k/(2p+r-2)}$. It follows from (2.9) that $f_\epsilon^{r/2} = 0$ near $\Omega \cap \partial B(x,2\rho)$. Then, by Lemmas 2.1 and 2.3 we have

$$\int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon(y,\tau)^{2p+r-2}\zeta^k \, dyds = \int_\tau^t \int_{\Omega(x,2\rho)} g_\epsilon(y,\tau)^{2p+r-2} \, dyds$$

$$\leq C \sup_{0<s<t} \left(\int_{\Omega(x,2\rho)} g_\epsilon(y,s)^{N(p-1)} \, dy\right)^{\frac{2}{N}} \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla g_\epsilon^{\frac{r}{2}}|^2 \, dyds$$

$$\leq C \sup_{0<s<t} \left(\rho^{\frac{r}{p-1}-N} \int_{\Omega(x,2\rho)} f_\epsilon(y,s)^r \, dy\right)^{\frac{2(p-1)}{r}}$$

$$\times \left[\int_\tau^t \int_{\Omega(x,2\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}\int_\tau^t \int_{\Omega(x,2\rho)} f_\epsilon^r \, dyds\right]$$

(2.11)

$$\leq C\left[\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[f_\epsilon](t)\right]^{\frac{2(p-1)}{r}}$$

$$\times \left[\sup_{x\in\overline{\Omega}}\int_\tau^t \int_{\Omega(x,\rho)} |\nabla f_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[f_\epsilon](t)\right]$$

for $0 < \tau < t \leq T$. Therefore, taking $\delta = [\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[f_\epsilon](t)]^{-(p-1)/r}$, by (2.10) and (2.11) we obtain (2.8), and the proof is complete. $\square$

3 Proof of Theorems 1.1 and 1.2 in the case $r > 1$.

Let v and w be $L^r_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0,T]$, where $0 < T < \infty$ and r is as in (1.6). Set $z := v - w$ and $z_\epsilon := \max\{z,0\} + \epsilon$ for $\epsilon \geq 0$. Then z_ϵ satisfies

$$\partial_t z_\epsilon \leq \Delta z_\epsilon \quad \text{in} \quad \Omega \times (0,T], \qquad \nabla z_\epsilon \cdot \nu(x) \leq a(x,t)z_\epsilon \quad \text{on} \quad \partial\Omega \times (0,T], \qquad (3.1)$$

in the weak sense (see e.g. [13, Chapter II]). Here

$$a(x,t) := \begin{cases} \dfrac{|v(x,t)|^{p-1}v(x,t) - |w(x,t)|^{p-1}w(x,t)}{v(x,t) - w(x,t)} & \text{if } v(x,t) \neq w(x,t), \\ p|v(x,t)|^{p-1} & \text{if } v(x,t) = w(x,t), \end{cases} \quad (3.2)$$

which satisfies

$$0 \leq a(x,t) \leq C(|v|^{p-1} + |w|^{p-1}) \quad \text{in} \quad \Omega \times (0,T]. \quad (3.3)$$

In this section we give some estimates of z, and prove Theorems 1.1 and 1.2 in the case $r > 1$.

We first give an L^∞_{loc} estimate of z_0 by using the Moser iteration method with the aid of (1.17). For related results, see [17].

Lemma 3.1 *Assume the same conditions as in Theorem 1.1. Let v and w be $L^r_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0,T]$, where $0 < T < \infty$ and $r \geq 1$. Set $z_0 := \max\{v - w, 0\}$ and $a = a(x,t)$ as in (3.2). Then there exists a constant C such that*

$$\|z_0(t)\|_{L^\infty(\Omega(x,R_1)\times(t_1,t))} \leq CD^{\frac{N+2}{2r}} \left(\int_{t_2}^t \int_{\Omega(x,R_2)} z_0^r \, dyds \right)^{1/r}, \quad (3.4)$$

$$\int_{t_1}^t \int_{\Omega(x,R_1)} |\nabla z_0|^2 \, dyds \leq CD \int_{t_2}^t \int_{\Omega(x,R_2)} z_0^2 \, dyds, \quad (3.5)$$

for all $x \in \overline{\Omega}$, $0 < R_1 < R_2 < \rho_*$ and $0 < t_2 < t_1 < t \leq T$, where

$$D := \|a\|_{L^\infty(\partial\Omega(x,R_2)\times(t_2,t))}^2 + (R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}.$$

Proof. Let $x \in \overline{\Omega}$, $0 < R_1 < R_2 < \rho_*$ and $0 < t_2 < t_1 < t \leq T$. For $j = 0, 1, 2, \ldots$, set

$$r_j := R_1 + (R_2 - R_1)2^{-j}, \quad \tau_j := t_1 - (t_1 - t_2)2^{-j}, \quad Q_j := \Omega(x,r_j) \times (\tau_j, t).$$

Let ζ_j be a piecewise smooth function in Q_j such that

$$\begin{aligned} &0 \leq \zeta_j \leq 1 \quad \text{in} \quad \mathbf{R}^N, \quad \zeta_j = 1 \quad \text{on} \quad Q_{j+1}, \\ &\zeta_j = 0 \quad \text{near} \quad \partial\Omega(x,r_j) \times [\tau_j, t] \cup \Omega(x,r_j) \times \{\tau_j\}, \\ &|\nabla\zeta_j| \leq \frac{2^{j+1}}{R_2 - R_1} \quad \text{and} \quad 0 \leq \partial_t\zeta_j \leq \frac{2^{j+1}}{t_1 - t_2} \quad \text{in} \quad Q_j. \end{aligned} \quad (3.6)$$

Let $\alpha_0 > 1$ and $\epsilon > 0$. For any $\alpha \geq \alpha_0$, multiplying (3.1) by $z_\epsilon^{\alpha-1}\zeta_j^2$ and integrating it on Q_j, we obtain

$$\frac{1}{\alpha} \sup_{\tau_j < s < t} \int_{\Omega(x,r_j)} z_\epsilon^\alpha \zeta_j^2 \, dy + \frac{\alpha-1}{2} \iint_{Q_j} z_\epsilon^{\alpha-2} |\nabla z_\epsilon|^2 \zeta_j^2 \, dyds$$

$$\leq \frac{4}{\alpha} \iint_{Q_j} z_\epsilon^\alpha \zeta_j |\partial_t\zeta_j| \, dyds + \frac{4}{\alpha-1} \iint_{Q_j} z_\epsilon^\alpha |\nabla\zeta_j|^2 \, dyds \quad (3.7)$$

$$+ 2 \int_{\tau_j}^t \int_{\partial\Omega(x,r_j)} a(y,s) z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds.$$

This calculation is somewhat formal, however it is justified by the same argument as in [29, Chapter III] (see also [13]). Then it follows that

$$\sup_{\tau_j < s < t} \int_{\Omega(x,r_j)} z_\epsilon^\alpha \zeta_j^2 \, dy + \iint_{Q_j} |\nabla[z_\epsilon^{\frac{\alpha}{2}} \zeta_j]|^2 \, dyds \le C \iint_{Q_j} z_\epsilon^\alpha \zeta_j \partial_t \zeta_j \, dyds$$
$$+ C \iint_{Q_j} z_\epsilon^\alpha |\nabla \zeta_j|^2 \, dyds + C\alpha \int_{\tau_j}^t \int_{\partial\Omega(x,r_j)} a(y,s) z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds \tag{3.8}$$

for all $j = 0, 1, 2, \ldots$ and $\alpha \ge \alpha_0$. On the other hand, by Lemma 2.3 we have

$$C\alpha \int_{\tau_j}^t \int_{\partial\Omega(x,r_j)} a(y,s) z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds \le C\alpha \|a\|_{L^\infty(Q_0)} \int_{\tau_j}^t \int_{\partial\Omega_j} z_\epsilon^\alpha \zeta_j^2 \, d\sigma ds$$
$$\le \frac{1}{2} \iint_{Q_j} |\nabla[z_\epsilon^{\frac{\alpha}{2}} \zeta_j]|^2 \, dyds + C\alpha^2 \|a\|_{L^\infty(Q_0)}^2 \iint_{Q_j} z_\epsilon^\alpha \zeta_j^2 \, dyds. \tag{3.9}$$

We deduce from (3.6), (3.8) and (3.9) that

$$\sup_{\tau_j < s < t} \int_{\Omega(x,r_j)} z_\epsilon^\alpha \zeta_j^2 \, dy + \iint_{Q_j} |\nabla[z_\epsilon^{\frac{\alpha}{2}} \zeta_j]|^2 \, dyds$$
$$\le C \left[\alpha^2 \|a\|_{L^\infty(Q_0)}^2 + \frac{2^{2j}}{(R_2 - R_1)^2} + \frac{2^j}{t_1 - t_2} \right] \iint_{Q_j} z_\epsilon^\alpha \, dyds \tag{3.10}$$

for all $j = 0, 1, 2, \ldots$ and $\alpha \ge \alpha_0$. This together with (1.4) implies that

$$\left(\iint_{Q_{j+1}} z_\epsilon^{\kappa\alpha} \, dyds \right)^{1/\kappa}$$
$$\le C \left[\alpha^2 \|a\|_{L^\infty(Q_0)}^2 + \frac{2^{2j}}{(R_2 - R_1)^2} + \frac{2^j}{t_1 - t_2} \right] \iint_{Q_j} z_\epsilon^\alpha \, dyds \tag{3.11}$$

for all $j = 0, 1, 2, \ldots$ and $\alpha \ge \alpha_0$, where $\kappa := 1 + 2/N$. Furthermore, by (3.10) with $\alpha = 2$ we have (3.5).

We prove (3.4) in the case $r \ge 2$. Setting

$$I_j := \|z_\epsilon\|_{L^{\alpha_j}(Q_j)}, \qquad \alpha_j := r\kappa^j,$$

by (3.11) we have

$$I_{j+1} \le C^{\frac{1}{\alpha_j}} \left[\alpha_j^2 \|a\|_{L^\infty(Q_0)}^2 + \frac{2^{2j}}{(R_2 - R_1)^2} + \frac{2^j}{t_1 - t_2} \right]^{\frac{1}{\alpha_j}} I_j \le C^{\frac{j}{\alpha_j}} (CD)^{\frac{1}{\alpha_j}} I_j \tag{3.12}$$

for all $j = 0, 1, 2, \ldots$, where $D := \|a\|_{L^\infty(Q_0)}^2 + (R_2 - R_1)^{-2} + (t_1 - t_2)^{-1}$. Since

$$\sum_{j=0}^\infty \frac{1}{\alpha_j} = \frac{1}{r} \sum_{j=0}^\infty \kappa^{-j} = \frac{1}{r(1 - \kappa^{-1})} = \frac{N+2}{2r}, \qquad \sum_{j=0}^\infty \frac{j}{\alpha_j} < \infty,$$

we deduce from (3.12) that

$$\|z_\epsilon\|_{L^\infty(Q_\infty)} = \lim_{j\to\infty} I_j \le C^{\sum_{j=0}^\infty \frac{j}{\alpha_j}} (CD)^{\sum_{j=0}^\infty \frac{1}{\alpha_j}} I_0 \le CD^{(N+2)/2r}\|z_\epsilon\|_{L^r(Q_0)},$$

which implies

$$\|z_\epsilon\|_{L^\infty(\Omega(x,R_1)\times(t_1,t))} \le CD^{\frac{N+2}{2r}}\left(\int_{t_2}^t\int_{\Omega(x,R_2)} z_\epsilon^r\, dy ds\right)^{1/r}, \tag{3.13}$$

where $r \ge 2$. Then, passing the limit as $\epsilon \to 0$, we obtain (3.4).

On the other hand, for the case $1 \le r < 2$, applying (3.13) with $r = 2$ to the cylinders Q_j and Q_{j+1}, we have

$$\|z_\epsilon\|_{L^\infty(Q_{j+1})} \le C\left((2^{2j}D)^{\frac{N+2}{2}}\iint_{Q_j} z_\epsilon^2\, dy ds\right)^{\frac{1}{2}}$$

$$\le Cb^j\|z_\epsilon\|_{L^\infty(Q_j)}^{1-r/2}\left(D^{(N+2)/2}\iint_{Q_j} z_\epsilon^r\, dy ds\right)^{\frac{1}{2}},$$

where $b = 2^{(N+2)/2}$. Then, for any $\nu > 0$, we have

$$\|z_\epsilon\|_{L^\infty(Q_{j+1})} \le \nu\|z_\epsilon\|_{L^\infty(Q_j)} + C\nu^{-\frac{2-r}{r}}b^{\frac{2}{r}j}D^{\frac{N+2}{2r}}\left(\iint_{Q_j} z_\epsilon^r\, dy ds\right)^{1/r}$$

$$\le \nu^{j+1}\|z_\epsilon\|_{L^\infty(Q_0)} + C\nu^{-\frac{2-r}{r}}\sum_{i=0}^j (\nu b^{\frac{2}{r}})^i D^{\frac{N+2}{2r}}\left(\iint_{Q_0} z_\epsilon^r\, dy ds\right)^{1/r}$$

for $j = 1, 2, \ldots$. Taking a sufficiently small ν if necessary, we see that

$$\|z_\epsilon\|_{L^\infty(Q_{j+1})} \le \nu^{j+1}\|z_\epsilon\|_{L^\infty(Q_0)} + CD^{\frac{N+2}{2r}}\left(\iint_{Q_0} z_\epsilon^r\, dy ds\right)^{1/r}$$

for $j = 1, 2, \ldots$. Passing to the limit as $j \to \infty$ and $\epsilon \to 0$, we obtain

$$\|z_0\|_{L^\infty(Q_\infty)} \le CD^{\frac{N+2}{2r}}\left(\iint_{Q_0} z_0^r\, dy ds\right)^{1/r},$$

which implies (3.4) in the case $1 \le r < 2$. Thus Lemma 3.1 follows. $\square$

Lemma 3.2 *Assume the same conditions as in Theorem 1.1. Let r satisfy (1.6) and $r > 1$. Let v be a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, T]$, where $T > 0$. Then there exists a positive constant Λ such that, if*

$$\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[v](T) \le \Lambda \tag{3.14}$$

for some $\rho \in (0, \rho_*/2)$, then

$$\Psi_{r,\rho}[v](t) \le 5M\Psi_{r,\rho}[v](\tau), \tag{3.15}$$

$$\sup_{x\in\overline\Omega}\int_\tau^t\int_{\partial\Omega(x,\rho)} |v|^{p+r-1}\, d\sigma ds \le C\Lambda^{\frac{p-1}{r}}\Psi_{r,\rho}[v](\tau), \tag{3.16}$$

for all $0 \leq \tau \leq t \leq T$ with $t - \tau \leq \mu\rho^2$, where C and μ are positive constants depending only on N, Ω, p and r.

Proof. Let $x \in \overline{\Omega}$ and let ζ and k be as in Lemma 2.4. By (3.14) we can take a sufficiently small $\epsilon > 0$ so that

$$\rho^{\frac{r}{p-1}-N} \Psi_{r,\rho}[v_\epsilon](T) \leq 2\Lambda, \tag{3.17}$$

where $v_\epsilon := \max\{\pm v, 0\} + \epsilon$. Similarly to (3.8), for any $0 < \tau < t \leq T$, multiplying (1.1) by $v_\epsilon^{r-1}\zeta^k$ and integrating it in $\Omega \times (\tau, t)$, we obtain

$$\int_{\Omega(x,2\rho)} v_\epsilon(y,s)^r \zeta^k \, dy \Big|_{s=\tau}^{s=t} + \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds$$
$$\leq C\rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} v_\epsilon^r \, dyds + C \int_\tau^t \int_{\partial\Omega(x,2\rho)} v_\epsilon^{p+r-1}\zeta^k \, d\sigma ds. \tag{3.18}$$

This together with $v \in C(\overline{\Omega} \times [\tau, T]) \cap L^\infty(\tau, T : L^\infty(\Omega))$ (see Definition 1.1) implies that

$$\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds < \infty. \tag{3.19}$$

Furthermore, by Lemma 2.4, (3.17) and (3.18) we have

$$\int_{\Omega(x,2\rho)} v_\epsilon(y,s)^r \zeta^k \, dy \Big|_{s=\tau}^{s=t} + \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds \leq C\rho^{-2} \int_\tau^t \int_{\Omega(x,2\rho)} v_\epsilon^r \, dyds$$
$$+ C(2\Lambda)^{\frac{p-1}{r}} \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \right] \tag{3.20}$$

for $0 < \tau < t \leq T$. Therefore, by Lemma 2.1, (1.17) and (3.20) we obtain

$$\sup_{x \in \overline{\Omega}} \int_{\Omega(x,2\rho)} v_\epsilon(y,t)^r \, dy + \sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds$$
$$\leq M \sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} v_\epsilon(y,\tau)^r \, dy + C\rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \tag{3.21}$$
$$+ C(2\Lambda)^{\frac{p-1}{r}} \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \right]$$

for $0 < \tau < t \leq T$. Taking a sufficiently small Λ if necessary, we deduce from (3.19) and (3.21) that

$$\sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} v_\epsilon(y,t)^r \, dy + \frac{1}{2} \sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dyds$$
$$\leq M \sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} v_\epsilon(y,\tau)^r \, dy + C\rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t).$$

Taking a sufficiently small $\mu \in (0,1]$, we obtain

$$
\Psi_{r,\rho}[v_\epsilon](t) + \frac{1}{2}\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla v_\epsilon^{\frac{r}{2}}|^2 \, dy ds
$$

$$
\leq 2M\Psi_{r,\rho}[v_\epsilon](\tau) + C\rho^{-2}(t-\tau)\Psi_{r,\rho}[v_\epsilon](t) \leq 2M\Psi_{r,\rho}[v_\epsilon](\tau) + \frac{1}{2}\Psi_{r,\rho}[v_\epsilon](t) \tag{3.22}
$$

for $0 < \tau < t \leq T$ with $t - \tau \leq \mu\rho^2$. This implies that

$$
\Psi_{r,\rho}[\max\{\pm v, 0\}](t) \leq \Psi_{r,\rho}[v_\epsilon](t) \leq 4M\Psi_{r,\rho}[v_\epsilon](\tau) \leq 5M\Psi_{r,\rho}[v](\tau) + C\epsilon^r\rho^N \tag{3.23}
$$

for $0 < \tau < t \leq T$ with $t - \tau \leq \mu\rho^2$. Furthermore, by Lemma 2.4, (3.22) and (3.23) we have

$$
\int_\tau^t \int_{\partial\Omega(x,\rho)} \max\{\pm v, 0\}^{p+r-1} \, d\sigma ds \leq \int_\tau^t \int_{\partial\Omega(x,\rho)} v_\epsilon^{p+r-1} \, d\sigma ds \tag{3.24}
$$

$$
\leq C\Lambda^{\frac{p-1}{r}}\Psi_{r,\rho}[v_\epsilon](\tau) \leq C\Lambda^{\frac{p-1}{r}}\Psi_{r,\rho}[v](\tau) + C\epsilon^r\rho^N.
$$

Since τ and ϵ is arbitrary, by (3.23) and (3.24) we obtain (3.15) and (3.16). Thus Lemma 3.2 follows. $\square$

Lemma 3.3 *Assume the same conditions as in Lemma* 3.1. *Let r satisfy* (1.6) *and $r > 1$. Then there exists a positive constant Λ such that, if*

$$
\rho^{\frac{r}{p-1}-N}\left(\Psi_{r,\rho}[v](T) + \Psi_{r,\rho}[w](T)\right) \leq \Lambda \tag{3.25}
$$

for some $\rho \in (0, \rho_*/2)$, then

$$
\Psi_{r,\rho}[z_0](t) \leq C\Psi_{r,\rho}[z_0](\tau) \tag{3.26}
$$

for $0 \leq \tau < t \leq T$ with $t - \tau \leq \mu\rho^2$, where C and μ are positive constants depending only on N, Ω, p and r.

Proof. Let $x \in \overline{\Omega}$ and ζ be as in Lemma 2.4. Let k be as in Lemma 2.4 and $\epsilon > 0$. Similarly to (3.18), we have

$$
\int_{\Omega(x,2\rho)} z_\epsilon(y,s)^r \zeta^k \, dy \Big|_{s=\tau}^{s=t} + \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla z_\epsilon^{\frac{r}{2}}|^2 \zeta^k \, dy ds
$$

$$
\leq C\rho^{-2}\int_\tau^t \int_{\Omega(x,2\rho)} z_\epsilon^r \, dy ds + C\int_\tau^t \int_{\partial\Omega(x,2\rho)} a(y,s)z_\epsilon^r\zeta^k \, d\sigma ds \tag{3.27}
$$

for all $0 < \tau < t \leq T$. This together with z_ϵ, $a \in C(\overline{\Omega} \times [\tau,T]) \cap L^\infty(\Omega \times (\tau,T))$ implies that

$$
\sup_{x\in\overline{\Omega}} \int_\tau^t \int_{\Omega(x,2\rho)} |\nabla z_\epsilon^{\frac{r}{2}}|^2 \, dy ds < \infty \tag{3.28}
$$

for $0 < \tau < t \le T$. On the other hand, by the Hölder inequality and (3.3) we have

$$\int_\tau^t \int_{\partial\Omega(x,2\rho)} a(y,\tau) z_\epsilon^r \zeta^k \, d\sigma ds \le C \left(\int_\tau^t \int_{\partial\Omega(x,2\rho)} (|v|^{p+r-1} + |w|^{p+r-1}) \, d\sigma ds \right)^{\frac{p-1}{p+r-1}}$$

$$\times \left(\int_\tau^t \int_{\partial\Omega(x,2\rho)} z_\epsilon^{p+r-1} \zeta^k \, d\sigma ds \right)^{\frac{r}{p+r-1}} . \tag{3.29}$$

Let Λ and μ be sufficiently small positive constants. Then, by Lemma 2.1, (3.16) and (3.25) we see that

$$\int_\tau^t \int_{\partial\Omega(x,2\rho)} (|v|^{p+r-1} + |w|^{p+r-1}) \, d\sigma ds$$

$$\le M \sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\partial\Omega(x,\rho)} (|v|^{p+r-1} + |w|^{p+r-1}) \, d\sigma ds \tag{3.30}$$

$$\le C\Lambda^{\frac{p-1}{r}} \{\Psi_{r,\rho}[v](\tau) + \Psi_{r,\rho}[w](\tau)\} \le C\Lambda^{\frac{p+r-1}{r}} \rho^{-\frac{r}{p-1}+N}$$

for all $0 < \tau < t \le T$ with $t - \tau \le \mu\rho^2$. Similarly, by Lemma 2.4 we obtain

$$\int_\tau^t \int_{\partial\Omega(x,2\rho)} z_\epsilon^{p+r-1} \zeta^k \, d\sigma ds \le C \left(\rho^{\frac{r}{p-1}-N} \Psi_{r,\rho}[z_\epsilon](t) \right)^{\frac{p-1}{r}}$$

$$\times \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla(z_\epsilon)^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](\tau) \right] \tag{3.31}$$

for all $0 < \tau < t \le T$ with $t - \tau \le \mu\rho^2$. Then we deduce from (3.29)–(3.31) that

$$\int_\tau^t \int_{\partial\Omega(x,2\rho)} a(y,t) z_\epsilon^r \zeta^k \, d\sigma ds$$

$$\le C\Lambda^{\frac{p-1}{r}} (\Psi_{r,\rho}[z_\epsilon](t))^{\frac{p-1}{p+r-1}}$$

$$\times \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla(z_\epsilon)^{\frac{r}{2}}|^2 \, dyds + \rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](t) \right]^{\frac{r}{p+r-1}} \tag{3.32}$$

$$\le C\Lambda^{\frac{p-1}{r}} \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla z_\epsilon^{\frac{r}{2}}|^2 \, dyds + \Psi_{r,\rho}[z_\epsilon](t) + \rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](\tau) \right]$$

for all $0 < \tau < t \le T$ with $t - \tau \le \mu\rho^2$. Therefore, by Lemma 2.1, (3.27) and (3.32) we have

$$\sup_{x \in \overline{\Omega}} \int_{\Omega(x,\rho)} z_\epsilon^r \, dy + \sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla z_\epsilon^{\frac{r}{2}}|^2 \, dyds$$

$$\le M\Psi_{r,\rho}[z_\epsilon](\tau) + C\rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](t)$$

$$+ C\Lambda^{\frac{p-1}{r}} \left[\sup_{x \in \overline{\Omega}} \int_\tau^t \int_{\Omega(x,\rho)} |\nabla z_\epsilon^{\frac{r}{2}}|^2 \, dyds + \Psi_{r,\rho}[z_\epsilon](t) + \rho^{-2}(t-\tau)\Psi_{r,\rho}[z_\epsilon](\tau) \right]$$

for all $0 < \tau < t \leq T$ with $t - \tau \leq \mu\rho^2$. Then, taking sufficiently small constants Λ and μ if necessary, we obtain

$$\Psi_{r,\rho}[z_\epsilon](t) \leq 4M\Psi_{r,\rho}[z_\epsilon](\tau)$$

for all $0 < \tau < t \leq T$ with $t - \tau \leq \mu\rho^2$. This implies (3.26), and the proof is complete. □

Now we are ready to complete the proof of Theorems 1.1 and 1.2 in the case $r > 1$.

Proof of Theorem 1.1 in the case $r > 1$. Let γ_1 be a sufficiently small positive constant and assume (1.7). Let $\{\varphi_n\}$ satisfy (1.13) and define T_n^* and T_n^{**} as in (1.17). Then it follows from (1.16) that

$$\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[u_n](t) \leq 6M\rho^{\frac{r}{p-1}-N}\Psi_{r,\rho}[u_n](0) \leq 6M(2\gamma_1)^r \qquad (3.33)$$

for all $0 \leq t \leq T_n^*$. Taking a sufficiently small γ_1 if necessary, by Lemma 3.2, (1.16) and (3.33), we can find a constant $\mu > 0$ such that

$$\Psi_{r,\rho}[u_n](t) \leq 5M\Psi_{r,\rho}[u_n](0) < 6M\Psi_{r,\rho}[u_n](0) \leq C\|\varphi\|_{r,\rho}^r \qquad (3.34)$$

for $0 \leq t \leq \min\{T_n^*, \mu\rho^2\}$. On the other hand, we apply Lemma 3.1 with $R_1 = \rho/2$, $R_2 = \rho$, $t_1 = t/2$ and $t_2 = t/4$ to obtain

$$\|u_n(t)\|_{L^\infty(\Omega(x,\rho/2))} \leq CD^{\frac{N+2}{2r}}\left(\int_{t/4}^t \int_{\Omega(x,\rho)} |u_n|^r \, dyds\right)^{1/r}, \qquad (3.35)$$

$$\int_{t/2}^t \int_{\Omega(x,\rho/2)} |\nabla u_n|^2 \, dyds \leq CD \int_{t/4}^t \int_{\Omega(x,\rho)} |u_n|^2 \, dyds, \qquad (3.36)$$

for all $x \in \overline{\Omega}$ and $t \in (0, T_n)$. where $D = \||u_n|^{p-1}\|_{L^\infty(\Omega(x,\rho)\times(t/4,t))}^2 + \rho^{-2} + t^{-1}$. By (1.17), (3.34) and (3.35) we have

$$\|u_n(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2r}}\|\varphi\|_{r,\rho} \leq C\gamma_1 t^{-\frac{1}{2(p-1)}}(\rho^{-2}t)^{-\frac{N}{2r}+\frac{1}{2(p-1)}}, \qquad (3.37)$$

$$\sup_{x\in\overline{\Omega}}\int_{t/2}^t \int_{\Omega(x,\rho)} |\nabla u_n|^2 \, dyds \leq C\rho^N\|u_n\|_{L^\infty(\Omega\times(t/4,t))}^2 \leq C\rho^N t^{-\frac{N}{r}}\|\varphi\|_{r,\rho}^2, \qquad (3.38)$$

for all $0 < t \leq \min\{\mu\rho^2, T_n^*, T_n^{**}\}$. Since $r \geq N(p-1)$, taking sufficiently small $\gamma_1 > 0$ and $\mu > 0$ if necessary, by (3.37) we have

$$(\rho^{-2}t)^{\frac{1}{2}} + t^{\frac{1}{2}}\|u_n(t)\|_{L^\infty(\Omega)}^{p-1} \leq \mu^{\frac{1}{2}} + (C\gamma_1)^{p-1}\mu^{-\frac{N(p-1)}{2r}+\frac{1}{2}} \leq 1$$

for $0 < t \leq \min\{\mu\rho^2, T_n^*, T_n^{**}\}$. This implies that $T_n > T_n^{**} > \min\{T_n^*, \mu\rho^2\}$ for $n = 1, 2, \ldots$. Then, by (3.34) we see that $T_n^* > \mu\rho^2$ for $n = 1, 2, \ldots$. Therefore, by (3.34), (3.37) and (3.38) we obtain

$$\|u_n(t)\|_{L^\infty(\Omega)} \leq Ct^{-\frac{N}{2r}}\|\varphi\|_{r,\rho}, \qquad (3.39)$$

$$\sup_{x\in\overline{\Omega}}\int_{t/2}^t \int_{\Omega(x,\rho)} |\nabla u_n|^2 \, dyds \leq C\rho^N t^{-\frac{N}{r}}\|\varphi\|_{r,\rho}^2, \qquad (3.40)$$

$$\sup_{0<t<\mu\rho^2} \|u_n(t)\|_{r,\rho} \leq C\|\varphi\|_{r,\rho}, \qquad (3.41)$$

for $0 < t \leq \mu\rho^2$ and $n = 1, 2, \ldots$.

Applying [12, Theorem 6.2] with the aid of (3.39), we see that u_n $(n = 1, 2, \ldots)$ are uniformly bounded and equicontinuous on $K \times [\tau, \mu\rho^2]$ for any compact set $K \subset \overline{\Omega}$ and $\tau \in (0, \mu\rho^2]$. Then, by the Ascoli-Arzelà theorem and the diagonal argument we can find a subsequence $\{u_{n'}\}$ and a continuous function u in $\Omega \times (0, \mu\rho^2]$ such that

$$\lim_{n' \to \infty} \|u_{n'} - u\|_{L^\infty(K \times [\tau, \mu\rho^2])} = 0$$

for any compact set $K \subset \overline{\Omega}$ and $\tau \in (0, \mu\rho^2]$. This together with (3.39) and (3.41) implies (1.8) and (1.9). Furthermore, by (3.40), taking a subsequence if necessary, we see that

$$\lim_{n' \to \infty} u_{n'} = u \quad \text{weakly in } L^2([\tau, \mu\rho^2] : W^{1,2}(\Omega \cap B(0, R)))$$

for any $R > 0$ and $0 < \tau < \mu\rho^2$. This implies that u satisfies (1.5).

On the other hand, since u_n is a $L^r_{uloc}(\Omega)$-solution of (1.1) (see (1.15)), we see that

$$u_n \in C([0, \mu\rho^2] : L^r_{uloc,\rho}(\Omega)).$$

Furthermore, by Lemma 3.3 and (3.33), taking a sufficiently small γ_1 if necessary, we have

$$\sup_{0 < \tau < \mu\rho^2} \|u_m(\tau) - u_n(\tau)\|_{r,\rho} \leq C\|u_m(0) - u_n(0)\|_{r,\rho}, \quad m, n = 1, 2, \ldots.$$

This means that $\{u_n\}$ is a Cauchy sequence in $C([0, \mu\rho^2] : L^r_{uloc,\rho}(\Omega))$, which implies

$$u \in C([0, \mu\rho^2] : L^r_{uloc,\rho}(\Omega)). \tag{3.42}$$

Therefore we see that u is a $L^r_{uloc}(\Omega)$-solution of (1.1) in $\Omega \times [0, \mu\rho^2]$ satisfying (1.8) and (1.9), and the proof of Theorem 1.1 for the case $r > 1$ is complete. $\square$

Proof of Theorem 1.2 in the case $r > 1$. Let v and w be $L^r_{uloc}(\Omega)$-solutions of (1.1) in $\Omega \times [0, T)$, where $T > 0$. Let γ_2 be a sufficiently small constant and assume (1.11). We can assume, without loss of generality, that $\rho \in (0, \rho_*/2)$. Since $v, w \in C([0, T] : L^r_{uloc,\rho}(\Omega))$, we can find a constant $T' \in (0, T)$ such that

$$\rho^{\frac{1}{p-1} - \frac{N}{r}} \left[\sup_{0 < \tau \leq T'} \|v(\tau)\|_{r,\rho} + \sup_{0 < \tau \leq T'} \|w(\tau)\|_{r,\rho} \right] \leq 2\gamma_2. \tag{3.43}$$

Furthermore, for any $T'' \in (T', T)$, since $v, w \in L^\infty(\Omega \times (T', T''))$, we see that

$$\tilde{\rho}^{\frac{1}{p-1} - \frac{N}{r}} \left[\sup_{T' < \tau \leq T''} \|v(\tau)\|_{r,\tilde{\rho}} + \sup_{T' < \tau \leq T''} \|w(\tau)\|_{r,\tilde{\rho}} \right] \leq \gamma_2 \tag{3.44}$$

for some $\tilde{\rho} \in (0, \rho)$. Since $v(x, 0) \leq w(x, 0)$ for almost all $x \in \Omega$, by (3.43) and (3.44) we apply Lemma 3.3 to obtain

$$\sup_{0 < \tau < \min\{\mu\tilde{\rho}^2, T''\}} \|(v(\tau) - w(\tau))_+\|_{r,\tilde{\rho}} \leq C\|(v(0) - w(0))_+\|_{r,\tilde{\rho}} = 0$$

for some constant $\mu > 0$. This implies that $v(x, t) \leq w(x, t)$ in $\Omega \times (0, \min\{\mu\tilde{\rho}^2, T''\}]$. Repeating this argument, we see that $v(x, t) \leq w(x, t)$ in $\Omega \times (0, T'']$. Finally, since T'' is arbitrary, we see that $v(x, t) \leq w(x, t)$ in $\Omega \times (0, T)$, and the proof is complete. $\square$

References

[1] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, **65**, Academic Press, 1975.

[2] D. Andreucci, New results on the Cauchy problem for parabolic systems and equations with strongly nonlinear sources, Manuscripta Math. **77** (1992), 127–159.

[3] D. Andreucci and E. DiBenedetto, On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **18** (1991), 363–441.

[4] J. M. Arrieta, A. N. Carvalho and A. Rodríguez-Bernal, Parabolic problems with nonlinear boundary conditions and critical nonlinearities, J. Differential Equations **156** (1999), 376–406.

[5] J. M. Arrieta, A. N. Carvalho and A. Rodríguez-Bernal, Attractors of parabolic problems with nonlinear boundary conditions. Uniform bounds, Comm. Partial Differential Equations **25** (2000), 1–37.

[6] J. M. Arrieta and A. Rodríguez-Bernal, Non well posedness of parabolic equations with supercritical nonlinearities, Commun. Contemp. Math. **6** (2004), 733–764.

[7] J. M. Arrieta, A. Rodríguez-Bernal, J. W. Cholewa and T. Dlotko, Linear parabolic equations in locally uniform spaces, Math. Models Methods Appl. Sci. **14** (2004), 253–293.

[8] M. Chlebík and M. Fila, From critical exponents to blow-up rates for parabolic problems, Rend. Mat. Appl. **19** (1999), 449–470.

[9] M. Chlebík and M. Fila, On the blow-up rate for the heat equation with a nonlinear boundary condition, Math. Methods Appl. Sci. **23** (2000), 1323–1330.

[10] M. Chlebík and M. Fila, Some recent results on blow-up on the boundary for the heat equation, in: Evolution Equations: Existence, Regularity and Singularities, Banach Center Publ., **52**, Polish Acad. Sci., Warsaw, (2000), 61–71.

[11] K. Deng, M. Fila, and H. A. Levine, On critical exponents for a system of heat equations coupled in the boundary conditions, Acta Math. Univ. Comenian **63** (1994), 169–192.

[12] E. DiBenedetto, Continuity of weak solutions to a general porous medium equation, Indiana Univ. Math. J. **32** (1983), 83–118.

[13] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[14] J. Fernández Bonder and J. D. Rossi, Life span for solutions of the heat equation with a nonlinear boundary condition, Tsukuba J. Math. **25** (2001), 215–220.

[15] M. Fila, Boundedness of global solutions for the heat equation with nonlinear boundary conditions, Comm. Math. Univ. Carol. **30** (1989), 479–484.

[16] M. Fila and P. Quittner, The blow-up rate for the heat equation with a nonlinear boundary condition, Math. Methods Appl. Sci. **14** (1991), 197–205.

[17] J. Filo and J. Kačur, Local existence of general nonlinear parabolic systems. Nonlinear Anal. **24** (1995), 1597–1618.

[18] V. A. Galaktionov and H. A. Levine, On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary, Israel J. Math. **94** (1996), 125–146.

[19] M.-H. Giga, Y. Giga, and J. Saal, *Nonlinear Partial Differential Equations, Asymptotic Behavior of Solutions and Self-Similar Solutions*, Progr. Nonlinear Differential Equations Appl., **79**, Birkhäuser Boston, Inc., Boston, MA, 2010.

[20] J.-S. Guo and B. Hu, Blowup rate for heat equation in Lipschitz domains with nonlinear heat source terms on the boundary, J. Math. Anal. Appl. **269** (2002), 28–49.

[21] J. Harada, Single point blow-up solutions to the heat equation with nonlinear boundary conditions, Differ. Equ. Appl. **5** (2013), 271–295.

[22] B. Hu, Nonexistence of a positive solution of the Laplace equation with a nonlinear boundary condition, Differential Integral Equations **7** (1994), 301–313.

[23] B. Hu, Nondegeneracy and single-point-blowup for solution of the heat equation with a nonlinear boundary condition, J. Math. Sci. Univ. Tokyo **1** (1994), 251–276.

[24] B. Hu, Remarks on the blowup estimate for solution of the heat equation with a nonlinear boundary condition, Differential Integral Equations **9** (1996), 891–901.

[25] B. Hu and H.-M. Yin, The profile near blowup time for solution of the heat equation with a nonlinear boundary condition, Trans. Amer. Math. Soc. **346** (1994), 117–135.

[26] K. Ishige, On the existence of solutions of the Cauchy problem for a doubly nonlinear parabolic equation, SIAM J. Math. Anal. **27** (1996), 1235–1260.

[27] K. Ishige and T. Kawakami, Global solutions of the heat equation with a nonlinear boundary condition, Calc. Var. Partial Differential Equations **39** (2010) 429–457.

[28] T. Kawakami, Global existence of solutions for the heat equation with a nonlinear boundary condition, J. Math. Anal. Appl. **368** (2010), 320–329.

[29] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, American Mathematical Society Translations, vol. 23, American Mathematical Society, Providence, RI, 1968.

[30] T.-Y. Lee and W.-M. Ni, Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem, Trans. Amer. Math. Soc. **333** (1992), 365–378.

[31] Y. Maekawa and Y. Terasawa, The Navier-Stokes equations with initial data in uniformly local L^p spaces, Differential Integral Equations **19** (2006), 369–400.

[32] M. Nakao, Global solutions for some nonlinear parabolic equations with nonmonotonic perturbations, Nonlinear Anal. **10** (1986), 299–314.

[33] P. Quittner, Liouville theorems for scaling invariant superlinear parabolic problems with gradient structure, preprint.

[34] P. Quittner and P. Souplet, *Superlinear Parabolic Problems, Blow-up, Global Existence and Steady States*, Birkhäuser Advanced Texts: Basler Lehrbücher Birkhäuser Verlag, Basel, 2007.

[35] P. Quittner and P. Souplet, Blow-up rate of solutions of parabolic problems with nonlinear boundary conditions, Discrete Contin. Dyn. Syst. Ser. S **5** (2012), 671–681.