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Global existence and blow-up of solutions for a nonlinear heat equation with exponential nonlinearity

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1 Introduction

We are concerned with a nonlinear heat equation;

$$\begin{cases} \partial_t u = \Delta u + f(u), & x \in \mathbf{R}^N, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $N \geq 1$ and u_0 is a continuous initial function on $\mathbf{R}^N$. The source term $f(u)$ denotes the nonlinear effect of problem (1.1) and typical examples of f which we deal with in this paper are

$$f(u) = u^p \ \ (p > 1), \qquad f(u) = e^u.$$

In the case of $f(u) = u^p$, we always assume u_0 to be a nonnegative function. Throughout this paper, we assume that there exist constants $\epsilon \in (0,2)$ and $C > 0$ such that

$$-C\exp(|x|^{2-\epsilon}) \leq u_0(x) \leq C \quad \text{in } \mathbf{R}^N. \tag{1.2}$$

We denote by $T(u_0)$ the maximal existence time of the unique classical solution for problem (1.1). Under the assumption (1.2), if $T(u_0) < \infty$, the solution u satisfies

$$\limsup_{t \nearrow T(u_0)} \sup_{x \in \mathbf{R}^N} u(x,t) = \infty.$$

Then we say that the solution blows up in finite time and call $T(u_0)$ the blow-up time of the solution. In this paper, we focus on the case where $f(u) = e^u$ and u_0 decays $-\infty$ at space infinity and consider the global in time solutions and blowing up solutions for problem (1.1). In particular, we study the optimal decay rate of u_0 for the solution to blow-up in finite time.

Case $f(u) = u^p$ with $p > 1$

We first introduce some known results concerning the global existence and blow-up of solutions for problem (1.1) with exponential nonlinearity. It is well known that, if $p \leq p_F := 1 + 2/N$, then problem (1.1) cannot possess a nontrivial nonnegative global in time solutions. In other words, if $u_0 \geq 0$ and $u_0 \not\equiv 0$, the solution must blow-up in finite time. On the other hand, if $p > p_F$, then there exist global in time solutions for problem (1.1). The exponent p_F is called the Fujita exponent. See [1]. Furthermore, Lee and Ni in [5] assumed $p > p_F$ and showed the following results:

- If $u_0(x)$ is of the form $\lambda\varphi(x)$ and $\lambda > 0$ is sufficiently small, where λ is a positive parameter and φ is a nonnegative function satisfying
$$\limsup_{|x|\to\infty} |x|^{\frac{2}{p-1}}\varphi(x) < \infty,$$
then $T(u_0) = \infty$.
- If
$$\liminf_{|x|\to\infty} |x|^{\frac{2}{p-1}}u_0(x)$$
is large enough, then $T(u_0) < \infty$.

From above results, the existence of global solutions for (1.1) depends on decay rate of u_0 at space infinity. This decay rate $|x|^{-2/(p-1)}$ is related to the one for stationary solutions for problem (1.1) and its stability properties. See [3]. The latter assertion of above results was improved by Wang in [10] for the case $N \geq 11$ and $p \geq p_{JL}$, where p_{JL} is the exponent which can be defined for $N \geq 11$ and is defined by
$$p_{JL} := \frac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-2}}, \qquad N \geq 11.$$
Indeed, he showed that, if $N \geq 11$ and $p \geq p_{JL}$, then $T(u_0) < \infty$ as long as
$$\liminf_{|x|\to\infty} |x|^{\frac{2}{p-1}}u_0(x) > L := \left(\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right)^{2/(p-1)}.$$
Remark 1.1 (i) *If $N \geq 11$ and $p \geq p_{JL}$, then there exists an initial function u_0 satisfying $T(u_0) = \infty$ and*
$$\lim_{|x|\to\infty} |x|^{\frac{2}{p-1}}u_0(x) = L.$$
In particular, the constant L in the result of [10] *gives optimal decay rate of u_0 for global existence and blow-up of solutions of* (1.1).
(ii) *The function $L|x|^{-2/(p-1)}$ is a singular stationary solution for problem* (1.1) *with $f(u) = u^p$.*

Recently, Naito in [7] improved Wang's result and proved that, if $p > p_F$, then $T(u_0) < \infty$ as long as
$$\liminf_{|x|\to\infty} |x|^{\frac{2}{p-1}}u_0(x) > l^*.$$
Here l^* is the constant related to the existence of forward self-similar solutions of (1.1) with $f(u) = u^p$. If solutions u have the form $u(x,t) = t^{-1/(p-1)}v(x/\sqrt{t})$ for some function v on $\mathbf{R}^N$, then such solutions are called forward self-similar solutions of (1.1) with $f(u) = u^p$. If v is radially symmetric, that is, $v = v(r)$ with $r = |x|$, the function v has to satisfy
$$v'' + \frac{N-1}{r}v' + \frac{r}{2}v' + \frac{1}{p-1}v + v^p = 0 \quad \text{in } (0,\infty). \tag{1.3}$$
As for the existence of forward self-similar solutions, Naito in [6] showed that there exists a constant $l^* > 0$ satisfying the following properties:

- If $0 < l < l^*$, then there exists a bounded solution of (1.3) with

$$\lim_{r\to\infty} r^{\frac{2}{p-1}} v(r) = l. \tag{1.4}$$

- If $l > l^*$, then there cannot exist any bounded solution of (1.3) with (1.4).

Since l^* corresponds with L if $N \geq 11$ and $p \geq p_{JL}$, the results obtained in [7] is an improvement of [10].

Case $f(u) = e^u$

Next we turn our attention to the case $f(u) = e^u$. In this case, Tello in [8] studied the stability and instability of stationary solutions for (1.1). It is shown that, for any $\alpha \in \mathbf{R}$, there exists the solution u_α of the ordinary differential equation;

$$u'' + \frac{N-1}{r}u' + e^u = 0 \quad \text{in } (0,\infty), \qquad u(0) = \alpha, \qquad u'(0) = 0.$$

Here we remark that $u_\alpha = u_\alpha(|x|)$ denotes a radially symmetric stationary solution for problem (1.1) with $f(u) = e^u$. Furthermore, if $N \geq 3$, every u_α satisfies

$$\lim_{|x|\to\infty} (2\log|x| + u_\alpha(|x|)) = \log(2N-4).$$

The function $\log(2N-4) - 2\log|x|$ denotes a singular stationary solution for problem (1.1) with $f(u) = e^u$. Therefore, as in the power type nonlinearity case, we expect that the constant $\log(2N-4)$ is related to the global existence and blow-up of solutions. The main purpose of this paper is to obtain the optimal decay rate of u_0 to classify the global existence and blow-up of solutions for problem (1.1) with $f(u) = e^u$. In particular, we discuss the relationship between the optimal decay rate of u_0 and forward self-similar solutions.

2 Main results

In this section we introduce our main results. In the rest of this paper, we focus on the case $f(u) = e^u$, that is,

$$\begin{cases} \partial_t u = \Delta u + e^u, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{2.1}$$

where $N \geq 3$ and $u_0 \in C(\mathbf{R}^N)$ satisfies (1.2).

Before stating the details, we introduce some results on forward self-similar solution of a heat equation with exponential nonlinear term. If solutions of (2.1) have the form $u(x,t) = -\log t + v(x/\sqrt{t})$ for some function v, then such solutions are called forward self-similar solution of (2.1) and if v is a radially symmetric function on $\mathbf{R}^N$, then v has to satisfy

$$v'' + \frac{N-1}{r}v' + \frac{r}{2}v' + e^v + 1 = 0 \quad \text{in } (0,\infty). \tag{2.2}$$

Concerning the existence of regular solutions for problem (2.2), it has been already proved that, for any $\alpha \in \mathbf{R}$, there exists a solution v_α of (2.2) satisfying $v_\alpha(0) = \alpha$ and $v'_\alpha(0) = 0$. Furthermore, for any $\alpha \in \mathbf{R}$, the limit

$$\lim_{r\to\infty} (2\log r + v_\alpha(r))$$

exists. See [2], [4] and [9]. For our purpose, we need to study more precise information of forward self-similar solutions of (2.1). For $l \in \mathbf{R}$, put

$$S_l := \left\{ v \in C^2([0,\infty)) : v \text{ is a solution of (2.2) satisfying } \lim_{r\to\infty} (2\log r + v(r)) = l \right\}.$$

For the structure of S_l, we have:

Theorem 2.1 *Let $N \geq 3$. Then there exists a constant $l^* \in \mathbf{R}$ such that*

$$S_l \begin{cases} = \emptyset & \text{if } l > l^*, \\ \neq \emptyset & \text{if } l < l^*. \end{cases}$$

Remark 2.1 (i) *If $N \geq 10$, then l^*, which is given in Theorem 2.1, corresponds with the constant log$(2N-4)$. On the other hand, if $3 \leq N \leq 9$, then we have $l^* > \log(2N-4)$.*
(ii) *For the case $3 \leq N \leq 9$, we have $S_{\log(2N-4)} = \infty$. Furthermore, there exist infinitely many α such that v_α intersects a singular solution $-2\log r + \log(2N-4)$.*

Using Theorem 2.1, we prove the following theorem.

Theorem 2.2 *Let $N \geq 3$. If*

$$\liminf_{|x|\to\infty} (2\log|x| + u_0(x)) > l^*,$$

then $T(u_0) < \infty$.

Remark 2.2 *There exists an initial function u_0 satisfying $T(u_0) = \infty$ and*

$$\lim_{|x|\to\infty} (2\log|x| + u_0(x)) = l^*.$$

Therefore the constant given in Theorem 2.1 gives optimal decay rate of u_0 classifying the global existence and blow-up of solutions for problem (2.1).

3 Outline of the proof of Theorem 2.2

In this section we explain how to apply Theorem 2.1 for the proof of Theorem 2.2. We only explain the outline of the proof of Theorem 2.2 for the case $3 \leq N \leq 9$. We first prepare one lemma on the construction of supersolutions.

Lemma 3.1 *Let $n \in \mathbf{N}$ with $n \geq 3$ and $\varphi_i = \varphi_i(|y|)$ (i=1,2,..., n) be regular radial symmetric supersolutions of* (2.2). *Assume that there exist constants $R_1 < R_2 < \cdots < R_{n-1}$ such that $\varphi_i(R_i) = \varphi_{i+1}(R_i)$ and $\varphi_i'(R_i) \geq \varphi_{i+1}'(R_i)$ for $i = 1, 2, \ldots, n-1$. Then the function $\overline{\varphi}$ defined by*

$$\overline{\varphi}(r) := \begin{cases} \varphi_1(r) & \textit{for} \quad r \in [0, R_1], \\ \varphi_{i+1}(r) & \textit{for} \quad r \in [R_i, R_{i+1}], \quad (i = 1, 2, \ldots, n-2) \\ \varphi_n(r) & \textit{for} \quad r \in [R_{n-1}, \infty), \end{cases}$$

is a supersolution of (2.2).

The proof of Theorem 2.2 is by contradiction. Assume that there exists a global in time solution for problem (2.2). Put

$$l_0 := \liminf_{|x| \to \infty} (2 \log |x| + u_0(x)).$$

By the assumption in Theorem 2.2 we have $l_0 > l^*$. Then, for any $l \in (l_0, l^*)$, by the similar argument as in [7] we can construct a subsolution $\underline{w}$ of (2.2) satisfying

$$\underline{w}(|x|) \leq u_0(x), \qquad \lim_{|x| \to \infty} (2 \log |x| + \underline{w}(|x|)) = l.$$

Here $\underline{w}$ is said to be a subsolution of (2.2) if

$$\underline{w}'' + \frac{N-1}{r}\underline{w}' + \frac{r}{2}\underline{w}' + e^{\underline{w}} + 1 \geq 0 \quad \text{in } (0, \infty).$$

Consider the solution $w = w(r, s)$ for the problem

$$\partial_s w = w'' + \frac{N-1}{r} w' + \frac{r}{2} w' + e^w + 1 \quad \text{in } (r, s) \in (0, \infty) \times (0, \infty) \tag{3.1}$$

with $w(\cdot, 0) = \underline{w}$. Then, since $\underline{w}$ is a subsolution of (2.2), $w(r, s)$ is non-decreasing in s for any $r \in (0, \infty)$. Therefore there exists a function $W \in C^2((0, \infty))$ such that

$$W(r) := \lim_{s \to \infty} w(r, s),$$

and the function W is a solution of (2.2). Furthermore, we see that there exists a constant $\tilde{l} \geq l$ such that

$$\lim_{r \to \infty} (2 \log r + W(r)) = \tilde{l}.$$

If $W(0) < \infty$, then $W \in C^2([0, \infty))$ and $W \in S_{\tilde{l}}$, which contradicts Theorem 2.1.

Next we consider the case $W(0) = \infty$. Unfortunately, we cannot prove non-existence of such W, however, we can get a contradiction even if such a function W exists. Indeed, if such W exists, then we have the following lemma.

Lemma 3.2 *Let $N \geq 3$. Assume that there exists a solution W of* (2.2) *such that*

$$\lim_{r\to\infty} W(r) = \infty, \qquad \lim_{r\to\infty} (2\log r + W(r)) > \log(2N-4).$$

Then there exists a sequence $\{r_k\} \subset (0,\infty)$ such that

$$\begin{aligned} &r_1 > r_2 > \cdots \to 0 \quad \textit{as } k \to \infty, \\ &W(r_k) = -2\log r_k + \log(2N-4), \\ &(-1)^k \left(W'(r_k) + \frac{2}{r_k} \right) < 0. \end{aligned}$$

Using Lemma 3.2, we derive a contradiction. By Remark 2.1 (i) we find infinitely many α such that the solution v_α of (2.2) with $v_\alpha(0) = \alpha$ and $v'_\alpha(0) = 0$ intersects a singular solution $-2\log r + \log(2N-4)$. Furthermore, its intersection point can be taken as small as possible if we take a sufficiently large α. Thus we can find α_* such that there exists a constant $r_* < r_2$ satisfying

$$v_{\alpha_*}(r_*) = -2\log r_* + \log(2N-4), \qquad v'_{\alpha_*}(r_*) > -\frac{2}{r_*}. \tag{3.2}$$

Therefore we can define the function φ by

$$\varphi(r) = \begin{cases} v_{\alpha_*}(r) & \text{if } \ 0 < r < r_*, \\ -2\log r + \log(2N-4) & \text{if } \ r_* < r < r_2, \\ W(r) & \text{if } \ r > r_2, \end{cases}$$

and the function φ is a supersolution of (2.2) by Lemmas 3.1 and 3.2 and (3.2). Then we consider the solution $\Phi = \Phi(r,s)$ of (3.1) with $\Phi(\cdot,0) = \varphi$. Since φ is a supersolution of (2.2), we see that $\Phi(r,s)$ is non-increasing in s for all $r \in (0,\infty)$. This implies that there exists a function

$$\tilde{\Phi}(r) := \lim_{s\to\infty} \Phi(r,s).$$

Then there exists a constant $\hat{l} \geq l > l^*$ such that $\tilde{\Phi} \in S_{\hat{l}}$, which contradicts Theorem 2.1. Therefore we get a contradiction.

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