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Solutions to the equation of the scalar-field type on a large spherical cap*

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1 Introduction

The ingredients of this paper are based on the joint works with C. Bandle (University of Basel), T. Kawakami (Osaka Prefecture University), A. Kosaka (Osaka City University) and H. Ninomiya (Meiji University) ([4, 5, 14]).

Our main interest lays in the structure of solutions to the scalar-field type equation

$$\Lambda u+\lambda(-u+|u|^{p-1}u)=0, \quad \text{in } \Omega(\varepsilon)\subset \mathbb{S}^{n} \tag{1.1}$$

with $p>1$, $n\geq 2$, and Λ is the Laplace-Beltrami operator defined by the standard metric on the usual unit sphere $\mathbb{S}^{n}$, $\Omega(\varepsilon)$ is a geodesic ball centered at the North Pole with its radius $\pi-\varepsilon$ with small $\varepsilon>0$.

We investigate the structure of solutions to (1.1) under the Neumann or the Dirichlet boundary condition including the non-azimuthal solutions. To investigate the whole structure of bifurcations, first we need to check the bifurcation points (eigenvalues). We consider the linear equation

$$\Lambda u+\lambda u=0, \quad \text{in } \Omega(\varepsilon)\subset \mathbb{S}^{n} \tag{1.2}$$

under the homogeneous Dirichlet, Neumann or additionally the Robin condition. In the whole sphere case, all the eigenvalues and their multiplicity of $-\Lambda$ are well-known as follows (see e.g., Chapter 2 of Shimakura [28]):

The k-th eigenvalue (counting from $k=0$) of $-\Lambda$ on $\mathbb{S}^{n}$ is

$$k(k+n-1)$$

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and its multiplicity is

$$(2k+n-1)\frac{(k+n-2)!}{(n-1)!k!}.$$

How are the eigenvalues affected by the perturbation of the domain? In the Euclidean space, there are works by S. Ozawa [24, 25, 26, 27], Courtois [12] and Lanza de Cristoforis [19]. They treated the Dirichlet problem (the first two authors) and the Neumann condition (the latter one) in a bounded domain rid of a small domain inside (the condition inside is the Neumann condition) under the condition that the eigenvalue is simple. The asymptotic behavior is estimated in terms of the capacity of the domain which is rid of the larger one. In our setting, the problem is like a "Neumann-Neumann" problem with multiplicity of eigenvalues.

We consider (1.2) in the polar coordinates. A point $(y_1, y_2, \ldots, y_{n+1}) \in \mathbb{S}^n$ in the polar coordinates is expressed as

$$\begin{cases} y_1 = \sin\theta_1 \cos\theta_2, \ y_{n+1} = \cos\theta_1, \\ y_k = \Big(\prod_{j=1}^{k} \sin\theta_j\Big) \cos\theta_{k+1}, \quad k = 2, .., n-2, \\ y_{n-1} = \Big(\prod_{j=1}^{n-1} \sin\theta_j\Big) \cos\phi, \\ y_n = \Big(\prod_{j=1}^{n-1} \sin\theta_j\Big) \sin\phi. \end{cases}$$

Also the spherical domain $\Omega(\varepsilon)$ in the polar coordinates is done as

$$\Omega(\varepsilon) := \{(\theta_1, \theta_2, \ldots, \theta_{n-1}, \phi) \,|\, 0 \le \theta_1 < \pi - \varepsilon, \ 0 \le \theta_j \le \pi, \ 0 \le \phi \le 2\pi\}$$

where $\varepsilon > 0$ is sufficiently small. θ_1 is called the azimuthal angle.

The expression of Λ in the polar coordinate is

$$\begin{aligned} \Lambda u &= \sum_{k=1}^{n-1} (\sin\theta_1 \ldots \sin\theta_{k-1})^{-2} (\sin\theta_k)^{k-n} \frac{\partial}{\partial\theta_k}\Big\{(\sin\theta_k)^{n-k}\frac{\partial u}{\partial\theta_k}\Big\} \\ &\quad + (\prod_{k=1}^{n-1} \sin\theta_k)^{-2} \frac{\partial^2 u}{\partial\phi^2}. \end{aligned}$$

We agree that $\sin\theta_1 \ldots \sin\theta_{k-1} \equiv 1$ if $k = 1$.

First, we discuss the linear problem in Section 2. In Section 3, the nonlinear problem under the Neumann condition is investigated. Imperfect bifurcations are shown in Sections 4 and 5. Concluding remarks are in Section 6.

2 Linear problem

In this section, we consider the linear problem. First, let $n = 2$. Then we solve

$$\frac{\partial^2 u}{\partial \theta^2} + \cot\theta \frac{\partial u}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \lambda u = 0 \tag{2.1}$$

by the separation of variables under the condition $\partial_{\mathbf{n}} u = 0$. Let us define $u(\theta, \phi) = \Phi(\theta)\Psi(\phi)$. According to the book by Tichmarsh [30] (also using the Weierstrass polynomial approximation theorem, see also Tichmarsh [31]), eigenfunctions must be of the form of separation of variables. For the Japanese, see §90, 91 in Yosida [32]. Then, we proceed as in the undergraduate ODE course to obtain the relation

$$\sin^2\theta(\Phi''(\theta) + (\cot\theta)\Phi'(\theta) + \lambda\Phi) = -\frac{\Psi''(\phi)}{\Psi(\phi)} = m^2$$

with $m = 0, 1, 2, \ldots$. We have $\Psi(\phi) = c_1 \cos m\phi + c_2 \sin m\phi$ with c_1 and c_2 being constants. In case of $m = 0$, we agree that $\Psi(\phi) \equiv 1$. Also, Φ satisfies the associated Legendre differential equation

$$\Phi''(\theta) + (\cot\theta)\Phi'(\theta) + (\lambda - \frac{m^2}{\sin^2\theta})\Phi = 0. \tag{2.2}$$

Let us define $\lambda = \nu(\nu+1)$ and take positive ν. Then, we see that the associated Legendre function of the first kind $P_\nu^m(\cos\theta)$ is the regular solution to (2.2) (cf. Beals and Wong [8] or Moriguchi, Udagawa and Hitotsumatsu [22]). ν is determined by the boundary condition

$$\frac{d}{d\theta} P_\nu^m(\cos(\pi - \varepsilon)) = 0.$$

To determine the relation between ν and ε, we employ properties of the Gauss hypergeometric functions since

$$P_\nu^m(t) = k_{m,\nu}(1 - t^2)^{m/2} F(m - \nu, m + \nu + 1, m + 1; \frac{1-t}{2})$$

with some $k_{m,\nu}$. Since $\varepsilon > 0$ is small, $t = \cos(\pi - \varepsilon) \approx -1$. Since the radius of convergence of $F(a, b, c; z)$ is $|z| = 1$, we need to check properties of the Gauss hypergeometric functions F (cf.Beals and Wong [8] or [22]). More precisely, we use the following inversion formula.

Let α, β be non-integer values and ℓ be a positive integer. The function

$U(\alpha, \beta, \ell, x)$ is defined as

$$
\begin{aligned}
&U(\alpha, \beta, \ell; x) \\
&= \frac{(-1)^\ell}{\Gamma(\alpha+1-\ell)\Gamma(\beta+1-\ell)(\ell-1)!} \times \\
&\Big[F(\alpha, \beta, \ell; x) \log x \\
&+ \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\ell)_n n!} \{\psi(\alpha+n) + \psi(\beta+n) - \psi(n+1) - \psi(\ell+n)\} x^n\Big] \\
&+ \frac{(\ell-2)!}{\Gamma(\alpha)\Gamma(\beta)} x^{1-\ell} \sum_{n=0}^{\ell-2} \frac{(\alpha+1-\ell)_n(\beta+1-\ell)_n}{(2-\ell)_n n!} x^n,
\end{aligned}
\tag{2.3}
$$

where $\psi(z)$ is the psi (or di-Gamma) function defined as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Then the conclusion in [8] in p. 276 is

$$F(\alpha, \beta, \ell; x) = \Gamma(\ell) U(\alpha, \beta, \alpha+\beta+1-\ell; 1-x) \tag{2.4}$$

provided $\alpha + \beta + 1 - \ell$ is a non-positive integer.

Using (2.4), we see that ν is expressed as

$$\nu = \nu(k, \mu) = k + \nu(\varepsilon), \quad \nu(\varepsilon) \approx o(1)$$

with some natural number k.

In our case, $\ell = 2m$ (if $m \geq 1$), we have

$$\lambda = \nu(\nu+1) \approx k(k+1) + c_{k,m}\varepsilon^{2m}$$

with some constant $c_{k,m}$ (we can compute this value exactly). When $m = 0$, then we see that the order is neither 0 nor $|\log \varepsilon|$, but ε. Moreover, for each $m \in [1, k]$, $P^m_{\nu(k,m)}(\cos\theta)(c_1 \cos m\phi + c_2 \sin m\phi)$ is the eigenfunctions corresponding to $\lambda \approx k(k+1) + c_{k,m}\varepsilon^{2m}$. For $m = 0$, $P^0_{\nu(k,0)}(\cos\theta)$ is the corresponding eigenfunction.

In the end, we have the following.

Theorem 2.1 *Let $n = 2$. If $\varepsilon > 0$ is small, then for each $k \in \mathbb{N}$, around $k(k+1)$, there exist $k+1$ eigenvalues $\lambda_{k,\varepsilon,m}$ ($m = 0, 1, \ldots, k$) to (2.2) under the homogeneous Neumann condition, such that*

$$\lambda_{k,\varepsilon,m} - k(k+1) \approx c_{k,m}\varepsilon^{\max\{2m,1\}}$$

where $c_{k,m}$ is a constant. Moreover, the multiplicity of $\lambda_{k,\varepsilon,m}$ is 2 if $m \geq 1$ and $\lambda_{k,\varepsilon,0}$ is 1.

Remark 2.1 *In the whole sphere $\mathbb{S}^2$ case, the multiplicity of the eigenvalue $k(k+1)$ is $2k+1$. Thus, by the presence of $\varepsilon > 0$, the eigenvalue $k(k+1)$ is split into $(k+1)$ eigenvalues. $c_{k,m} < 0$ if $m \geq 1$ and $c_{k,0} > 0$. The domain monotonicity property (see e.g., Ni [23]) fails. Among them, the largest one is simple and others are of multiplicity 2. Similar results are obtained for a generic dimension.*

For the three dimensional case, the polar coordinates yield

$$\begin{cases} y_1 = \cos\phi\sin\varphi\sin\theta, \\ y_2 = \sin\phi\sin\varphi\sin\theta, \\ y_3 = \cos\varphi\sin\theta, \\ y_4 = \cos\theta. \end{cases}$$

$$\Omega_\varepsilon = \{(\theta,\varphi,\phi)\,|\,0 \leq \theta < \pi - \varepsilon,\ 0 \leq \varphi \leq \pi,\ 0 \leq \phi \leq 2\pi\}.$$

Λu is expressed as

$$\begin{aligned}\Lambda u &= \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}(\sin^2\theta\frac{\partial u}{\partial\theta}) + \frac{1}{\sin^2\theta\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi\frac{\partial u}{\partial\varphi}) \\ &\quad + \frac{1}{\sin^2\theta\sin^2\varphi}\frac{\partial^2 u}{\partial\phi^2}.\end{aligned}$$

Then separating variables

$$u(\theta,\varphi,\phi) = U(\theta)V(\varphi)W(\phi),$$

we get

$$\begin{cases} \frac{1}{U}\frac{\partial}{\partial\theta}(\sin^2\theta\frac{\partial U}{\partial\theta}) + \lambda\sin^2\theta = \ell, \\ -\frac{1}{V\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi\frac{\partial V}{\partial\varphi}) - \frac{1}{W\sin^2\varphi}\frac{\partial^2 W}{\partial\phi^2} = \ell, \end{cases} \tag{2.5}$$

for some constant ℓ. Again, the second equation of (2.5) is reduced to

$$\begin{cases} \sin\varphi\frac{d}{d\varphi}(\sin\varphi\frac{dV}{d\varphi}) + \ell(\sin^2\varphi)V = LV, \\ -\frac{d^2W}{d\phi^2} = LW, \end{cases} \tag{2.6}$$

for some constant L. Since W is periodic, $L = m^2$ with $m = 0, 1, 2, \ldots$ and thus we have

$$W = c_1 \cos m\phi + c_2 \sin m\phi$$

with c_1 and c_2 being constants. We set $W \equiv 1$ when $m = 0$. Then V satisfies

$$V''(\varphi) + (\cot\varphi)V'(\varphi) + (\ell - \frac{m^2}{\sin^2\varphi})V = 0. \tag{2.7}$$

Put $t = \cos\varphi$ and $P(t) = V(\varphi)$. It is a solution of the associated Legendre differential equation. Let $\ell = \hat{\nu}(\hat{\nu}+1)$. Then a solution is expressed as $P(t) = P_{\hat{\nu}}^m(t)$. Since the solutions are regular at $t = \pm 1$, $\hat{\nu}$ must be a positive integer q and

$$\hat{\nu} = q \geq m \tag{2.8}$$

and $P_q^m(t)$ is indeed a polynomial. By (2.5) with $\ell = q(q+1)$, U satisfies

$$U''(\theta) + 2(\cot\theta)U'(\theta) + \{\lambda - \frac{q(q+1)}{\sin^2\theta}\}U = 0, \tag{2.9}$$

which is called the "hyper-sphere differential equation". (2.9) is equivalent to

$$\{(\sin^2\theta)U'\}' + \lambda(\sin^2\theta)U - q(q+1)U = 0. \tag{2.10}$$

Let $t = \cos\theta$ and $U(\theta) = \tilde{U}(t)/(1-t^2)^{1/4}$. Then $\tilde{U}(t)$ satisfies the associated Legendre differential equation

$$(1-t^2)\tilde{U}''(t) - 2t\tilde{U}'(t) + \{\lambda + \frac{3}{4} - \frac{q(q+1)+1/4}{1-t^2}\}\tilde{U}(t) = 0. \tag{2.11}$$

We see that $\tilde{U}(t)$ is expressed as $\tilde{U}(t) = P_\nu^\alpha(t)$ with

$$\nu(\nu+1) = \lambda + \frac{3}{4}, \quad \alpha^2 = q(q+1) + \frac{1}{4} = (q + \frac{1}{2})^2.$$

To obtain a regular solution at $t = 1$, we take the negative sign[1] and U can be written as

$$U(\theta) = \frac{P_\nu^{-(q+1/2)}(\cos\theta)}{\sqrt{\sin\theta}}.$$

Thus the eigenfunctions $-\Lambda$ are of the form

$$\Phi = \frac{P_\nu^{-(q+1/2)}(\cos\theta)}{\sqrt{\sin\theta}} P_q^m(\cos\varphi)(c_1\cos m\phi + c_2 \sin m\phi). \tag{2.12}$$

[1]We may use the associated Legendre function of the second kind $Q_\nu^{(q+1/2)}(\cos\theta)$ here. This is essentially the same as $P_\nu^{-(q+1/2)}(\cos\theta)$.

We determine $\nu = \nu(\varepsilon)$ as a function of ε so that the Neumann boundary condition is satisfied, i.e.

$$\frac{d}{d\theta}U(\theta)\Big|_{\theta=\pi-\varepsilon} = 0.$$

We again express $P_\nu^{-(q+1/2)}(\cos\theta)$ in terms of the Gauss hypergeometric functions.

As above, unfortunately, the treatment of the Gauss hypergeometric functions depends on whether the subscript is an integer or a half-integer. In this case, we use the following inversion formula instead of (2.3):

$$\begin{aligned} &F(-\nu, \nu+1, q+\frac{3}{2}; \frac{1-x}{2}) \\ &= \frac{a_q}{\Gamma(q+\nu+\frac{3}{2})\Gamma(q-\nu+\frac{1}{2})}F(-\nu, \nu+1, -q+\frac{1}{2}; \frac{x+1}{2}) \\ &+\frac{b_q}{\Gamma(-\nu)\Gamma(\nu+1)}(\frac{t_\varepsilon+1}{2})^{q+1/2}\times \\ &\qquad F(q+\nu+\frac{3}{2}, q-\nu+\frac{1}{2}, q+\frac{3}{2}; \frac{x+1}{2}). \end{aligned} \tag{2.13}$$

Here we define

$$a_q := \Gamma(q+\frac{3}{2})\Gamma(q+\frac{1}{2}) \text{ and } b_q := \Gamma(q+\frac{3}{2})\Gamma(-q-\frac{1}{2}).$$

In general, (2.13) is used for the odd dimensional case.

In the case $n = 3$, $\nu \approx k + 1/2$ with a nonnegative integer k. More precisely, we have the following.

Theorem 2.2 *Let $n = 3$. If $\varepsilon > 0$ is small, then for each $k \in \mathbb{N}$, around $k(k+2)$, there exist $(k+1)$ distinct eigenvalues $\lambda_{k,\varepsilon,q}$ $(q = 0, 1, \ldots, k)$ to (2.2) under the homogeneous Neumann condition, such that*

$$\lambda_{k,\varepsilon,q} - k(k+2) \approx c_{k,q}\varepsilon^{\max\{2q+1,3\}},$$

with $c_{k,0} > 0$ and with $c_{k,q} < 0$ for $q = 1, 2, \ldots, k$. Moreover, the multiplicity of $\lambda_{k,\varepsilon,q}$ is $(2q+1)$ if $q \geq 1$ and that of $\lambda_{k,\varepsilon,0}$ is 1.

Remark 2.2 *In case of $q = 0$ and $q = 1$, the order of λ in terms of ε is the same ε^3. However, in case of $q = 0$, $c_{k,0} > 0$ while $c_{k,1} < 0$. This fact also shows that the domain-monotonicity property of eigenvalues does not necessarily hold in the Neumann case.*

Remark 2.3 *In case of $n = 3$, the azimuthal Neumann eigenvalue is closer to the whole sphere one than the Dirichlet case (the Neumann case is of ε^3 order while the Dirichlet one ε order, see Kosaka [17] or Macdonald [21]).*

Remark 2.4 *Similarly to $n = 2$ case, in the whole sphere $\mathbb{S}^3$ case, the multiplicity of the eigenvalue $k(k+2)$ is $(k+1)^2$. Thus, by the presence of $\varepsilon > 0$, the eigenvalue $k(k+2)$ is split into $(k+1)$ eigenvalues. Near $k(k+2)$, the largest one is simple and the multiplicities of others are $2q+1$. Also note that*

$$1 + \sum_{q=1}^{k}(2q+1) = (k+1)^2.$$

Thus, the total multiplicity of the split eigenvalues is preserved.

In a general dimension, we have the following.

Theorem 2.3 *Let $n \geq 4$. If $\varepsilon > 0$ is small, then for each $k \in \mathbb{N}$, around $k(k+n-1)$, there exist $(k+1)$ distinct eigenvalues $\lambda_{k,\varepsilon,m}$ $(m = 0, 1, \ldots, k)$ under the Neumann condition, such that*

$$\lambda_{k,\varepsilon,m} - k(k+n-1) \approx c_{k,m,n}\varepsilon^{n_m}, \tag{2.14}$$

where

$$n_m = \begin{cases} 2m + n - 2, & \text{if } n \text{ is even,} \\ \max\{2m+n-2, n\}, & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, the multiplicity of $\lambda_{k,\varepsilon,m}$ is

$$\frac{(2m+n-2)(m+n-3)!}{(n-2)!m!}$$

if $m \geq 1$ and that of $\lambda_{k,\varepsilon,0}$ is 1. In this case also, the total multiplicity is preserved.

As a slight generalization, we can consider the Robin boundary condition: $(\cos\sigma)\partial_{\mathbf{n}}u + (\sin\sigma)u = 0$ on $\partial\Omega_\varepsilon$ with $\sigma \in (0, \pi/2)$ (due to Kabeya, Kawakami, Kosaka and Ninomiya [14]). If $\sigma = 0$ implies the Neumann condition and $\sigma = \pi/2$ does the Dirichlet one.

Theorem 2.4 *Let $n = 2$. Fix $\sigma \in (0, \pi/2)$. For any $k \in \mathbb{N}$, there exist $(k+1)$ distinct eigenvalues $\lambda_{\varepsilon,k,m,\sigma}$ $(m = 0, 1, \ldots, k)$ of $-\Lambda$ exist near $k(k+1)$. As $\varepsilon \to +0$, the following asymptotic expansion hold:*

$$\lambda_{\varepsilon,k,m,\sigma} - k(k+1) \approx c_{k,m,\sigma,2}\varepsilon^{\max\{2m,1\}} \quad (m = 0, 1, 2, \ldots, k),$$

Remark 2.5 *In $n = 2$ case, if $m \geq 1$, then the leading term is exactly the same as that under the Neumann condition (independent of σ). However, if $m = 0$, the leading term depends on σ. Indeed,*

$$\lambda_{\varepsilon,k,m,\sigma} \approx k(k+1) + \frac{2k+1}{2}(\tan\sigma)\varepsilon.$$

The difference from the Neumann condition case appears, under which the order is ε^2. Under the Dirichlet case, the order is $|\log\varepsilon|^{-1}$.

Theorem 2.5 *Let $n = 3$. Fix $\sigma \in (0, \pi/2]$. For any $k \in \mathbb{N}$, there exist $(k+1)$ distinct eigenvalues $\lambda_{\varepsilon,k,q,\sigma}$ $(q = 0, 1, \ldots, k)$ of $-\Lambda$ exist near $k(k+2)$. As $\varepsilon \to +0$, the following asymptotic expansions hold:*
(i) $\sigma \in (0, \pi/2)$:

$$\lambda_{\varepsilon,k,q,\sigma} - k(k+2) \approx c_{k,q,\sigma,3}\varepsilon^{\max\{2q+1,2\}}.$$

(ii) $\sigma = \pi/2$:

$$\lambda_{\varepsilon,k,q,\sigma} - k(k+2) \approx c_{k,q,\sigma,3}\varepsilon^{2q+1}$$

Remark 2.6 *Similarly to the two dimensional case, when $q \geq 1$, the leading term in the right-hand side is not dependent on $\sigma \in [0, \pi/2)$. When $q = 0$, the leading term of eigenvalues depends on σ. Indeed,*

$$\lambda_{\varepsilon,k,q,\sigma} \approx k(k+2) + \frac{2(k+1)^2}{\pi}(\tan\sigma)\varepsilon^2.$$

If $q = 0$ and $\sigma = 0$ (the Neumann condition), the asymptotic order is ε^3, while under the Dirichlet one, the order is of ε.

3 Non-azimuthal solution – Neumann Problem

In this section, we consider the nonlinear problem

$$\Lambda u + \lambda(-u + |u|^{p-1}u) = 0$$

in Ω_ε under the homogeneous Neumann condition. For simplicity, we consider case $n = 2$ and the bifurcation points are the eigenvalues as we have seen before. Thus, we discuss solutions to the following problem

$$\begin{cases} \Lambda u + \lambda(-u + |u|^{p-1}u) = 0, & \text{in } \Omega(\varepsilon), \\ \partial_{\mathbf{n}} u = 0, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.1}$$

Theorem 3.1 (Local bifurcation) *Let $p > 1$ and $n = 2$. For ϵ sufficiently small the nonlinear problem* (3.1) *has a non-trivial solution (which is close to* 1*) near $(p-1)\lambda = \lambda_{k,\varepsilon,m}$. If $m = 0$ it depends only on θ, while solutions depending on both θ and φ depending bifurcate from $(p-1)\lambda = \lambda_{k,\varepsilon,m}$ if $m \geq 1$.*

This result is obtained by means of the theory of bifurcation. Although the eigenvalues has even multiplicity, we can prove the bifurcation according to Chapter 7 of Chow and Hale [11] or Section 5.5 in the book by Ambrosetti and Prodi [1]. Eventually, we need to solve a system of algebraic equations which appear from the Lyapunov-Schimidt reduction. For a general dimension, almost the same statement holds and this is reported in [5].

As for the Dirichlet case, Kosaka [17] obtained the similar results.

4 Imperfect Bifurcation

In this section, we quickly review the results by Bandle, Kabeya andNinomiya [4].

As an ODE problem, (1.1) is reduced to

$$\frac{1}{\sin^{n-1}\theta_1}\frac{d}{d\theta_1}(\sin^{n-1}\theta_1\frac{du}{d\theta_1}) + \lambda(-u + |u|^{p-1}u) = 0 \tag{4.1}$$

and we impose the Dirichlet boundary condition

$$u(\pi - \varepsilon) = 0. \tag{4.2}$$

For sub-critical case ($(n-2)p < (n+2)$), the usual variational methods works on the problems on the spherical cap. There are many works on the nonlinear elliptic problems. Contributors are among others, Bandle and Peletier [6], Bandle, Brillard, Flucher [3], Bandle and Benguria [2], Brezis and Peletier [6], Bandle and Wei [7] .

There are results analogous to Gidas, Ni and Nirenberg [13]. Kumaresan and Prajapat [18] and Brock and Prajapat [10]. Positive solutions to $\Lambda u + f(u) = 0$ under the homogeneous Dirichlet problem necessarily depend only on the azimuthal angle if the domain is a geodesic ball centered at the north pole contained in the northern hemisphere.

There are numerical results by Stingelin [29]. He showed the bifurcation diagrams to (1) with $n = 3$ and $p = 5$. The diagram resembles to what Kabeya, Morishita, Ninomiya [15] have shown. Especially, we want to know what will happen to the diagram and the profile of solutions when $\varepsilon \to 0$.

As a one dimensional problem, several abstract theorems on the imperfect bifurcations have been obtained by P. Liu, J. Shi and Y. Wang [20].

Similar phenomena for the exponential type nonlinearity was shown in Kan and Miyamoto [16].

As a step to non-azimuthal solutions, which are discussed in the next section, we review the result in [4]. Note that (4.1) has $u \equiv 1$ as a solution (neglecting the boundary condition). Then a solution must be approximated by the solution to the linearized equation (around $u \equiv 1$)

$$\Lambda u + (p-1)\lambda u = 0$$

under the condition $u(\pi - \varepsilon) = 0$. The corresponding eigenpairs are denoted by $(\varphi_{j,\varepsilon}, \lambda_{j,\varepsilon})$.

Let $n \geq 2$ and choose q so that

$$\max\left\{\frac{n}{2}, \left(1 - \frac{1}{p}\right) n\right\} \leq q < n \tag{4.3}$$

(q is taken very close to n).

We introduce

$$\mathcal{W} := W^{1,q}_{0,\mathrm{az}}(\Omega_\varepsilon),$$

which is the completion of $C^\infty_{0,\mathrm{az}}(\Omega_\varepsilon)$ (C^∞_0 functions depending only on θ_1) with respect to the following norm $\|\Phi\|_{\mathcal{W}}$

$$\|\Phi\|_{\mathcal{W}} := \left(\int_{\Omega_\varepsilon} |\Phi_{\theta_1}|^q \, dS + \int_{\Omega_\varepsilon} |\Phi|^q \, dS\right)^{\frac{1}{q}}.$$

We have the following theorem on imperfect bifurcations.

Theorem 4.1 ([4]) *Assume $p > 1$, $n \geq 3$ and let $\lambda_j := (j-1)(j+n-2)/(p-1)$. Then for $j = 2, 3, \ldots$ there exist small positive numbers ε_*, ζ_* and, for any $0 < \varepsilon < \varepsilon_*$ and for any j, a one-dimensional C^1-manifold $\mathcal{S}_\varepsilon(j) \subset (\lambda_j - \zeta_*, \lambda_j + \zeta_*) \times \mathcal{W}$, with the following property.*

(1) *The elements $v(\theta_1; \varepsilon)$ of $\mathcal{S}_\varepsilon(j)$ are solutions of* (1.1).

(2) *They are of the form*

$$v(\theta_1; \varepsilon) = \rho_\varepsilon + w_\varepsilon + s\varphi_{j,\varepsilon} + h_{(s;\varepsilon)},$$

where $\rho_\varepsilon \in C^\infty$ is such that $0 \leq \rho_\varepsilon \leq 1$, $\rho_\varepsilon(\theta_1) = 1$ if $\theta_1 \leq \pi - 2\varepsilon$ and $\rho_\varepsilon(\theta_1) = 0$ if $\theta_1 \geq \pi - \varepsilon$ and s is a small parameter satisfying the relation (4.4) *in* (5). *Moreover w_ε and $h_{(s;\varepsilon)}$ belong to $\mathcal{W}$ and are small in the sense that $\|w_\varepsilon\|_{\mathcal{W}} \to 0$ as $\varepsilon \to 0$ and $\|h_{(s;\varepsilon)}\|_{\mathcal{W}} \to 0$ as $\varepsilon \to 0$ and $s \to 0$.*

More precisely there hold

(3) $\|w_\varepsilon\|_W = O(\varepsilon^{(n-q)/q})$, $w_\varepsilon(\theta_1) \to 0$ *locally uniformly on* $[0,\pi)$ *as* $\varepsilon \to 0$ *and* $\|h_{(s;\varepsilon)}\|_W = O(\varepsilon^{n\min\{p-1,1\}/pq}|s| + |s|^{\min\{p,2\}})$ *for* $|s| \le s_*(\varepsilon)$.

(4) $s_*(\varepsilon) = o(\varepsilon^{(n-2)/\min\{2,p\}})$.

(5) *The relation between* s *and* λ *is determined implicitly from the equation* $H_\varepsilon(s,\lambda) = 0$ *which for small* ε *and* s *is given by*

$$H_\varepsilon(s,\kappa) = s\kappa + \eta(\varepsilon) + O(\varepsilon^n|s| + |s|^{\min\{2,p\}}). \tag{4.4}$$

Here $\kappa = (p-1)(\lambda-\lambda_{j,\varepsilon})$ *varies within* $\kappa = O(\varepsilon^{(n-2)(\min\{2,p\}-1)/\min\{2,p\}})$, $\eta(\varepsilon)$ *depends only on* ε *and satisfies*

$$\eta(\varepsilon) = (-1)^j b_{n,j}\varepsilon^{n-2} + O(\varepsilon^{n-1})$$

with a positive constant $b_{n,j}$.

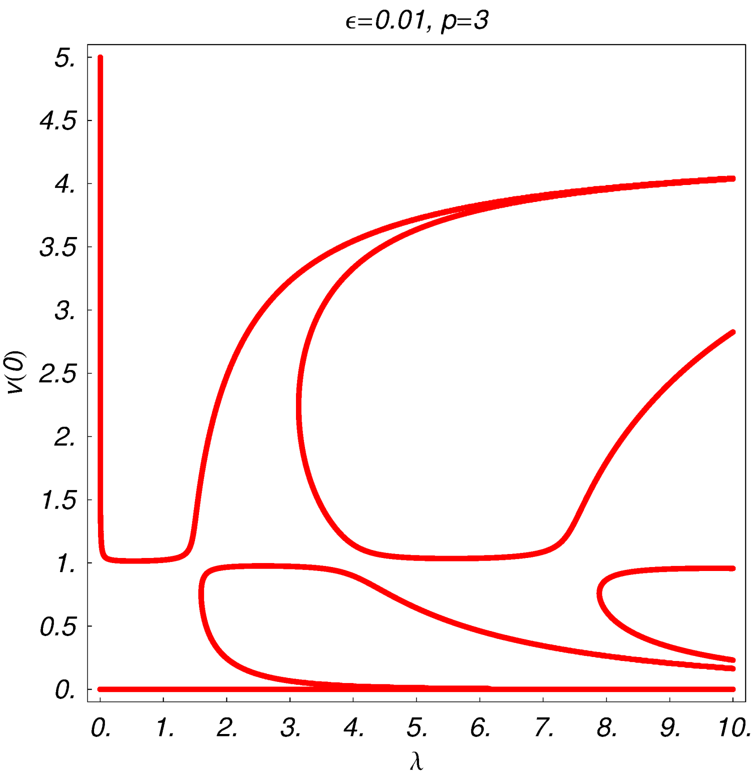

図 1: The diagram of (1) with $n = 3$ and $p = 3$.

Then a solution near $\lambda = \lambda_{j,\varepsilon}$ is expressed as

$$v(\theta_1;\varepsilon) = \rho_\varepsilon + s(\varepsilon)\varphi_{j,\varepsilon} + h(s;\varepsilon;\lambda),$$

where ρ_ε is a compensation function which converges to 1 as $\varepsilon \to 0$ locally uniformly, $s(\varepsilon) \to 0$ also as $\varepsilon \to 0$ and $h(s;\varepsilon;\lambda) \perp \varphi_{j,\varepsilon}$ in $L^2(\Omega_\varepsilon; \sin^{n-1}\theta\, d\theta)$. This expression is valid for λ with $|\lambda - \lambda_{j,\varepsilon}| \leq O(\varepsilon^{n-2})$ if $n \geq 3$ and with $|\lambda - \lambda_{j,\varepsilon}| \leq O(\log(1/\varepsilon))$ for $n = 2$.

The solution is close to 1 and satisfies the Dirichlet problem.

Remark 4.1 *By the existence of $\eta(\varepsilon) \neq 0$, $H_\varepsilon(s,\kappa) = 0$ represents two hyperbolae in the $\kappa - s$ plane. Moreover, from (5) of Theorem, they look like "$\kappa s = 1$" type if j is odd and "$\kappa s = -1$" type if j is even, which coincides with the numerical computations (see Figure in the previous page).*

Remark 4.2 *In the whole sphere case($\varepsilon = 0$), it follows that $\eta(0) = 0$ and the diagram exhibits the branches which are connected with the constant solution.*

5 Non-azimuthal solution – Dirichlet case

In Section 4, we see that the imperfect bifurcations occur for azimuthal solutions. What will happen if we take non-azimuthal solutions into account? Instead of the Neumann condition, we consider the linearized problem under the Dirichlet problem

$$\begin{cases} \Lambda v + (p-1)\lambda u_\varepsilon^{p-1} v = 0 \text{ in } \Omega_\varepsilon \subset \mathbb{S}^n, \\ v = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{5.1}$$

where u_ε is the azimuthal solution. $u_\varepsilon \approx 1$ but not identically equal to 1. The eigenvalues and corresponding eigenfunctions to (5.1) will be perturbed. What we are sure now is the following.

Theorem 5.1 (Imperfect including non-azimuthal solutions) *$n = 2$ and $\varepsilon > 0$ is very small. Then near $\lambda = k(k+1)/(p-1)$, the azimuthal imperfect bifurcations occur but non-azimuthal solutions (depending also on the longitude variable) bifurcate from the imperfectly bifurcating branch. The bifurcating point is also close to $\lambda = k(k+1)/(p-1)$.*

Consider the Rayleigh quotient corresponding to (5.1)

$$\frac{\displaystyle\int_{\Omega_\varepsilon} |\nabla v|^2\, dS}{\displaystyle\int_{\Omega_\varepsilon} (1+\xi_\varepsilon)|v|^2\, dS}$$

where $1 + \xi_\varepsilon$ is the solution on the imperfectly bifurcating branch and v is taken in $H_0^1(\Omega_\varepsilon)$.

Suppose that $\xi_\varepsilon \equiv 0$ (although this is impossible indeed). Then we can have the accurate estimate of the perturbed eigenvalues. The upper estimate of ξ_ε is of $\varepsilon^{(n-q)/q}$ order. So in this perturbation affects much on the asymptotic behavior of eigenvalues. The existence of azimuthal solution is assured only smaller range of λ for $n = 2$ for the moment.

6 Concluding Remarks

In this final section, we give two remarks.

Remark 6.1 *We can assure Theorem 5.1 in the $n = 2$ case only, however, we believe that non-azimuthal solutions bifurcate from the azimuthal solutions branch despite the dimension and the order of the eigenvalues.*

Since we rely on the properties of special functions, our technique does not apply to the cases of two or more holes except small holes at the North Pole and the South Pole. Rough preliminary calculations indicate that the larger hole dominates the asymptotic behavior of the eigenvalues. More precisely, we have the following.

Remark 6.2 *Let the domain Ω is defined as*

$$\Omega(\delta,\varepsilon) := \{(\theta_1,\theta_2,\dots,\theta_{n-1},\phi)\,|\,0<\delta<\theta_1<\pi-\varepsilon,\ 0\le\theta_j\le\pi,\ 0\le\phi\le 2\pi\}$$

with small $\delta > 0$ and $\varepsilon > 0$. If $\delta = o(\varepsilon)$, then at least azimuthal eigenvalues have the same asymptotic behaviors as in Section 2 with

$$\Omega(\varepsilon) = \{(\theta_1,\theta_2,\dots,\theta_{n-1},\phi)\,|\,0\le\theta_1<\pi-\varepsilon,\ 0\le\theta_j\le\pi,\ 0\le\phi\le 2\pi\}.$$

If $\delta = c\varepsilon + o(\varepsilon)$ with $c > 0$, then azimuthal eigenvalues have the same asymptotic behaviors as in Section 2 with

$$\Omega(\delta+\varepsilon) := \{(\theta_1,\theta_2,\dots,\theta_{n-1},\phi)\,|\,0\le\theta_1<\pi-(\delta+\varepsilon),\ 0\le\theta_j\le\pi,\ 0\le\phi\le 2\pi\}.$$

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