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Hadamard variational formula for the Stokes equations

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Introduction.

Let $\Omega\subset\mathbb{R}^{n}(n\geq 2)$ be a bounded domain with the smooth boundary $\partial\Omega$. We consider the eigenvalue problem for the Stokes equations on Ω with the Dirichlet boundary condition.

$$\begin{cases}-\Delta \boldsymbol{V}+\nabla P=\lambda \boldsymbol{V} & \text{in } \Omega,\\ \operatorname{div} \boldsymbol{V}=0 & \text{in } \Omega,\\ \boldsymbol{V}=0 & \text{on } \partial\Omega,\end{cases}$$

where λ is the eigenvalue of the Stokes equations, while $\boldsymbol{V}=\boldsymbol{V}(x)=(V^{1}(x),\cdots,V^{n}(x))$ and $P=P(x)$ denote the corresponding eigenfunctions for the velocity and the pressure at $x=(x^{1},\cdots,x^{n})\in\Omega_{\varepsilon}$, respectively. The purpose of this paper is to construct the Hadamard variational formula for the multiple eigenvalue of the Stokes equations with the Dirichlet boundary condition. For a small real parameter ε, we regard Ω_{ε} as the smoothly perturbed domain from $\Omega=\Omega_{0}$. We consider the eigenvalue problem for the Stokes equations on Ω_{ε} under the general smooth perturbation. For the eigenvalue $\lambda(\varepsilon)$ of the Stokes equations on Ω_{ε}, the eigenfunctions $\{\boldsymbol{V}_{\varepsilon}, P_{\varepsilon}\}$ satisfies

$$\begin{cases}-\Delta \boldsymbol{V}_{\varepsilon}+\nabla P_{\varepsilon}=\lambda(\varepsilon)\boldsymbol{V}_{\varepsilon} & \text{in } \Omega_{\varepsilon},\\ \operatorname{div} \boldsymbol{V}_{\varepsilon}=0 & \text{in } \Omega_{\varepsilon},\\ \boldsymbol{V}_{\varepsilon}=0 & \text{on } \partial\Omega_{\varepsilon}.\end{cases} \tag{0.1}$$

The spectral set of the Stokes operator in smoothly bounded domains consists of a descrete sequence of positive numbers (see, e.g., Courant-Hilbert [1], Ladyzhenskaya [12] and Temam [16]). If we arrange them in increasing order with counting multiplicity, we have

$$\lambda_{1}(\varepsilon)\leq\lambda_{2}(\varepsilon)\leq\lambda_{3}(\varepsilon)\leq\cdots\leq\lambda_{j}(\varepsilon)\leq\cdots\rightarrow\infty,$$

and the j-th eigenvalue of (0.1) is denoted by $\lambda_{j}(\varepsilon)$. Our aim is to establish the representation formula for the first variation of the eigenvalue $\lambda(\varepsilon)$ with respect to ε. More precisely, abbreviating $\lambda_k(0) = \lambda_k$, we investigate the asymptotic behavior of the convergence

$$(0.2) \qquad \lim_{\varepsilon\to 0} \lambda_k(\varepsilon) = \lambda_k \quad \text{(continuous dependence on domain)}$$

and prove the representation formula for $\delta\lambda_k := \lim_{\varepsilon\to 0} \varepsilon^{-1}(\lambda_k(\varepsilon) - \lambda_k)$. Furthermore, we deal with not only a simple eigenvalue but also a general multiple eigenvalue.

The Hadamard variational formula for the first eigenvalue of the Laplace operator with the Dirichlet boundary condition was first introduced by Hadamard [6]. For a smooth function ρ on $\partial\Omega$, we indicate Ω_ε by the perturbed domain such that the boundary $\partial\Omega_\varepsilon = \{x + \varepsilon\rho(x)\nu_x \ ; \ x \in \partial\Omega\}$, where ν_x is the unit outer normal vector to $\partial\Omega$ and he gave the representation formula as

$$\delta\lambda_1 = -\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu_x}(x)\right)^2 \rho(x)\, d\sigma_x,$$

where u is the eigenfunction corresponding to the first eigenvalue λ_1 of the Laplace operator with $\|u\|_{L^2(\Omega)} = 1$, and then Garabedian-Schiffer [3] gave the rigorous proof for that. Moreover, Ozawa [14] established the formula for multiple eigenvalues by investigating the formula for the trace of the fundamental solutions of the heat equation with the Dirichlet boundary condition. Such a perturbation problem for the usual Laplace operator had been analyzed for the another boundary condition or for a non-smooth domains (cf. Ozawa [15], Grinfeld [4], [5], Kozlov [9], Kozlov-Nazarov [10]). On the other hand, in the case of the Stokes equations, there are few results on the variation of eigenvalues with multiplicity for the perturbed domain.

For the case of the Stokes equations, the difficulty occurs in treating the divergence free condition. In the previous works for the Green function of the Stokes equations [11], [17], [18], the Stokes equations (0.1) was transformed by the volume preserving diffeomorphism $\Phi_\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ as in Inoue-Wakimoto [7]. Here, we succeed to get rid of such a restriction by making use of the *piola transform* (cf. Marsden-Hughes [13]).

For this paper, the essential problem is to investigate ε-dependence of the eigenvalue $\lambda(\varepsilon)$ and the corresponding eigenfunction u_ε. The method of Garabedian-Schiffer [3] is to expand the eigenfunction u_ε by the Fredholm theory. They made use of the analyticity of the Green function G_ε with respect to the parameter ε, whose method essentially depends on the simplicity of the first eigenvalue for the Laplace equations. On the other hand, for the case of the Stokes equations, since the simplicity of the first eigenvalue is still an open problem, we need to study the variational formula not only for the simple eigenvalue but also for the multiple ones. For that purpose, we analyze an ε-dependence of the multiple eigenvalues of the Stokes equations by means of the Min-Max principle. Approximating the eigenfunction $u_k(\varepsilon)$ analytically with respect to the parameter ε, we are able to obtain an detailed ε-dependence of the eigenvalue $\lambda_k(\varepsilon)$ even though λ_k is a multiple eigenvalue.

The paper is organized as follows. In Section 1, we introduce the assumption for the diffeomorphism Φ_ε, and then state our main result. Section 2 is devoted to the analysis of asymptotic behavior as $\varepsilon \to 0$ of the eigenvalue $\lambda(\varepsilon)$ and the corresponding eigenfunction $\{\boldsymbol{V}_\varepsilon, P_\varepsilon\}$. For that purpose, we introduce the min-max principle for the Stokes equations. We finally construct representation formula for the eigenvalue $\lambda(\varepsilon)$ in Section 3.

1 Results.

To state our result, we first introduce an assumption on the perturbation Ω_ε of domains from Ω.

Assumption. For a real parameter ε, there is a diffeomorphism $\Phi_\varepsilon : \overline{\Omega} \to \overline{\Omega}_\varepsilon$ satisfying the following conditions.

(A.1) $\Phi_\varepsilon = (\phi_\varepsilon^1, \phi_\varepsilon^2, \cdots, \phi_\varepsilon^n) \in C^\infty(\overline{\Omega})^n$.

(A.2) $\Phi_0(x) = x \quad$ for all $x \in \overline{\Omega}$.

(A.3) There exists $S = (S^1, S^2, \cdots, S^n) \in C^\infty(\overline{\Omega})^n$ such that $K(x;\varepsilon) := \Phi_\varepsilon(x) - x - S(x)\varepsilon$ satisfies

$$\sup_{x\in\overline{\Omega}} |K(x;\varepsilon)| + \sup_{x\in\overline{\Omega}} |\nabla K(x;\varepsilon)| = O(\varepsilon^2) \quad \text{as } \varepsilon \to 0.$$

We next introduce some function spaces. The space $L^2_\sigma(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to the L^2-norm $\|\cdot\|_2$, and the space $H^1_{0,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to the H^1-norm $\|\cdot\|_{H^1}$, i.e., $\|\phi\|_{H^1} = \|\nabla\phi\|_2$. Here, the space $C^\infty_{0,\sigma}(\Omega)$ denotes the set of all C^∞ divergence free vector fields with compact support in Ω.

Let $\lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \cdots$ be the set of eigenvalues of (0.1) counting the multiplicity. For any natural number $k \in \mathbb{N}$, let $\{\lambda_k(\varepsilon)\}_{k=1}^\infty$ be the set of the eigenvalues of (0.1) arranged in the increasing order, and let $\{\boldsymbol{V}_{\varepsilon,k}, P_{\varepsilon,k}\}_{k=1}^\infty$ be the corresponding eigenfunctions for the velocity and the pressure, respectively, which is the complete orthogonal system in $L^2_\sigma(\Omega_\varepsilon)$. Note that

$$(\boldsymbol{V}_{\varepsilon,k}, \boldsymbol{V}_{\varepsilon,l}) = \delta(k,l) = \begin{cases} 1 & (k = l), \\ 0 & (l \ne l), \end{cases} \qquad \text{(Kronecker delta).} \tag{1.1}$$

Moreover, for every $k \in \mathbb{N}$, we abbreviate

$$\lambda_k = \lambda_k(0), \qquad V_k = V_{0,k}, \qquad P_k = P_{0,k}, \tag{1.2}$$

and the multiplicity of the eigenvalue λ_k is denoted by n_k, i.e., $n_k = \dim N(\lambda_k)$, where $N(\lambda_k)$ is the eigenspace corresponding to λ_k. Without loss of generality, we may assume that

$$\cdots \le \lambda_{k-1} < \lambda_k = \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{k+n_k-1} < \lambda_{k+n_k} \le \cdots. \tag{1.3}$$

We study the perturbation of the eigenvalues λ_l $(k \le l \le k + n_k - 1)$.

Now we can state our result.

Theorem 1.1. *Let $\lambda_1(\varepsilon) \le \lambda_2(\varepsilon) \le \dots$ be the eigenvalues of (0.1) counting the multiplicity. Then the following limit value*

$$\lim_{\varepsilon \downarrow 0} \frac{\lambda_l(\varepsilon) - \lambda_l}{\varepsilon} \quad (:= \delta\lambda_l)$$

exists for every $k \le l \le k + n_k - 1$, and agrees in increasing order to the eigenvalues of the real symmetric matrix

$$\left(- \int_{\partial\Omega} \sum_{i=1}^{n} \frac{\partial V_{l_1}^i}{\partial \nu_x}(x) \frac{\partial V_{l_2}^i}{\partial \nu_x}(x)(S(x) \cdot \nu_x)\, d\sigma_x \right)_{k \le l_1, l_2 \le k+n_k-1} .$$

Here $\{V_l\}_{l=k}^{k+n_k-1}$ is a corresponding orthonormal system of the eigenspace of λ_k. The unit outer normal vector to $\partial\Omega$ at $x \in \partial\Omega$ is denoted by $\nu_x = (\nu_x^1, \cdots, \nu_x^d)$, and σ_x is the surface element of $\partial\Omega$. In particular, if λ_k is a simple eigenvalue, then it holds that

$$\delta\lambda_k = - \int_{\partial\Omega} \sum_{i=1}^{n} \left(\frac{\partial V_k^i}{\partial \nu_x}(x) \right)^2 (S(x) \cdot \nu_x)\, d\sigma_x .$$

Remark 1.1. *The case of $\varepsilon \uparrow 0$ can be handled in the same way. However, the limit value $\varepsilon^{-1}(\lambda_l(\varepsilon) - \lambda_l)$ for $\varepsilon \uparrow 0$ and $\varepsilon \downarrow 0$ may not coincide.*

Remark 1.2. *The monotonicity of the eigenvalue for the domain perturbation follows, i.e., we have that if $S(x) \cdot \nu_x \ge 0$ for all $x \in \partial\Omega$, then it holds that $\delta\lambda_l \le 0$.*

2 0-th Order approximation of the eigenvalue.

In this section, we consider the continuity of the eigenvalue $\lambda_k(\varepsilon)$ with respect to ε. Indeed, it holds that;

Lemma 2.1. *Let $\lambda_k(\varepsilon)$ be the k-th eigenvalue of the Stokes equations (0.1) in Ω_ε. Then it holds that*

$$\lim_{\varepsilon \to 0} \lambda_k(\varepsilon) = \lambda_k \qquad k \in \mathbb{N}. \tag{2.1}$$

Remark 2.1. *We take a limit of the left hand side of (2.1) as $\varepsilon \to 0$ both from below and above.*

The following Min-Max principle plays an important role to prove Lemma 2.1.

Proposition 2.1. *For any natural number $k \in \mathbb{N}$, let $\lambda_k(\varepsilon)$ be the k-th eigenvalue of the Stokes equations (0.1) in Ω_ε. For each $k \in \mathbb{N}$,*

$$\lambda_k(\varepsilon) = \sup_{\dim X \le k-1,\ X \subset L^2_\sigma(\Omega_\varepsilon)} \left(\inf\{ R_\varepsilon(\boldsymbol{u}) \ ; \ \boldsymbol{u} \in H^1_{0,\sigma}(\Omega_\varepsilon),\ \boldsymbol{u} \ne 0,\ \boldsymbol{u} \in X^\perp \} \right),$$

where $X^{\perp} := \{\boldsymbol{u} \in L^2_\sigma(\Omega_\varepsilon)\ ;\ (\boldsymbol{\psi}, \boldsymbol{u})_{L^2_\sigma(\Omega_\varepsilon)} = 0,\ \boldsymbol{\psi} \in X\}$ *and the functional* R_ε *is defined by*

$$R_\varepsilon(\boldsymbol{u}) := \left(\int_{\Omega_\varepsilon} \sum_{i,j=1}^{n} 2|e^{ij}(\boldsymbol{u})|^2\, d\tilde{x}\right) \Big/ \left(\int_{\Omega_\varepsilon} \sum_{i=1}^{n} |u^i|^2\, d\tilde{x}\right).$$

with

$$e^{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial u^i}{\partial x^j} + \frac{\partial u^j}{\partial x^i}\right), \qquad i, j = 1, \cdots, n.$$

Remark 2.2. *The functional* R_ε *is well known as the Rayleigh quotient. (cf. Evans [2])*

We next introduce the several useful identities about the expansion of the diffeomorphism Φ_ε in Assumption with respect to ε.

Proposition 2.2. *Let* Φ_ε *be as in Assumption. Suppose that* $\{a_{\varepsilon,ij}\}_{i,j=1,\cdots,n}$ *and* $\{a_\varepsilon^{ij}\}_{i,j=1,\cdots,n}$ *are defined by*

$$a_\varepsilon^{ij} := \sum_{l=1}^{n} \frac{\partial x^i}{\partial \tilde{x}^l}\frac{\partial x^j}{\partial \tilde{x}^l}, \quad a_{\varepsilon,ij} := \sum_{l=1}^{n} \frac{\partial \tilde{x}^l}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}, \quad i, j = 1, \cdots, d, \tag{2.2}$$

respectively. Then it holds that

$$\frac{\partial \tilde{x}^i}{\partial x^j} = \delta(i,j) + \varepsilon\frac{\partial S^i}{\partial x^j} + O(\varepsilon^2), \quad \frac{\partial x^i}{\partial \tilde{x}^j} = \delta(i,j) - \varepsilon\frac{\partial S^i}{\partial x^j} + O(\varepsilon^2),$$
$$a_{\varepsilon,ij} = \delta(i,j) + \varepsilon\delta a_{ij} + O(\varepsilon^2), \quad a_\varepsilon^{ij} = \delta(i,j) + \varepsilon\delta a^{ij} + O(\varepsilon^2), \quad i, j = 1, \cdots, n,$$

as $\varepsilon \to 0$*, where* $\tilde{x} = \Phi_\varepsilon(x)$ *for* $x \in \overline{\Omega}$*,* $\{\delta a_{ij}\}_{i,j=1,\cdots,n}$ *and* $\{\delta a^{ij}\}_{i,j=1,\cdots,n}$ *are defined by*

$$\delta a_{ij} := \left(\frac{\partial S^i}{\partial x^j} + \frac{\partial S^j}{\partial x^i}\right), \quad \delta a^{ij} := -\left(\frac{\partial S^i}{\partial x^j} + \frac{\partial S^j}{\partial x^i}\right), \quad i, j = 1, \cdots, n.$$

Furthermore, the Jacobian J_ε *as*

$$J_\varepsilon(x) := \det\left(\frac{\partial \phi_\varepsilon^i}{\partial x^j}(x)\right)_{1 \le i,j \le n}, \qquad x \in \overline{\Omega} \tag{2.3}$$

is also expressed by

$$J_\varepsilon = 1 + \varepsilon\delta J + O(\varepsilon^2),$$

as $\varepsilon \to 0$*, where* δJ *is defined by*

$$\delta J := \sum_{i=1}^{n} \frac{\partial S^i}{\partial x^i}.$$

The proof is immediate consequence of (A.3). So we omit it.

Remark 2.3. *We immediately have by Proposition 2.2 that*

$$\frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^l} = O(\varepsilon), \quad \frac{\partial J_\varepsilon}{\partial x^i} = O(\varepsilon), \quad i,j,l = 1, \cdots, d, \quad \text{as } \varepsilon \to 0. \tag{2.4}$$

Furthermore, we introduce the *piola identity*, whose methods was based on Marsden-Hughes [13].

Lemma 2.2. *For a real parameter ε and any solenoidal vector function $\boldsymbol{U}_\varepsilon \in C^\infty(\Omega_\varepsilon)$, we define the vector function $\boldsymbol{u}_\varepsilon \in C^\infty(\Omega)$ by*

$$\boldsymbol{u}_\varepsilon(x) := J_\varepsilon \sum_{j=1}^n \frac{\partial x}{\partial \tilde{x}^j} U_\varepsilon^j(\tilde{x}) \quad \text{for } \tilde{x} \in \Omega_\varepsilon,$$

where $\tilde{x} = \Phi_\varepsilon(x)$ and J_ε is the Jacobian as in (2.3). Then, it holds that

$$\operatorname{div}_x \boldsymbol{u}_\varepsilon(x) = 0 \qquad \text{for all } x \in \Omega.$$

For the reference, see Marsden-Hughes [13, Chapter I, Sec.7.20].

We next consider the limit of the eigenfunction $V_{\varepsilon,k}$ for $\varepsilon \to 0$.

Lemma 2.3. *For any natural number $k \in \mathbb{N}$ and any arbitrary sequence $\{\varepsilon(m)\}_{m=1}^\infty$ satisfying $\varepsilon(m) \to 0$ as $m \to \infty$, there exist a subsequence $\{\tau(m)\}_{m=1}^\infty \subset \{\varepsilon(m)\}_{m=1}^\infty$ and a function $\boldsymbol{v}_{0,k} \in H_{0,\sigma}^1(\Omega)$ such that $\boldsymbol{v}_{\tau(m),k}$ strongly converges to $\boldsymbol{v}_{0,k}$ in $L_\sigma^2(\Omega)$ as $m \to \infty$, and $\boldsymbol{v}_{\tau(m),k}$ weakly converges to $\boldsymbol{v}_{0,k}$ in $H_{0,\sigma}^1(\Omega)$ as $m \to \infty$, where $\{\boldsymbol{v}_{\tau(m),k}\}_{m=1}^\infty$ is defined by*

$$\boldsymbol{v}_{\tau(m),k}(x) := J_\varepsilon \sum_{j=1}^n \frac{\partial x}{\partial \tilde{x}^j} V_{\tau(m)}^j(\tilde{x}) \quad \text{for } \tilde{x} \in \Omega_\varepsilon.$$

Furthermore, there exists $p_{0,k} \in H^1(\Omega)$ such that

$$-\Delta_x \boldsymbol{v}_{0,k}(x) + \nabla_x p_{0,k}(x) = \lambda_k \boldsymbol{v}_{0,k}(x) \tag{2.5}$$

for all $x \in \Omega$.

For the proof, see Jimbo-Ushikoshi [8].

3 Construction of the representation formula.

In this section, we prove the main theorem 1.1.

3.1 Expression by the integral form.

In this subsection, we assure the existence of $\delta\lambda_k$, and construct the expression for $\delta\lambda_k$. For that purpose, we introduce the following useful identity.

Lemma 3.1. *Let $\{\boldsymbol{V}_p\}_{k\le p\le k+n_k-1}$ be the eigenfunctions corresponding to the eigenvalue λ_k (see (1.2)). Then it holds that*

$$\int_\Omega \left(\delta F(\boldsymbol{V}_p,\boldsymbol{V}_q)(x) + \lambda_l \sum_{i,j=1}^n (\delta a^{ij} - \delta(i,j)\delta J) V_p^i(x) V_q^j(x)\right) dx \tag{3.1}$$
$$= \int_{\partial\Omega} \sum_{i=1}^n \left(\frac{\partial V_p^i}{\partial \nu_x}(x)\frac{\partial V_q^i}{\partial \nu_x}(x)\right)(S(x)\cdot\nu_x)\, d\sigma_x$$

for $k \le l,p,q \le k+n_k-1$, where the bilinear operator δF is defined by

(3.2)

$$\delta F(\boldsymbol{u},\boldsymbol{w})(x)$$
$$:= \sum_{i,j=1}^n \left\{ \sum_{l=1}^n \left(\delta a^{jl} \frac{\partial u^i}{\partial x^j}(x)\frac{\partial w^i}{\partial x^l}(x) + \frac{\partial}{\partial x^l}\left(\frac{\partial S^i}{\partial x^j} u^j(x)\right)\frac{\partial w^i}{\partial x^l} + \frac{\partial u^i}{\partial x^l}\frac{\partial}{\partial x^l}\left(\frac{\partial S^i}{\partial x^j} w^j(x)\right)\right)\right.$$
$$\left. + \delta J \frac{\partial u^i}{\partial x^j}(x)\frac{\partial w^i}{\partial x^j}(x) + \frac{\partial}{\partial x^j}\left(\delta J u^i(x)\right)\frac{\partial w^i}{\partial x^j} + \frac{\partial u^i}{\partial x^j}\frac{\partial}{\partial x^j}\left(\delta J w^i(x)\right)\right\}$$

with the variable coefficient $\{\delta a^{ij}\}_{i,j=1,\cdots,n}$ and δJ as in Proposition 2.2. Moreover, $\nu_x = (\nu_x^1,\cdots,\nu_x^n)$ is the unit outer normal to $\partial\Omega$ at $x\in\partial\Omega$, σ_x denotes the surface element of $\partial\Omega$ and $\{S^i\}_{i=1,\cdots,n}$ is the vector functions introduced by (A.3).

Proof. For any $k \le l,p,q \le k+n_k-1$, we prove Lemma 3.1 by integration by parts and applying Proposition 2.2 to the left hand side of (3.1). □

3.2 Proof of Theorem 1.1.

Since the eigenfunction $\{\boldsymbol{V}_{\varepsilon,l}, P_{\varepsilon,l}\}_{l_2=k}^{k+n_k-1}$ satisfies the Stokes equations, we see that for any number $k \le l_1, l_2 \le k+n_k-1$,

$$\int_{\Omega_\varepsilon} \sum_{i=1}^n \nabla_{\tilde{x}} V_{\varepsilon,l_1}^i(\tilde{x}) \nabla_{\tilde{x}} \eta_{\varepsilon,l_2}^i(\tilde{x})\, d\tilde{x} = \lambda_{l_1}(\varepsilon) \int_{\Omega_\varepsilon} \sum_{i=1}^n V_{\varepsilon,l_1}^i(\tilde{x}) \eta_{\varepsilon,l_2}^i(\tilde{x})\, d\tilde{x} \tag{3.3}$$

for arbitrary function $\eta_{\varepsilon,l_2} \in H_{0,\sigma}^1(\Omega_\varepsilon)$. By changing varables and by integration by parts of (3.3), Lemma 2.3 and Lemma 3.1 completes the proof of Theorem 1.1. □

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