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Variational formulae and estimates for decomposed Möbius energies

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1 Introduction

O'Hara proposed the shape optimization problem of knots in [8]: What is the canonical configuration of knot in its knot type ? To this problem he introduced the so-called *O'Hara's energy*

$$\mathcal{E}_{(\alpha,p)}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^3}^{\alpha}} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^{\alpha}} \right)^p ds_1 ds_2,$$

where $\boldsymbol{f}$ is a knots, *i.e.*, a closed curve without self-intersection in $\mathbb{R}^3$. $\mathcal{L}$ is total length of the closed curve, s_i's are arc-length parameters, and $\mathscr{D}$ is the distance along the curve. The exponents α and p are positive constants.

The energy has scaling invariance if and only of $\alpha p = 2$. Therefore in considering the variational problem for $\alpha p \neq 2$, we have to deal with it under length constrain. O'Hara showed in [9] that minimizers of $\mathcal{E}_{(\alpha,p)}$ (with the constraints when $\alpha p \neq 2$) exists in every knot types if and only if $\alpha p > 2$. In this sense the case $\alpha p = 2$ is critical.

Freedman-He-Wang [2] considered one of the critical cases $(\alpha, p) = (2, 1)$, and showed that the energy $\mathcal{E}_{(2,1)}$ has not only scaling invariance but also the invariance under the Möbius transformations. For this the energy $\mathcal{E}_{(2,1)}$ is called the *Möbius energy*. Furthermore they proved that there exists a minimizer in every *prime* knot type by using the Möbius invariance skillfully. Kusner-Sulliban [7] conjectured that there does not exists in every *composite* knot type, and it is still an open problem.

In this article the Möbius energy is dealt with, and we denote it simply by $\mathcal{E}$. The energy can be defined for a closed curve in $\mathbb{R}^n$:

$$\mathcal{E}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^{2}} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^{2}} \right) ds_1 ds_2.$$

Needless to say, for the study of variational problem, we must determine the proper class of functions where we work, and derive the variational formulae on it. Blatt [1] showed that $\mathcal{E}(\boldsymbol{f}) < \infty$ if and only if $\boldsymbol{f}$ belongs to $H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and has bi-Lipschitz continuity. It is known that the energy of this class of curves can be decomposed into three parts ([4]):

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4,$$

where

$$\begin{aligned}
\mathcal{E}_i(\boldsymbol{f}) &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f})\, ds_1 ds_2, \\
\mathscr{M}_1(\boldsymbol{f}) &= \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2}, \\
\mathscr{M}_2(\boldsymbol{f}) &= \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4} \\
&\qquad \times \det \begin{pmatrix} \boldsymbol{\tau}(s_1)\cdot\boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 \end{pmatrix},
\end{aligned}$$

and $\boldsymbol{\tau}$ is the unit tangent vector field along the curve. The first decomposed energy $\mathcal{E}_1$ is an analogue of the Gagliardo semi-norm of $\boldsymbol{\tau} = \boldsymbol{f}'$ in the fractional Sobolev space $H^{1/2}$. This implies the natural domain of $\mathcal{E}$ is $H^{3/2} \cap H^{1,\infty}$, which was shown by Blatt [1]. The integrand $\mathscr{M}_2$ of the second one has the determinant structure, which implies a cancellation of integrand. Note that each term in the density of $\mathcal{E}$ is not integrable.

The inveriance of $\mathcal{E}$ under the Möbius transformations implies that of $\mathcal{E}_1 + \mathcal{E}_2$. In [4] the Möbius invariance of each $\mathcal{E}_i$ has been proved.

We are interested in the first and second variational formulae of decomposed energies $\mathcal{E}_i$. The variational formulae for the Möbius energy were obtained by several authors, for example by He [3], in the integration of Caucy's principal value:

$$\lim_{\varepsilon \to +0} \iint_{|s_1 - s_2| \geqq \varepsilon} \cdots\, ds_1 ds_2.$$

In this article we study the absolute integrability of the first and second variational formulae in the proper domain of $\mathcal{E}$. Direct calculation produces a lot of terms which are not integrable even in the sense of Cauchy's principal value. By combining several terms appropriately, the integrability recovers, however, it is a quite hard job. Using the decomposition which was given above, we can calculate the variational formulae which have absolutely integrable integrands relatively easily and systematically. One can find their explicit expressions, and can show the estimates. Furthermore we derive the explicit expression of L^2-*gradient* of each decomposed energy. Since the only sketch of proof is given here, see joint papers [5, 6] with Aya Ishizeki of Saitama University for details.

2 The explicit expressions of variational formulae and estimates

Let $\mathscr{F}(\boldsymbol{f})$ be a geometric quantity determined by the closed curve $\boldsymbol{f}$, and let $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ be functions from $\mathbb{R}/\mathcal{L}\mathbb{Z}$ to $\mathbb{R}^n$. We use δ and δ^2 to mean

$$\delta\mathscr{F}(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{d}{d\varepsilon}\mathscr{F}(\boldsymbol{f}+\varepsilon\boldsymbol{\phi})\Big|_{\varepsilon=0},$$

$$\delta^2\mathscr{F}(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\mathscr{F}(\boldsymbol{f}+\varepsilon_1\boldsymbol{\phi}+\varepsilon_2\boldsymbol{\psi})\Big|_{\varepsilon_1=\varepsilon_2=0}.$$

The first variation $\mathscr{G}_i$ and the second variation $\mathscr{H}_i$ are given by

$$\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\,ds_1ds_2 = \delta(\mathscr{M}_i(\boldsymbol{f})\,ds_1ds_2)[\boldsymbol{\phi}],$$

$$\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]\,ds_1ds_2 = \delta^2(\mathscr{M}_i(\boldsymbol{f})\,ds_1ds_2)[\boldsymbol{\phi},\boldsymbol{\psi}].$$

Then we have the explicit expressions of variations. To give the statement, we introduce several notations. For a function $\boldsymbol{v}$ on $\mathbb{R}/\mathcal{L}\mathbb{Z}$, we put

$$\boldsymbol{v}_i = \boldsymbol{v}(s_i), \quad \Delta\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2.$$

The operations Q, $\tilde{Q}_i$, S and S_i are defined as

$$Q\boldsymbol{v} = \Delta\boldsymbol{v}', \quad \tilde{Q}_i\boldsymbol{v} = (-1)^{i-1}2\{\boldsymbol{v}_i' - R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v}\},$$

$$R\boldsymbol{v} = \frac{|\Delta s|\Delta\boldsymbol{v}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}\Delta s}, \quad \hat{R}\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v}_1'+\boldsymbol{v}_2'),$$

$$S(\boldsymbol{v},\boldsymbol{w}) = \hat{\boldsymbol{v}}\cdot Q\boldsymbol{w} + Q\boldsymbol{v}\cdot\hat{R}\boldsymbol{w}, \quad \tilde{S}_i(\boldsymbol{v},\boldsymbol{w}) = R\boldsymbol{v} + \tilde{Q}_i\boldsymbol{w} + \tilde{Q}_i\boldsymbol{v}\cdot R\boldsymbol{w},$$

Proposition 2.1 *We have*

$$\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{Q\boldsymbol{f}\cdot Q\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{2\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}},$$

$$\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}] = -\frac{\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}$$

and

$$\mathscr{H}_1(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] = \frac{Q\boldsymbol{\phi}\cdot Q\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{S(\boldsymbol{f},\boldsymbol{\phi})S(\boldsymbol{f},\boldsymbol{\psi})}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}$$
$$- \frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{2\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{2\mathscr{M}_1(\boldsymbol{f})\Delta\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}},$$

$$\begin{aligned}\mathscr{H}_2(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] &= -\frac{\tilde{Q}_1\boldsymbol{\phi}\cdot\tilde{Q}_2\boldsymbol{\psi}+\tilde{Q}_2\boldsymbol{\phi}\cdot\tilde{Q}_1\boldsymbol{\psi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\\ &+\frac{\tilde{S}_1(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_2(\boldsymbol{f},\boldsymbol{\psi})+\tilde{S}_2(\boldsymbol{f},\boldsymbol{\phi})\tilde{S}_1(\boldsymbol{f},\boldsymbol{\psi})}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\\ &-\frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-\frac{2\mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\psi}]\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}-\frac{2\mathscr{M}_2(\boldsymbol{f})\Delta\boldsymbol{\phi}\cdot\Delta\boldsymbol{\psi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}.\end{aligned}$$

This is proven as follows. Put

$$\mathscr{M}_i(\boldsymbol{f})=\frac{\mathscr{N}_i(\boldsymbol{f})}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},$$

then it is easy to see

$$\mathscr{N}_1(\boldsymbol{f})=\frac{1}{2}Q\boldsymbol{f}\cdot Q\boldsymbol{f},\quad \mathscr{N}_2(\boldsymbol{f})=-\frac{1}{2}\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{f}.$$

Therefore to obtain the expressions of $\mathscr{G}_i$ and $\mathscr{H}_i$, we need those of variations of $\mathscr{N}_i$, $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2$, and ds_1ds_2. Firstly we begin with the variations of basic quantities.

Lemma 2.1 *The following first variational formulae hold.*

1. $\delta\boldsymbol{\tau}[\boldsymbol{\phi}]=\boldsymbol{\phi}'-(\boldsymbol{\tau}\cdot\boldsymbol{\phi}')\boldsymbol{\tau}$.
2. $\delta\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2[\boldsymbol{\phi}]=2\Delta\boldsymbol{\tau}\cdot\Delta\boldsymbol{\phi}'-\|\Delta\boldsymbol{\tau}\|_{\mathbb{R}^n}^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')$.
3. $\delta\left(\frac{1}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2}\right)[\boldsymbol{\phi}]=-\frac{2\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^4}$.
4. $\delta(ds_j)[\boldsymbol{\phi}]=\boldsymbol{\tau}_j\cdot\boldsymbol{\phi}_j'\,ds_j$.

Since the proof is not difficult, we omit it. As a consequence of this lemma we obtain

$$\begin{aligned}\delta\mathscr{N}_1(\boldsymbol{f})[\boldsymbol{\phi}]&=Q\boldsymbol{f}\cdot Q\boldsymbol{\phi}-(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\mathscr{N}_1(\boldsymbol{f}),\\ \delta\mathscr{N}_2(\boldsymbol{f})[\boldsymbol{\phi}]&=-\frac{1}{2}(\tilde{Q}_1\boldsymbol{f}\cdot\tilde{Q}_2\boldsymbol{\phi}+\tilde{Q}_2\boldsymbol{f}\cdot\tilde{Q}_1\boldsymbol{\phi})-(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\mathscr{N}_2(\boldsymbol{f}).\end{aligned}$$

From Lemma 2.1,

$$\begin{aligned}\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\,ds_1ds_2&=\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}]\,ds_1ds_2+\mathscr{M}_i(\boldsymbol{f})\,\delta(ds_1ds_2)[\boldsymbol{\phi}]\\ &=\{\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}]+\mathscr{M}_i(\boldsymbol{f})(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2')\}\,ds_1ds_2,\end{aligned}$$

i.e.,

$$\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]=\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}]+\mathscr{M}_i(\boldsymbol{f})(\boldsymbol{\tau}_1\cdot\boldsymbol{\phi}_1'+\boldsymbol{\tau}_2\cdot\boldsymbol{\phi}_2').$$

Now we are in the position to calculate $\mathscr{G}_i$. Using Lemma 2.1 again, we have

$$\begin{aligned}\delta\mathscr{M}_i(\boldsymbol{f})[\boldsymbol{\phi}] &= \frac{\delta\mathscr{N}_i(\boldsymbol{f})[\boldsymbol{\phi}]}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}} + \mathscr{N}_i(\boldsymbol{f})\,\delta\left(\frac{1}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}\right)[\boldsymbol{\phi}] \\ &= \frac{\delta\mathscr{N}_i(\boldsymbol{f})[\boldsymbol{\phi}]}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}} - \frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|^2_{\mathbb{R}^n}}.\end{aligned}$$

Inserting the expressions of $\delta\mathscr{N}_i$, we obtain Proposition 2.1.

Similarly we can obtain the expressions of $\mathscr{H}_i$. See [5]. □

We can show estimates of variational formulae in several function spaces.

Theorem 2.1 *We put $X = H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, and $Y = H^{\frac{1}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap L^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Assume that there exists a positive constant λ such that $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geqq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))|$.*

1. *If $\boldsymbol{f}$, $\boldsymbol{\phi}$, and $\boldsymbol{\psi} \in X$, then $\mathscr{M}_i(\boldsymbol{f})$, $\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}]$, and $\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]$ belong to $L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Furthermore there exists a positive constant C depending on $\|\boldsymbol{f}'\|_Y$ and λ such that*

$$\|\mathscr{M}_i(\boldsymbol{f})\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C,$$

$$\|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{\phi}'\|_Y,$$

$$\|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]\|_{L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{\phi}'\|_Y\|\boldsymbol{\psi}'\|_Y.$$

2. *If $\boldsymbol{f}$, $\boldsymbol{\phi}$, and $\boldsymbol{\psi} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathscr{M}_i(\boldsymbol{f})$, $\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}]$, and $\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]$ belong to $L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Furthermore there exists a positive constant C depending on $\|\boldsymbol{f}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{R})}$, λ and $\mathcal{L}$ such that*

$$\|\mathscr{M}_i(\boldsymbol{f})\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C,$$

$$\|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{\phi}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{R})},$$

$$\|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{\phi}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{R})}\|\boldsymbol{\psi}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{R})}.$$

3. *If $\boldsymbol{f}$, $\boldsymbol{\phi}$, and $\boldsymbol{\psi} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then $\mathscr{M}_i(\boldsymbol{f})$, $\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}]$, $\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]$ can be extended on the diagonal set $\{(s_1, s_2) \,|\, s_1 \equiv s_2 \pmod{\mathcal{L}\mathbb{Z}}\}$ such that these functions are continuous everywhere. The limits of sum vanish on the diagonal set:*

$$\lim_{(s_1,s_2)\to(s,s)} (\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})) = 0,$$

$$\lim_{(s_1,s_2)\to(s,s)} (\mathscr{G}_1(\boldsymbol{f})[\boldsymbol{\phi}] + \mathscr{G}_2(\boldsymbol{f})[\boldsymbol{\phi}]) = 0,$$

$$\lim_{(s_1,s_2)\to(s,s)} (\mathscr{H}_1(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}] + \mathscr{H}_2(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]) = 0.$$

Furthermore there exists a positive constant C depending on $\|\boldsymbol{f}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{R})}$, λ and $\mathcal{L}$ such that

$$\|\mathscr{M}_i(\boldsymbol{f})\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C,$$

$$\|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{\phi}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{R})},$$

$$\|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi},\boldsymbol{\psi}]\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{\phi}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{R})}\|\boldsymbol{\psi}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{R})}.$$

Estimates in Theorem 2.1 follows from next lemma.

Lemma 2.2 *1. For $\boldsymbol{v} \in X$, the following estimate holds:*

$$\left\| \frac{Q\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq \|\boldsymbol{v}'\|_Y.$$

2. For $\boldsymbol{v} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, the following estimate holds:

$$\left\| \frac{Q\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq \|\boldsymbol{v}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

3. Assume $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. If we set $Q\boldsymbol{v}|_{s=s_1=s_2} = \boldsymbol{v}''$, then $Q\boldsymbol{v}$ is continuous everywhere and

$$\left\| \frac{Q\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq \|\boldsymbol{v}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

4. Assume that $\boldsymbol{f} \in X$ and that $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geqq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))|$. Then there exists a positive constant C depending on $\|\boldsymbol{f}'\|_Y$ and λ such that

$$\left\| \frac{\tilde{Q}_i\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{v}'\|_Y$$

holds for all $\boldsymbol{v} \in X$.

5. Assume that $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and that $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geqq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))|$. Then there exists a positive constant C depending on $\|\boldsymbol{f}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}$, λ, and $\mathcal{L}$ such that

$$\left\| \frac{\tilde{Q}_i\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{v}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}$$

holds for all $\boldsymbol{v} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

6. *Assume that* $\boldsymbol{f} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ *and that* $\boldsymbol{f}$ *has no self-intersections. For* $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, $\tilde{Q}_i\boldsymbol{v}$ *is continuous everywhere by setting* $\tilde{Q}_i\boldsymbol{v}\Big|_{s=s_1=s_2} = \boldsymbol{v}''$. *If we further assume that* $\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n} \geqq \lambda^{-1}|\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))|$, *then there exists a positive constant* C *depending on* $\|\boldsymbol{f}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}$, λ, *and* $\mathcal{L}$ *such that*

$$\left\| \frac{\tilde{Q}_i\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} \leqq C\|\boldsymbol{v}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}$$

holds for all $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Proof. Without loss of generality, we may assume that $|s_1 - s_2| \leqq \frac{\mathcal{L}}{2}$, and we use $|\Delta s|$ instead of $\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))$ for simplicity. The assetion for $Q\boldsymbol{v}$ is almost the definition of norms. Indeed, we immediately have

$$\left\| \frac{Q\boldsymbol{v}}{\Delta s} \right\|_{L^2((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} = [\boldsymbol{v}']_{H^{\frac{1}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})} \leqq \|\boldsymbol{v}'\|_Y$$

and

$$\left\| \frac{Q\boldsymbol{v}}{\Delta s} \right\|_{L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} = \|\boldsymbol{v}'\|_{\mathrm{Lip}} \leqq \|\boldsymbol{v}'\|_{C^{0,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

If $\boldsymbol{v} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then it is easy to see

$$\lim_{(s_1,s_2)\to(s,s)} \frac{Q\boldsymbol{v}}{\Delta s} = \boldsymbol{v}''(s),$$

and

$$\left\| \frac{Q\boldsymbol{v}}{\Delta s} \right\|_{C^0((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)} = \max_{|\Delta s|\leqq\frac{\mathcal{L}}{2}} \left\| \frac{1}{s_1 - s_2}\int_{s_2}^{s_1} \boldsymbol{v}''(s)\,ds \right\|_{\mathbb{R}^n} \leqq \|\boldsymbol{v}'\|_{C^1(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

To show the assetion for $\tilde{Q}_i\boldsymbol{v}$, we decompose $\frac{(-1)^{i-1}}{2}\tilde{Q}_i\boldsymbol{v} = \boldsymbol{v}_i' - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v}$ into

$$\boldsymbol{v}_i' - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v} = \left(\boldsymbol{v}_i' - \frac{\Delta\boldsymbol{v}}{\Delta s}\right) + \left(\frac{\Delta\boldsymbol{v}}{\Delta s} - R\boldsymbol{v}\right) + (1 - R\boldsymbol{f}\cdot\boldsymbol{\tau}_i)R\boldsymbol{v} = V_1 + V_2 + V_3.$$

We must show L^2, L^∞ and C^0 estimates for each $V_i/\Delta s$. Since these are rather complicated, see readers should refer [5]. □

We now give the proof of Theorem 2.1. Let $\bar{Q}$ be Q or $\tilde{Q}_i$. Then we have

$$|\mathscr{M}_i(\boldsymbol{f})| \leqq \frac{\lambda^2}{2}\left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1),\boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n},$$

for both $i = 1, 2$. Similarly, from Proposition 2.1, it derived that

$$
\begin{aligned}
|\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]| &\leqq \lambda^2 \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \\
&\quad + 2\lambda |\mathscr{M}_i(\boldsymbol{f})| \left\| \frac{\Delta\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \\
&\leqq \lambda^2 \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \\
&\quad + 2\lambda |\mathscr{M}_i(\boldsymbol{f})| \|\boldsymbol{\phi}\|_{\mathrm{Lip}}.
\end{aligned}
$$

Let $\bar{R}$ be $\hat{R}$ or R, and let $\bar{S}$ be S or $\tilde{S}_i$. Then the definition of these operations yields

$$
\begin{aligned}
&\left| \frac{\bar{S}(\boldsymbol{v}, \boldsymbol{w})}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right| \\
&\leqq \|\bar{R}\boldsymbol{v}\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{w}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\bar{R}\boldsymbol{w}\|_{\mathbb{R}^n} \\
&\leqq \lambda \left(\|\boldsymbol{v}\|_{\mathrm{Lip}} \left\| \frac{\bar{Q}\boldsymbol{w}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{v}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\boldsymbol{w}\|_{\mathrm{Lip}} \right).
\end{aligned}
$$

Therefore, Proposition 2.1 implies

$$
\begin{aligned}
&|\mathscr{H}_i(\boldsymbol{f})[\boldsymbol{\phi}, \boldsymbol{\psi}]| \\
&\leqq \lambda^2 \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \left\| \frac{\bar{Q}\boldsymbol{\psi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \\
&\quad + \lambda^4 \left(\|\boldsymbol{f}\|_{\mathrm{Lip}} \left\| \frac{\bar{Q}\boldsymbol{\phi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\boldsymbol{\phi}\|_{\mathrm{Lip}} \right) \\
&\qquad \times \left(\|\boldsymbol{f}\|_{\mathrm{Lip}} \left\| \frac{\bar{Q}\boldsymbol{\psi}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} + \left\| \frac{\bar{Q}\boldsymbol{f}}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))} \right\|_{\mathbb{R}^n} \|\boldsymbol{\psi}\|_{\mathrm{Lip}} \right) \\
&\quad + 2\lambda^2 |\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]| \|\boldsymbol{\psi}\|_{\mathrm{Lip}} + 2\lambda^2 |\mathscr{G}_i(\boldsymbol{f})[\boldsymbol{\psi}]| \|\boldsymbol{\phi}\|_{\mathrm{Lip}} \\
&\quad + 2\lambda^2 |\mathscr{M}_i(\boldsymbol{f})| \|\boldsymbol{\phi}\|_{\mathrm{Lip}} \|\boldsymbol{\psi}\|_{\mathrm{Lip}}.
\end{aligned}
$$

Consequently, the estimates in Theorem 2.1 are easily derived from Lemma 2.2. If $\boldsymbol{f} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, then Lemma 2.2 yields

$$
\lim_{(s_1, s_2) \to (s, s)} (\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f})) = \frac{1}{2} \|\boldsymbol{f}''(s)\|_{\mathbb{R}^n}^2 - \frac{1}{2} \|\boldsymbol{f}''(s)\|_{\mathbb{R}^n}^2 = 0.
$$

Similarly, we can show that both the limits of $\mathscr{G}_1(\boldsymbol{f}) + \mathscr{G}_2(\boldsymbol{f})$ and $\mathscr{H}_1(\boldsymbol{f}) + \mathscr{H}_2(\boldsymbol{f})$ vanish. □

3 The L^2-gradient

Theorem 2.1 shows that $\delta\mathcal{E}_i(\boldsymbol{f})[\cdot]$ is a linear form on the space $X = H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap W^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. If $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$, it seems that the first variation can be expanded into $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ as a linear form by integration by parts formally.

Indeed, the principal term of $\delta\mathcal{E}_1(\boldsymbol{f})$ is

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{(\boldsymbol{f}'(s_1) - \boldsymbol{f}'(s_2)) \cdot (\boldsymbol{\phi}'(s_1) - \boldsymbol{\phi}'(s_2))}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} ds_1 ds_2.$$

By bi-Lipschitz continuity we replace the denominator with $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2$ and then

$$\begin{aligned} &\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{(\boldsymbol{f}'(s_1) - \boldsymbol{f}'(s_2)) \cdot (\boldsymbol{\phi}'(s_1) - \boldsymbol{\phi}'(s_2))}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} ds_1 ds_2 \\ &\qquad = 2\pi \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})} (-\Delta_s)^{\frac{1}{4}} \boldsymbol{f}' \cdot (-\Delta_s)^{\frac{1}{4}} \boldsymbol{\phi}' \, ds. \end{aligned}$$

Here $\Delta_s = \partial_s^2$ is the Laplace operator with respecto to s, not $\Delta s = s_1 - s_2$. Integrating by parts formally, we obtain

$$2\pi \int_{(\mathbb{R}/\mathcal{L}\mathbb{Z})} (-\Delta_s)^{\frac{3}{2}} \boldsymbol{f} \cdot \boldsymbol{\phi} \, ds.$$

It seems to be meaningful for $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $\boldsymbol{\phi} \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Indeed we can justify this not only for the principal term but also all terms, including $\delta\mathcal{E}_2$.

Here we define a new operation T_i^k in order to describe the L^2-gradient of $\delta\mathcal{E}_i$ as

$$T_i^k \boldsymbol{f} := \left(\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}} \right)^k \frac{\Delta \boldsymbol{f}}{\Delta s} - \boldsymbol{\tau}_i.$$

Theorem 3.1 *Let $\boldsymbol{f} \in H^3$ and be bi-Lipschitz. Then for $\boldsymbol{\phi} \in L^2$, it holds that*

$$\delta\mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \langle L_i \boldsymbol{f} + \boldsymbol{N}_i(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2},$$

where

$$\begin{aligned} L_1 \boldsymbol{f} &= 2\pi(-\Delta_s)^{\frac{3}{2}} \boldsymbol{f} - 4 \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \operatorname{si}(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \varphi_k + \frac{8}{\mathcal{L}} \Delta_s (\boldsymbol{f} - \check{\boldsymbol{f}}), \\ L_2 \boldsymbol{f} &= -\frac{4}{3}\pi(-\Delta_s)^{\frac{3}{2}} \boldsymbol{f} + \frac{8}{3} \sum_{k \in \mathbb{Z}} \left| \frac{2\pi k}{\mathcal{L}} \right|^3 \operatorname{si}(|k\pi|) \langle \boldsymbol{f}, \varphi_k \rangle_{L^2} \varphi_k \\ &\quad + \frac{16}{3\mathcal{L}} \Delta_s \check{\boldsymbol{f}} + \frac{128}{3\mathcal{L}^3} (\boldsymbol{f} - \check{\boldsymbol{f}}), \end{aligned}$$

$$\mathrm{si}(t) = -\int_t^\infty \frac{\sin\lambda}{\lambda}\,d\lambda, \quad \varphi_k(s) = \frac{1}{\mathcal{L}}\exp\left(\frac{2\pi i k s}{\mathcal{L}}\right), \quad \check{\boldsymbol{f}}(s) = \boldsymbol{f}(s+\tfrac{\mathcal{L}}{2}),$$

$$\begin{aligned}
\boldsymbol{N}_1(\boldsymbol{f})(s_1) = &-2\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{2}{(\Delta s)^2}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)\Delta\boldsymbol{\tau} - \mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}_1 \right\} ds_2 \\
&-4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\mathscr{M}_1(\boldsymbol{f})}{\Delta s}T_1^2\boldsymbol{f} + \frac{1}{\Delta s}\left\{ \mathscr{M}_1(\boldsymbol{f}) - \frac{1}{2}\|\boldsymbol{\kappa}_1\|_{\mathbb{R}}^2 \right\}\boldsymbol{\tau}_1 \right] ds_2, \\
\boldsymbol{N}_2(\boldsymbol{f})(s_1) = &-4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} \left\{ (T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}_1)T_2^0\boldsymbol{f} + (T_2^2\boldsymbol{f}\cdot\boldsymbol{\tau}_2)T_1^0\boldsymbol{f} \right\} ds_2 \\
&-4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)^3} \left[(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1)T_2^0\boldsymbol{f} + (T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}_2)T_1^4\boldsymbol{f} \right. \\
&\qquad \left. +2\left\{ (T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}_2)+1 \right\}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}_1)T_1^4\boldsymbol{f} \right] ds_2 \\
&-4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \frac{1}{(\Delta s)^3} \left[T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}_1 - T_2^4\boldsymbol{f}\cdot\boldsymbol{\tau}_2 \right. \\
&\qquad +2\left\{ (T_2^0\boldsymbol{f}\cdot\boldsymbol{\tau}_2)+1 \right\}(T_1^2\boldsymbol{f}\cdot\boldsymbol{\tau}_1) + T_1^0\boldsymbol{f}\cdot\boldsymbol{\tau}_2 \\
&\qquad \left. -\frac{(\Delta s)^2}{6}\|\boldsymbol{\kappa}_1\|_{\mathbb{R}^n}^2 \right] \boldsymbol{\tau}_1 ds_2 \\
&-4\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left[\frac{\mathscr{M}_2(\boldsymbol{f})}{\Delta s}T_1^2\boldsymbol{f} + \frac{1}{\Delta s}\left\{ \mathscr{M}_2(\boldsymbol{f}) + \frac{1}{2}\|\boldsymbol{\kappa}_1\|_{\mathbb{R}^n}^2 \right\}\boldsymbol{\tau}_1 \right] ds_2.
\end{aligned}$$

Furthermore it holds for $\alpha \in (0, \frac{1}{2})$ *that*

$$\|\boldsymbol{N}_i(\boldsymbol{f})\|_{L^2} \leqq C_\alpha(\|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Our strategy for proving the above theorem is as follows. Since $C^\infty(\mathbb{R}/\mathcal{L}\mathbb{Z})$ is dense both in $H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$, we may assume $\boldsymbol{f}$ and $\boldsymbol{\phi}$ are sufficiently smooth. According to the previous section, the first variation $\delta\mathcal{E}_i(\cdot)[\cdot]$ can be expressed as

$$\begin{aligned}
\delta\mathcal{E}_i(\boldsymbol{f})[\boldsymbol{\phi}] &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathcal{G}_i(\boldsymbol{f})[\boldsymbol{\phi}]\,ds_1 ds_2 \\
&= \sum_{j=1}^{2} \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{ij}(\boldsymbol{f},\boldsymbol{\phi})(s_1,s_2)\,ds_1 ds_2,
\end{aligned}$$

where

$$\begin{aligned}
G_{i1}(\boldsymbol{f},\boldsymbol{\phi}) &= \frac{Q_{i1}\boldsymbol{f}\cdot Q_{i2}\boldsymbol{\phi} + Q_{i2}\boldsymbol{f}\cdot Q_{i1}\boldsymbol{\phi}}{2\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2} \\
G_{i2}(\boldsymbol{f},\boldsymbol{\phi}) &= -\frac{2\mathscr{M}_i(\boldsymbol{f})\Delta\boldsymbol{f}\cdot\Delta\boldsymbol{\phi}}{\|\Delta\boldsymbol{f}\|_{\mathbb{R}^n}^2},
\end{aligned}$$

$$Q_{11} = Q_{12} = Q, \quad Q_{2j}\boldsymbol{v} = 2\{\boldsymbol{v}_j' - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_j)R\boldsymbol{v}\}.$$

We decompose these operations Q_{ij} as $Q_{ij}\boldsymbol{v} = \tilde{Q}_{ij}\boldsymbol{v} + \bar{Q}_{ij}\boldsymbol{v}$, where

$$\tilde{Q}_{1j} = Q_{1j} = Q, \quad \bar{Q}_{1j} = 0, \quad \tilde{Q}_{2j}\boldsymbol{v} = 2\left(\boldsymbol{v}_j' - \frac{\Delta \boldsymbol{v}}{\Delta s}\right),$$

$$\bar{Q}_{2j}\boldsymbol{v} = 2\left\{\frac{\Delta \boldsymbol{v}}{\Delta s} - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_j)R\boldsymbol{v}\right\} = 2\left\{1 - (R\boldsymbol{f}\cdot\boldsymbol{\tau}_j)\frac{|\Delta s|}{\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}}\right\}\frac{\Delta \boldsymbol{v}}{\Delta s},$$

and then we have

$$G_{i1}(\boldsymbol{f},\boldsymbol{\phi}) = \sum_{k=1}^{3} G_{i1k}(\boldsymbol{f},\boldsymbol{\phi}),$$

$$G_{i11}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{\tilde{Q}_{i1}\boldsymbol{f}\cdot\tilde{Q}_{i2}\boldsymbol{\phi} + \tilde{Q}_{i2}\boldsymbol{f}\cdot\tilde{Q}_{i1}\boldsymbol{\phi}}{2(\Delta s)^2},$$

$$G_{i12}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{1}{2}\mathscr{M}(\boldsymbol{f})(\tilde{Q}_{i1}\boldsymbol{f}\cdot\tilde{Q}_{i2}\boldsymbol{\phi} + \tilde{Q}_{i2}\boldsymbol{f}\cdot\tilde{Q}_{i1}\boldsymbol{\phi}),$$

$$G_{i13}(\boldsymbol{f},\boldsymbol{\phi}) = \frac{\tilde{Q}_{i1}\boldsymbol{f}\cdot\bar{Q}_{i2}\boldsymbol{\phi} + \bar{Q}_{i1}\boldsymbol{f}\cdot\tilde{Q}_{i2}\boldsymbol{\phi} + \bar{Q}_{i1}\boldsymbol{f}\cdot\bar{Q}_{i2}\boldsymbol{\phi}}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2} + \frac{\tilde{Q}_{i2}\boldsymbol{f}\cdot\bar{Q}_{i1}\boldsymbol{\phi} + \bar{Q}_{i2}\boldsymbol{f}\cdot\tilde{Q}_{i1}\boldsymbol{\phi} + \bar{Q}_{i2}\boldsymbol{f}\cdot\bar{Q}_{i1}\boldsymbol{\phi}}{2\|\Delta \boldsymbol{f}\|_{\mathbb{R}^n}^2}.$$

G_{i11} is linear with respect to $\boldsymbol{f}$, however, G_{i12}, G_{i13} and G_{i2} are not. Then we would like to write

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{i11}(\boldsymbol{f},\boldsymbol{\phi})\,ds_1ds_2 = \langle L_i\boldsymbol{f},\boldsymbol{\phi}\rangle_{L^2},$$

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} (G_{i12}(\boldsymbol{f},\boldsymbol{\phi}) + G_{i13}(\boldsymbol{f},\boldsymbol{\phi}) + G_{i2}(\boldsymbol{f},\boldsymbol{\phi}))\,ds_1ds_2 = \langle \boldsymbol{N}_i(\boldsymbol{f}),\boldsymbol{\phi}\rangle_{L^2},$$

where L_i and $\boldsymbol{N}_i$ are linear and nonlinear operations from H^3 to L^2 and to estimate them.

For the linear parts L_i we use the Fourier expansion

$$\boldsymbol{f} = \sum_{k\in\mathbb{Z}} \varphi_k \boldsymbol{a}_k, \quad \boldsymbol{\phi} = \sum_{k\in\mathbb{Z}} \varphi_k \boldsymbol{b}_k,$$

where $\{\varphi_k\}$ is a complete orthogonal basis of $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Using

$$\varphi_k'(s) = \frac{2\pi ik}{\mathcal{L}}\varphi_k(s) \quad \text{and} \quad \varphi_k(s+h) = \exp\left(\frac{2\pi ikh}{\mathcal{L}}\right)\varphi_k(s),$$

we have

$$\boldsymbol{f}'(s_1) - \boldsymbol{f}'(s_1+h) = \sum_{k\in\mathbb{Z}} \frac{2\pi ik}{\mathcal{L}}\left\{1 - \exp\left(\frac{2\pi ikh}{\mathcal{L}}\right)\right\}\varphi_k(s_1)\boldsymbol{a}_k,$$

$$\boldsymbol{\phi}'(s_1) - \boldsymbol{\phi}'(s_1+h) = \sum_{k\in\mathbb{Z}} \frac{2\pi ik}{\mathcal{L}}\left\{1 - \exp\left(\frac{2\pi ikh}{\mathcal{L}}\right)\right\}\varphi_k(s_1)\boldsymbol{b}_k.$$

Consequentlty the orthogonality implies

$$
\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{111}(\boldsymbol{f},\boldsymbol{\phi})\,ds_1ds_2 \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}} \frac{(\boldsymbol{f}'(s_1)-\boldsymbol{f}'(s_1+h))\cdot(\boldsymbol{\phi}'(s_1)-\boldsymbol{\phi}'(s_1+h))}{h^2}\,dhds_1 \\
&= 2\sum_{k\in\mathbb{Z}}\left(\frac{2\pi k}{\mathcal{L}}\right)^2\left\{\int_{-\frac{\mathcal{L}}{2}}^{\frac{\mathcal{L}}{2}}\frac{1-\cos\left(\frac{2\pi kh}{\mathcal{L}}\right)}{h^2}\,dh\right\}ds_1\langle\boldsymbol{a}_k,\boldsymbol{b}_k\rangle_{\mathbb{C}} \\
&= 2\sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^3\left(\int_{-\pi|k|}^{\pi|k|}\frac{1-\cos\lambda}{\lambda^2}\,d\lambda\right)\langle\boldsymbol{a}_k,\boldsymbol{b}_k\rangle_{\mathbb{C}}.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_{211}(\boldsymbol{f},\boldsymbol{\phi})\,ds_1ds_2 \\
&= \sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^3\left[\int_{-\pi|k|}^{\pi|k|}\frac{4\{\lambda^2\cos\lambda-2\lambda\sin\lambda+2(1-\cos\lambda)\}}{\lambda^4}\,d\lambda\right]\langle\boldsymbol{a}_k,\boldsymbol{b}_k\rangle_{\mathbb{C}}.
\end{aligned}
$$

Thus we arrive at

$$
L_i\boldsymbol{f} = \sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^3\left(\int_{-\pi|k|}^{\pi|k|} z_i(\lambda)\,d\lambda\right)\langle\boldsymbol{f},\varphi_k\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}\varphi_k,
$$

where

$$
z_1(\lambda) = \frac{2(1-\cos\lambda)}{\lambda^2},\quad z_2(\lambda) = \frac{4\{\lambda^2\cos\lambda-2\lambda\sin\lambda+2(1-\cos\lambda)\}}{\lambda^4}.
$$

It is not difficult to see

$$
\int_{-\pi|k|}^{\pi|k|} z_i(\lambda)\,d\lambda = a_i\left(\pi+2\mathrm{si}(|k\pi|)\right)+2b_i|k\pi|z_i(|k\pi|),
$$

where

$$
a_1 = 2,\quad b_1 = -1,\quad a_2 = -\frac{4}{3},\quad b_2 = -\frac{1}{3}.
$$

From the definition of z_i it follows

$$
\begin{aligned}
|k\pi|z_1(|k\pi|) &= \frac{2\{1-\cos(|k\pi|)\}}{|k\pi|} = \frac{2\{1-(-1)^k\}}{|k\pi|}, \\
|k\pi|z_2(|k\pi|) &= \frac{4[-|k\pi|^2\cos(|k\pi|)+2|k\pi|\sin(|k\pi|)-2\{1-\cos(|k\pi|)\}]}{|k\pi|^3} \\
&= \frac{4[-(-1)^k|k\pi|^2-2\{1-(-1)^k\}]}{|k\pi|^3}.
\end{aligned}
$$

Combining these with

$$\sum_{k\in\mathbb{Z}}\langle \boldsymbol{f}, \varphi_k\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}\varphi_k = \boldsymbol{f},$$

$$\sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^2\langle \boldsymbol{f}, \varphi_k\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}\varphi_k = -\Delta_s\boldsymbol{f},$$

$$\sum_{k\in\mathbb{Z}}(-1)^k\langle \boldsymbol{f}, \varphi_k\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}\varphi_k = \check{\boldsymbol{f}},$$

$$\sum_{k\in\mathbb{Z}}\left|\frac{2\pi k}{\mathcal{L}}\right|^2(-1)^k\langle \boldsymbol{f}, \varphi_k\rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}\varphi_k = -\Delta_s\check{\boldsymbol{f}}$$

we obtain the expressions of linear parts L_i as in Theorem 3.1.

Since the derivation of nonlinear parts $\boldsymbol{N}_i$ is much more complicated, we give only the sketch. For detail, see [6].

We use $G(\boldsymbol{f},\boldsymbol{\phi})$ as one of $G_{i1k}(\boldsymbol{f},\boldsymbol{\phi})$ ($k = 2, 3$) or $G_{i2}(\boldsymbol{f},\boldsymbol{\phi})$ and they have in the form

$$G(\boldsymbol{f},\boldsymbol{\phi}) = \boldsymbol{G}_A(\boldsymbol{f})\cdot\Delta\boldsymbol{\phi}' + \boldsymbol{G}_B(\boldsymbol{f})\cdot\Delta\boldsymbol{\phi} + \boldsymbol{G}_C(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_1) + \boldsymbol{G}_D(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_2).$$

The following is obtained from an easy calculation.

Lemma 3.1 *The following relations hold.*

1.

$$\begin{aligned}&\iint_{|s_1-s_2|\geqq\varepsilon}\boldsymbol{\zeta}(s_1,s_2)\cdot\boldsymbol{\phi}'(s_1)\,ds_1ds_2\\ &= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{\zeta}(s,s+\varepsilon)-\boldsymbol{\zeta}(s,s-\varepsilon))\cdot\boldsymbol{\phi}(s)\,ds\\ &\quad-\iint_{|s_1-s_2|\geqq\varepsilon}\frac{\partial}{\partial s_1}\boldsymbol{\zeta}(s_1,s_2)\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2\end{aligned}$$

2.

$$\begin{aligned}&\iint_{|s_1-s_2|\geqq\varepsilon}\boldsymbol{\zeta}(s_1,s_2)\cdot\boldsymbol{\phi}'(s_2)\,ds_1ds_2\\ &= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{\zeta}(s+\varepsilon,s)-\boldsymbol{\zeta}(s-\varepsilon,s))\cdot\boldsymbol{\phi}(s)\,ds\\ &\quad-\iint_{|s_1-s_2|\geqq\varepsilon}\frac{\partial}{\partial s_2}\boldsymbol{\zeta}(s_1,s_2)\cdot\boldsymbol{\phi}(s_2)\,ds_1ds_2\end{aligned}$$

3.

$$\begin{aligned}&\iint_{|s_1-s_2|\geqq\varepsilon}\boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\phi}\,ds_1ds_2\\ &= \iint_{|s_1-s_2|\geqq\varepsilon}(\boldsymbol{\zeta}(s_1,s_2)-\boldsymbol{\zeta}(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2\end{aligned}$$

4.

$$\iint_{|s_1-s_2|\geqq\varepsilon} \boldsymbol{\zeta}(s_1,s_2)\cdot\Delta\boldsymbol{\phi}'\,ds_1ds_2$$
$$=\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{\zeta}(s,s+\varepsilon)-\boldsymbol{\zeta}(s+\varepsilon,s)-\boldsymbol{\zeta}(s,s-\varepsilon)+\boldsymbol{\zeta}(s-\varepsilon,s))\cdot\boldsymbol{\phi}(s)\,ds$$
$$-\iint_{|s_1-s_2|\geqq\varepsilon}\frac{\partial}{\partial s_1}(\boldsymbol{\zeta}(s_1,s_2)-\boldsymbol{\zeta}(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2.$$

From Lemma 3.1, we obtain

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G(\boldsymbol{f},\boldsymbol{\phi})\,ds_1ds_2$$
$$=\lim_{\varepsilon\to+0}\iint_{|s_1-s_2|\geqq\varepsilon} G(\boldsymbol{f},\boldsymbol{\phi})\,ds_1ds_2$$
$$=\lim_{\varepsilon\to+0}\left(\iint_{|s_1-s_2|\geqq\varepsilon}\boldsymbol{G}_A(\boldsymbol{f})\cdot\Delta\boldsymbol{\phi}'ds_1ds_2+\iint_{|s_1-s_2|\geqq\varepsilon}\boldsymbol{G}_B(\boldsymbol{f})\cdot\Delta\boldsymbol{\phi}\,ds_1ds_2\right.$$
$$+\iint_{|s_1-s_2|\geqq\varepsilon}\boldsymbol{G}_C(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_1)\,ds_1ds_2$$
$$\left.+\iint_{|s_1-s_2|\geqq\varepsilon}\boldsymbol{G}_D(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_2)\,ds_1ds_2\right)$$
$$=\lim_{\varepsilon\to+0}\left\{\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{G}_A(\boldsymbol{f})(s,s+\varepsilon)-\boldsymbol{G}_A(\boldsymbol{f})(s+\varepsilon,s)\right.$$
$$-\boldsymbol{G}_A(\boldsymbol{f})(s,s-\varepsilon)-\boldsymbol{G}_A(\boldsymbol{f})(s-\varepsilon,s))\cdot\boldsymbol{\phi}(s)\,ds$$
$$-\iint_{|s_1-s_2|\geqq\varepsilon}\frac{\partial}{\partial s_1}(\boldsymbol{G}_A(\boldsymbol{f})(s_1,s_2)-\boldsymbol{G}_A(\boldsymbol{f})(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2$$
$$+\iint_{|s_1-s_2|\geqq\varepsilon}(\boldsymbol{G}_B(\boldsymbol{f})(s_1,s_2)-\boldsymbol{G}_B(\boldsymbol{f})(s_2,s_1))\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2$$
$$+\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{G}_C(\boldsymbol{f})(s,s+\varepsilon)-\boldsymbol{G}_C(\boldsymbol{f})(s,s-\varepsilon))\cdot\boldsymbol{\phi}(s)\,ds$$
$$-\iint_{|s_1-s_2|\geqq\varepsilon}\frac{\partial}{\partial s_1}\boldsymbol{G}_C(\boldsymbol{f})(s_1,s_2)\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2$$
$$+\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}(\boldsymbol{G}_D(\boldsymbol{f})(s+\varepsilon,s)-\boldsymbol{G}_D(\boldsymbol{f})(s-\varepsilon,s))\cdot\boldsymbol{\phi}(s)\,ds$$
$$\left.-\iint_{|s_1-s_2|\geqq\varepsilon}\frac{\partial}{\partial s_2}\boldsymbol{G}_D(\boldsymbol{f})(s_1,s_2)\cdot\boldsymbol{\phi}(s_2)\,ds_1ds_2\right\}$$
$$=(\dagger).$$

Here we shall prove that if $\boldsymbol{f} \in H^3$, then

$$(\dagger) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left\{ -\frac{\partial}{\partial s_1} (\boldsymbol{G}_A(\boldsymbol{f})(s_1, s_2) - \boldsymbol{G}_A(\boldsymbol{f})(s_2, s_1)) + \boldsymbol{G}_B(\boldsymbol{f})(s_1, s_2) - \boldsymbol{G}_B(\boldsymbol{f})(s_2, s_1) - \frac{\partial}{\partial s_1} \boldsymbol{G}_C(\boldsymbol{f})(s_1, s_2) - \frac{\partial}{\partial s_1} \boldsymbol{G}_D(\boldsymbol{f})(s_2, s_1) \right\} \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2 = (\ddagger)$$

holds. Furthermore, we shall check that

$$\boldsymbol{N}(\boldsymbol{f})(s) = \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ -\frac{\partial}{\partial s} (\boldsymbol{G}_A(\boldsymbol{f})(s, s_2) - \boldsymbol{G}_A(\boldsymbol{f})(s_2, s)) + \boldsymbol{G}_B(\boldsymbol{f})(s, s_2) - \boldsymbol{G}_B(\boldsymbol{f})(s_2, s) - \frac{\partial}{\partial s} \boldsymbol{G}_C(\boldsymbol{f})(s, s_2) - \frac{\partial}{\partial s} \boldsymbol{G}_D(\boldsymbol{f})(s_2, s) \right\} ds_2$$

is well-defined at $\mathscr{L}^1$-a.e. $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$ in order to use Fubini's theorem which induces

$$(\ddagger) = \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \boldsymbol{N}(\boldsymbol{f})(s) \cdot \boldsymbol{\phi}(s)\, ds = \langle \boldsymbol{N}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})}.$$

We also shall show that $\boldsymbol{N}(\boldsymbol{f})$ is a lower order term whose order is less than 3 and

$$\|\boldsymbol{N}(\boldsymbol{f})\|_{L^2} \leqq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}, \lambda).$$

We use $\boldsymbol{N}_{i1k}$ or $\boldsymbol{N}_{i2}$ as counterparts of $\boldsymbol{N}$ when $G = G_{i1k}$ or G_{i2}, respectively.

Here we give a detail for $\boldsymbol{N}_{112}$ only. For this we need the following facts. We omit their proofs.

Lemma 3.2 *Let $\boldsymbol{\kappa} \in L^\infty$. It holds that*

$$T_i^0 \boldsymbol{f} = \mathcal{O}(\Delta s), \quad T_i^0 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) = \mathcal{O}(\Delta s)^2, \quad T_i^0 \boldsymbol{f} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} = \mathcal{O}(\Delta s)^2.$$

Suppose that $\boldsymbol{f}$ is bi-Lipschitz and $\boldsymbol{\kappa} \in L^\infty$. For $k \geqq 1$,

$$T_i^k \boldsymbol{f} = \mathcal{O}(\Delta s), \quad T_i^k \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j) = \mathcal{O}(\Delta s)^2 \quad T_i^k \boldsymbol{f} \cdot \frac{\Delta \boldsymbol{f}}{\Delta s} = \mathcal{O}(\Delta s)^2$$

holds.

Lemma 3.3 *It holds that*

$$\frac{\partial}{\partial s_j} \mathscr{M}(\boldsymbol{f}) = \frac{2(-1)^j}{(\Delta s)^3} T_j^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_j).$$

It is easy to see

$$G_{112}(\boldsymbol{f}, \boldsymbol{\phi}) = \boldsymbol{g}_{112}(\boldsymbol{f}) \cdot \Delta\boldsymbol{\phi}',$$

where

$$\boldsymbol{g}_{112}(\boldsymbol{f})(s_1, s_2) = \mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{f}'.$$

In what follows, we denote $\boldsymbol{g}_{112}(\boldsymbol{f})(s_1, s_2)$ by $\boldsymbol{g}_{112}(s_1, s_2)$ in short.

Lemma 3.4 *Let* $\alpha \in (0, \frac{1}{2})$. *If* $\boldsymbol{f} \in H^{3-\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$,

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{R})^2} G_{112}(\boldsymbol{f}, \boldsymbol{\phi})(s_1, s_2)\, ds_1 ds_2 = \langle \boldsymbol{N}_{112}(\boldsymbol{f}), \boldsymbol{\phi} \rangle_{L^2}$$

follows, where

$$\boldsymbol{N}_{112}(\boldsymbol{f})(s_1) = 2\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} \left\{ \frac{2}{(\Delta s)^3}(T_1^4 \boldsymbol{f} \cdot \boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau} - \mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1) \right\} ds_2,$$

$$\|\boldsymbol{N}_{112}(\boldsymbol{f})\|_{L^2} \leqq C(\|\boldsymbol{f}\|_{H^{3-\alpha}}).$$

Proof. Using Lemma 3.1, we have

$$\begin{aligned}
&\iint_{|s_1 - s_2| \geqq \varepsilon} G_{112}(\boldsymbol{f}, \boldsymbol{\phi})\, ds_1 ds_2 \\
&= \iint_{|s_1 - s_2| \geqq \varepsilon} \boldsymbol{g}_{112}(\boldsymbol{f}) \cdot \Delta\boldsymbol{\phi}'\, ds_1 ds_2 \\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}} (\boldsymbol{g}_{112}(s, s+\varepsilon) - \boldsymbol{g}_{112}(s+\varepsilon, s) \\
&\qquad - \boldsymbol{g}_{112}(s, s-\varepsilon) + \boldsymbol{g}_{112}(s-\varepsilon, s)) \cdot \boldsymbol{\phi}(s)\, ds \\
&\quad - \iint_{|s_1 - s_2| \geqq \varepsilon} \frac{\partial}{\partial s_1}(\boldsymbol{g}_{112}(s_1, s_2) - \boldsymbol{g}_{112}(s_2, s_1)) \cdot \boldsymbol{\phi}(s_1)\, ds_1 ds_2 \\
&= (*).
\end{aligned}$$

Remarking that

$$\boldsymbol{g}_{112}(s_1, s_2) = \mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{f}' = \mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{\tau} = \mathcal{O}(\Delta s),$$

we show

$$\boldsymbol{g}_{112}(s, s+\varepsilon) - \boldsymbol{g}_{112}(s+\varepsilon, s) - \boldsymbol{g}_{112}(s, s-\varepsilon) + \boldsymbol{g}_{112}(s-\varepsilon, s) = \mathcal{O}(\varepsilon) \quad (\varepsilon \to 0).$$

$\mathcal{O}(\varepsilon)$ is uniform with regard to $s \in \mathbb{R}/\mathcal{L}\mathbb{Z}$, and in what follows, we use this notation in this meaning. Since

$$\boldsymbol{g}_{112}(s_1, s_2) - \boldsymbol{g}_{112}(s_2, s_1) = 2\mathscr{M}(\boldsymbol{f})\Delta\boldsymbol{\tau}$$

holds, we use Lemmas 3.2–3.3 and then we have

$$
\begin{aligned}
&\frac{\partial}{\partial s_1}\{\boldsymbol{g}_{112}(s_1,s_2)-\boldsymbol{g}_{112}(s_2,s_1)\}\\
&\quad = 2\left(\frac{\partial \mathscr{M}(\boldsymbol{f})}{\partial s_1}\Delta\boldsymbol{\tau}+\mathscr{M}(\boldsymbol{f})\frac{\partial \Delta\boldsymbol{\tau}}{\partial s_1}\right)\\
&\quad = 2\left\{-\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau}+\mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\}\\
&\quad = \mathcal{O}(\Delta s)^{-3+2+1}+\mathcal{O}(1)\\
&\quad = \mathcal{O}(1) \quad \text{as} \quad \Delta s\to 0
\end{aligned}
$$

which insists $\frac{\partial}{\partial s_1}\{\boldsymbol{g}_{112}(s_1,s_2)-\boldsymbol{g}_{112}(s_2,s_1)\}\in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. Therefore it is absolutely integrable and then we use Fubini's theorem in order to get

$$
\begin{aligned}
(*) &\to -\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} 2\left\{-\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau}+\mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\}\cdot\boldsymbol{\phi}(s_1)\,ds_1ds_2\\
&= \int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left[2\int_{\mathbb{R}/\mathcal{L}\mathbb{Z}}\left\{\frac{2}{(\Delta s)^3}(T_1^4\boldsymbol{f}\cdot\boldsymbol{\tau}(s_1))\Delta\boldsymbol{\tau}-\mathscr{M}(\boldsymbol{f})\boldsymbol{\kappa}(s_1)\right\}ds_2\right]\cdot\boldsymbol{\phi}(s_1)\,ds_1\\
&= \langle \boldsymbol{N}_{112}(\boldsymbol{f}),\boldsymbol{\phi}\rangle_{L^2}
\end{aligned}
$$

by tending $\varepsilon\to+0$ in $(*)$. Since

$$\|\boldsymbol{\kappa}\|_{L^\infty}=\|\boldsymbol{f}''\|_{L^\infty}\leqq C_\alpha\|\boldsymbol{f}\|_{H^{3-\alpha}}$$

for $\alpha\in\left(0,\frac{1}{2}\right)$, the bound of the integrand of $\boldsymbol{N}_{112}$ follows from Lemma 3.2. Thus we have

$$\|\boldsymbol{N}_{112}(\boldsymbol{f})\|_{L^2}\leqq C\|\boldsymbol{f}\|_{H^{3-\alpha}}.$$

□

Though the proof is complicated, a similar result holds for G_{212}. Since $G_{113}=0$, we have nothing to do. We must deal with G_{213} very carefully. It is decomposed as

$$G_{213}(\boldsymbol{f},\boldsymbol{\phi})=\boldsymbol{G}_{213B}(\boldsymbol{f})\cdot\Delta\boldsymbol{\phi}+\boldsymbol{G}_{213C}(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_1)+\boldsymbol{G}_{213D}(\boldsymbol{f})\cdot\boldsymbol{\phi}'(s_2),$$

where

$$\begin{aligned}
G_{213B}(f) = \frac{2}{(\Delta s)\|\Delta f\|_{\mathbb{R}^n}^2} \bigg\{ & (T_1^2 f \cdot \tau(s_1)) \left(T_1^0 f + \frac{\Delta f}{\Delta s} \right) \\
& + (T_2^2 f \cdot \tau(s_2)) \left(T_2^0 f + \frac{\Delta f}{\Delta s} \right) \\
& + 2(T_1^2 f \cdot \tau(s_1))(T_2^2 f \cdot \tau(s_2)) \frac{\Delta f}{\Delta s} \bigg\}, \\
G_{213C}(f) = -\frac{2}{\|\Delta f\|_{\mathbb{R}^n}^2} & (T_2^2 f \cdot \tau(s_2)) \frac{\Delta f}{\Delta s}, \\
G_{213D}(f) = -\frac{2}{\|\Delta f\|_{\mathbb{R}^n}^2} & (T_1^2 f \cdot \tau(s_1)) \frac{\Delta f}{\Delta s}.
\end{aligned}$$

Since calculations of G_{213B}, G_{213C}, G_{213D} are not closed in each term to recover the absolute integrability, we have to combine them appropriately. We need a simiar treatment for G_{i2}, but the stuation is less serious than that of G_{213}. For details, see [6]. □

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