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Existence of a ground state of a model of relativistic quantum electrodynamics with cutoffs for all values of coupling constants

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1 Introduction

In this article we review a result of spectral analysis of a model in quantum electrodynamics in [7]. Quantum electrodynamics describes the interaction system of electrons, positrons and photons. We consider a system of a Dirac field coupled to a quantized radiation field in the Coulomb gauge. We define the state space as a Hilbert space, and the full Hamiltonian on the Hilbert space. The state space is defined by $\mathcal{H}_{\mathrm{QED}} = \mathcal{H}_{\mathrm{Dirac}} \otimes \mathcal{H}_{\mathrm{rad}}$, where $\mathcal{H}_{\mathrm{Dirac}}$ is a fermion Fock space and $\mathcal{H}_{\mathrm{rad}}$ a boson Fock space. The full Hamiltonian is of the form

$$H_{\mathrm{QED}} = H_{\mathrm{Dirac}} \otimes I + I \otimes H_{\mathrm{rad}} + \kappa_{\mathrm{I}} H_{\mathrm{I}} + \kappa_{\mathrm{II}} H_{\mathrm{II}}.$$

Here H_{Dirac} and H_{rad} are energy Hamiltonians of the Dirac field and the radiation field, respectively. H_{I} and H_{II} are interactions between the Dirac field and the radiation field, and $\kappa_{\mathrm{I}} \in \mathbf{R}$ and $\kappa_{\mathrm{II}} \in \mathbf{R}$ are coupling constants. By imposing ultraviolet cutoffs on the field's operator and spatial cutoffs on the interactions, H_{QED} is self-adjoint and bounded from below on the Hilbert space. We analyze the property of the infimum of the spectrum. If the infimum of the spectrum of is eigenvalue, we say that the ground state exists. The infimum of the free Hamiltonian $H_0 = H_{\mathrm{Dirac}} \otimes I + I \otimes H_{\mathrm{rad}}$ is eigenvalue, but it is embedded in the continuous spectrum. The eigenvalue embedded in the continuous spectrum is not stable when the interaction turns on. Hence the existence of the ground state of H_{QED} is non-trivial. Since the mid-1990s, the spectral analysis and scattering theory for the system of particles coupled to quantum fields have been developed. In particular the analysis of the embedded eigenvalue has been successfully analyzed. By applying the methods for the system of particles coupled to quantum fields to H_{QED}, we prove that H_{QED} has a ground state for all values of the coupling constants.

2 Dirac Fields and Quantized Radiation Fields

First we consider the Dirac fields. The state space for the Dirac field is defined by $\mathcal{H}_{\mathrm{Dirac}} = \mathcal{F}_{\mathrm{f}}(L^2(\mathbf{R}^3_{\mathbf{p}};\mathbf{C}^4))$ where $\mathcal{F}_{\mathrm{f}}(L^2(\mathbf{R}^3_{\mathbf{p}};\mathbf{C}^4))$ denotes the fermion Fock space over the Hilbert space $L^2(\mathbf{R}^3_{\mathbf{p}};\mathbf{C}^4)$. The energy Hamiltonian is defined by

$$H_{\mathrm{Dirac}} = d\Gamma_{\mathrm{f}}(\omega_M),$$

where $d\Gamma_{\mathrm{f}}(\omega_M)$ denotes the second quantization of the multiplication operator $\omega_M = \sqrt{\mathbf{p}^2+M^2}$, $M>0$. Physically, the constant $M>0$ denotes the rest mass of electron. The Dirac field operator $\psi(\mathbf{x}) = (\psi_l)_{l=1}^4$ with the ultraviolet cutoff $\chi_{\mathrm{D}} = \chi_{\mathrm{D}}(\mathbf{p})$ is defined by

$$\psi_l(\mathbf{x}) = \sum_{s=\pm 1/2} \left(b_s(\frac{\chi_{\mathrm{D}}\, u^l_{s,\mathbf{x}}}{\sqrt{(2\pi)^3\omega_M}}) + d^*_s(\frac{\chi_{\mathrm{D}}\, \tilde{v}^l_{s,\mathbf{x}}}{\sqrt{(2\pi)^3\omega_M}}) \right),$$

where $b_s(f)$, $f\in L^2(\mathbf{R}^3)$, is the annihilation operator of electrons and $d^*_s(g)$, $g\in L^2(\mathbf{R}^3)$, the creation operator of positrons, $u^l_{s,\mathbf{x}}(\mathbf{p}) = u^l_s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}$ and $\tilde{v}^l_{s,\mathbf{x}}(\mathbf{p}) = v^l_s(-\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}$ with spinors u^l_s and v^l_s. Creation operators and annihilation operators for the Dirac field satisfy the canonical anti-commutation relations :

$$\begin{aligned}
&\{b_s(g), b^*_{s'}(h)\} = \{d_s(g), d^*_{s'}(h)\} = \delta_{s,s'}(g,h),\\
&\{b_s(f), b_{s'}(g)\} = \{d_s(f), d_{s'}(g)\} = \{b_s(g), d^*_{s'}(g)\} = 0,
\end{aligned}$$

where $\{X,Y\} = XY+YX$.

Formally, the distribution kernels of annihilation operators for the Dirac field are expressed by $b_s(\mathbf{p})$ and $d_s(\mathbf{p})$. The distribution kernels of creation operators are also expressed by $b^*_s(\mathbf{p})$ and $d^*_s(\mathbf{p})$. Then the energy Hamiltonian and the field operators are denoted by

$$\begin{aligned}
H_{\mathrm{Dirac}} &= \sum_{s=\pm 1/2} \int_{\mathbf{R}^3} \omega_M(\mathbf{p})\,(b^*_s(\mathbf{p})b_s(\mathbf{p}) + d^*_s(\mathbf{p})d_s(\mathbf{p}))\,d\mathbf{p},\\
\psi_l(\mathbf{x}) &= \sum_{s=\pm 1/2} \int_{\mathbf{R}^3} \frac{\chi_{\mathrm{D}}(\mathbf{p})}{\sqrt{(2\pi)^3\omega_M(\mathbf{p})}} \left(u^l_s(\mathbf{p})b_s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + v^l_s(-\mathbf{p})d^*_s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}} \right) d\mathbf{p}.
\end{aligned}$$

Next we consider the quantized radiation field in the Coulomb gauge. The state space is defined by where $\mathcal{H}_{\text{rad}} = \mathcal{F}_{\text{b}}(L^2(\mathbf{R}^3_{\mathbf{k}} \times \{1,2\}))$ where $\mathcal{F}_{\text{b}}(L^2(\mathbf{R}^3_{\mathbf{k}} \times \{1,2\}))$ denotes the boson Fock space over the Hilbert space $L^2(\mathbf{R}^3_{\mathbf{k}} \times \{1,2\})$. The energy Hamiltonian is defined by

$$H_{\text{rad}} = d\Gamma_{\text{b}}(\omega),$$

where $d\Gamma_{\text{b}}(\omega)$ denotes the second quantization of the multiplication operator $\omega(\mathbf{k}) = |\mathbf{k}|$. Here note that mass of photon is zero. The radiation field operator $A_j(\mathbf{x}) = (A_j(\mathbf{x}))_{j=1}^3$ with the ultraviolet cutoff $\chi_{\text{rad}} = \chi_{\text{rad}}(\mathbf{k})$ is defined by

$$A_j(\mathbf{x}) = \sum_{r=1,2} \left(a_r(\frac{\chi_{\text{rad}}\, e^j_{r,\mathbf{x}}}{\sqrt{2(2\pi)^3\omega}}) + a^*_r(\frac{\chi_{\text{rad}}\, e^j_{r,\mathbf{x}}}{\sqrt{2(2\pi)^3\omega}}) \right),$$

where $a_r(h)$, $h \in L^2(\mathbf{R}^3)$, and $a^*_r(h')$, $h' \in L^2(\mathbf{R}^3)$, denote the annihilation operator and the creation operator of photons, respectively, and $e^j_{r,\mathbf{x}}(\mathbf{k}) = e^j_r(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}$ with polarization vector e^j_r. Creation operators and annihilation operators for the radiation field satisfy the canonical commutation relations :

$$[a_r(f), a^*_{r'}(g)] = \delta_{r,r'}(f,g),$$
$$[a_r(f), a_{r'}(g)] = [a^*_r(f), a^*_{r'}(g)] = 0,$$

where $[X,Y] = XY - YX$.

The distribution kernels of annihilation operator and creation operator of the radiation field are also expressed by $a_r(\mathbf{k})$ and $a^*_r(\mathbf{k})$. Then, the energy Hamiltonian and the field operators of the radiation field are denoted by

$$H_{\text{rad}} = \sum_{r=1,2} \int_{\mathbf{R}^3} \omega(\mathbf{k}) a^*_r(\mathbf{k}) a_r(\mathbf{k}) d\mathbf{k},$$
$$A_j(\mathbf{x}) = \sum_{r=1,2} \int_{\mathbf{R}^3} \frac{\chi_{\text{rad}}(\mathbf{k})\, e^j_r(\mathbf{k})}{\sqrt{2(2\pi)^3\omega(\mathbf{k})}} \left(a_r(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_r(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} \right) d\mathbf{k}.$$

The quantization of the radiation field depends on the gauge. The quantization in the Lorentz gauge, it need indefinite metric in the Hilbert space.

3 Main Theorem and Outline of the Proof

We define the state space and the total Hamiltonians for the interaction system of a Dirac field coupled to the radiation field. The state space is defined by $\mathcal{H}_{\mathrm{QED}} = \mathcal{H}_{\mathrm{Dirac}} \otimes \mathcal{H}_{\mathrm{rad}}$. The full Hamiltonian is defined by

$$H_{\mathrm{QED}} = H_{\mathrm{Dirac}} \otimes I + I \otimes H_{\mathrm{rad}} + \kappa_{\mathrm{I}} H_{\mathrm{I}} + \kappa_{\mathrm{II}} H_{\mathrm{II}},$$

where H_{I} and H_{II} are given by

$$H_{\mathrm{I}} = \sum_{j=1}^{3} \int_{\mathbf{R}^3} \chi_{\mathrm{I}}(\mathbf{x}) (\psi^*(\mathbf{x}) \alpha_j \psi(\mathbf{x}) \otimes A_j(\mathbf{x})) d\mathbf{x},$$

$$H_{\mathrm{II}} = \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{\chi_{\mathrm{II}}(\mathbf{x}) \chi_{\mathrm{II}}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} (\psi^*(\mathbf{x}) \psi(\mathbf{x}) \psi^*(\mathbf{y}) \psi(\mathbf{y}) \otimes I) \, d\mathbf{x} d\mathbf{y}.$$

Here α_j, $j = 1,2,3$, are 4×4 Dirac matrices which satisfy $\{\alpha_j, \alpha_l\} = 2\delta_{j,l}$, and $\chi_{\mathrm{I}} = \chi_{\mathrm{I}}(\mathbf{x})$ and $\chi_{\mathrm{II}} = \chi_{\mathrm{II}}(\mathbf{x})$ are spatial cutoffs.

First we consider the self-adjointness of the Hamiltonians. H_{Dirac} and H_{rad} are self-adjoint operator with bounded from below, and hence $H_0 = H_{\mathrm{Dirac}} \otimes I + I \otimes H_{\mathrm{rad}}$ is self-adjoint and bounded from below. To prove the self-adjointness of H_{QED}, we assume the following condition ;

(A.1 ; Ultraviolet Cutoff for the Dirac field)

$$\int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{rad}}(\mathbf{k})|^2}{\omega(\mathbf{k})^k} d\mathbf{k} < \infty, \; k = 1,2,$$

(A.2 ; Ultraviolet Cutoff for the radiation field)

$$\int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{D}}(\mathbf{p}) u_s^l(\mathbf{p})|^2}{\omega_M(\mathbf{p})} d\mathbf{p} < \infty, \; \int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{D}}(\mathbf{p}) v_s^l(-\mathbf{p})|^2}{\omega_M(\mathbf{p})} d\mathbf{p} < \infty$$

(A.3 ; Spatial Cutoff)

$$\int_{\mathbf{R}^3} |\chi_{\mathrm{I}}(\mathbf{x})| d\mathbf{x} < \infty, \qquad \int_{\mathbf{R}^3 \times \mathbf{R}^3} \frac{|\chi_{\mathrm{II}}(\mathbf{x}) \chi_{\mathrm{II}}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|} d\mathbf{x} d\mathbf{y} < \infty.$$

By **(A.1)** - **(A.3)**, it holds that for $\Psi \in \mathcal{D}(H_0)$,

$$\|H_{\mathrm{I}}\Psi\| \leq L_{\mathrm{I}}\|H_0^{1/2}\Psi\| + R_{\mathrm{I}}\|\Psi\|,$$
$$\|H_{\mathrm{II}}\Psi\| \leq R_{\mathrm{II}}\|\Psi\|,$$

where $L_I \geq 0, R_{\mathrm{I}} \geq 0$ and $L_{\mathrm{II}} \geq 0$ are some constants. Then it is proven that H_{I} is relatively bounded to $H_0 = H_{\mathrm{Dirac}} \otimes I + I \otimes H_{\mathrm{rad}}$ with infinitely small bound. We also see that H_{II} is bounded. Hence it holds that H_{QED} is self-ajoint and bounded from below by Kato-Rellich Theorem in [6].

Next we consider spectrum of the Hamiltonians. The spectrum of the H_{Dirac} and H_{rad} are as follows:

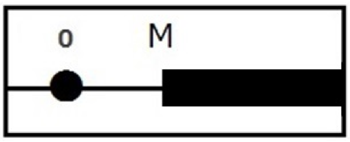

Figure 1 : Spectrum of H_{Dirac}

Figure 2 :Spectrum of H_{rad}

Then the spectrum of $H_0 = H_{\mathrm{Dirac}} \otimes I + I \otimes H_{\mathrm{rad}}$ is as follows ;

Figure 3 : Spectrum of H_0

Thus we see that the infimum of the spectrum of H_0 is eigenvalue, but it is embedded in the continuous spectrum. Hence the existence of the ground state of H_{QED} is not trivial. The existence of a ground state of H_{QED} for sufficiently small values of coupling constants is proven in [6]. We prove the existence of a ground state of H_{QED} for all values of coupling constants. To prove this, we assume the following conditions:

(A.4 ; Spatial Localization)

$$\int_{\mathbf{R}^3} |\mathbf{x}||\chi_{\mathrm{I}}(\mathbf{x})|d\mathbf{x} < \infty, \quad \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|\chi_{\mathrm{II}}(\mathbf{x})\chi_{\mathrm{II}}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}|\mathbf{x}|d\mathbf{x}d\mathbf{y} < \infty,.$$

(A.5 ; Infrared Regularity condition)

$$\int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{rad}}(\mathbf{k})|^2}{\omega(\mathbf{k})^5} d\mathbf{k} < \infty.$$

(A.6)

$$\int_{\mathbf{R}^3} \frac{|\partial_{k^j}\chi_{\mathrm{rad}}(\mathbf{k})|^2}{\omega(\mathbf{k})} d\mathbf{k} < \infty, \int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{rad}}(\mathbf{k})\partial_{k^j}e_r^i(\mathbf{k})|^2}{\omega(\mathbf{k})} d\mathbf{k} < \infty.$$

(A.7)

$$\int_{\mathbf{R}^3} \frac{|(\partial_{p^j}\chi_{\mathrm{D}}(\mathbf{p}))u_s^l(\mathbf{p})|^2}{\omega_M(\mathbf{p})} d\mathbf{p} < \infty, \int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{D}}(\mathbf{p})\partial_{p^j}u_s^l(\mathbf{p})|^2}{\omega_M(\mathbf{p})} d\mathbf{p} < \infty,$$

$$\int_{\mathbf{R}^3} \frac{|(\partial_{p^j}\chi_{\mathrm{D}}(\mathbf{p}))v_s^l(-\mathbf{p})|^2}{\omega_M(\mathbf{p})} d\mathbf{p} < \infty, \int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{D}}(\mathbf{p})\partial_{p^j}v_s^l(-\mathbf{p})|^2}{\omega_M(\mathbf{p})} d\mathbf{p} < \infty$$

The conditions **(A.4)**- **(A.7)** are used when we estimate the derivative bound of the annihilation operators for the Dirac field and the radiation field. In particular **(A.5)** implies that we neglect the influence of low-energy photons, which cause the infrared divergent problem. Following Theorem 1 is the main result in [7].

Theorem 1 ([7])
Suppose **(A.1)-(A.7)**. Then H_{QED} has a ground state for all values of coupling constants.

The strategy of the proof of Theorem 1 consists of two steps, and these are subsequently explained.

4 Outline of Proof of Theorem 1

[1st step]
Let

$$H_m = H_{\text{Dirac}} \otimes I + I \otimes H_{\text{rad},m} + \kappa_{\text{I}} H_{\text{I}} + \kappa_{\text{II}} H_{\text{II}},$$

where $H_{\text{rad},m} = d\Gamma_{\text{b}}(\omega_m)$ with $\omega_m(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}, m > 0$. Physically, $m > 0$ denotes the artificial mass of the photon. The infimum of the spectrum of $H_0(m) = H_{\text{Dirac}} \otimes I + I \otimes H_{\text{rad},m}$ is discrete eigenvalue. It is proven that H_m has purely discrete spectrum in $[E_0(H_m), E_0(H_m) + m)$. And then, H_m has a ground state. The outline of the proof is as follows. We use Weyl's sequence method in [3] and partition of unity on Fock space in [2]. Let $\lambda \in \sigma_{\text{ess}}(H_m)$. Then by Weyl's theorem, there exists a Weyl sequence $\{\Psi_n\}_{n=1}^{\infty}$ for $\mathcal{D}(H_m)$ and λ. Then by this sequence and partition of unity of Dirac field and radiation field, we can show that $\lambda \geq E_0(H_m) + m$. Then we obtain that $\sigma_{\text{ess}}(H_m) \subset [E_0(H_m) + m, \infty)$, and the proof is obtained.

[2nd step]
From 1st step, we see that H_m has the ground state. Let Ψ_m, $m > 0$, be the normalized ground state of H_m, i.e. $H_m \Psi_m = E_0(H_m)\Psi_m$, $\|\Psi_m\| = 1$. Since $\|\Psi_m\| = 1$, $m > 0$, there exists a subsequence $\{\Psi_{m_j}\}$ such that the weak limit of Ψ_{m_j} as $j \to \infty$ exists. To prove the weak limits of Ψ_{m_j} as $j \to \infty$ is a non-zero vector, we consider the same strategy of [4] and use derivative bound method for annihilation operators in [5]. Here in particular, we need both a boson derivative bound for the radiation field and a fermion derivative bound for Dirac fields.

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