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Golden-Thompson Type Inequalities and Functional Integral Approach to Boson-Fermion Systems

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Abstract

Some aspects of Golden-Thompson type inequalities for a boson system with finite degrees of freedom are reviewed. Then a general class of boson-fermion systems with finite degrees of freedom, including supersymmetric ones, is considered. Functional integral representations for the partition function as well as related objects of a boson-fermion system are derived and applied to obtain Golden-Thompson type inequalities. The present article is intended to be a review paper, but, includes new results which are extensions of some results obtained in a previous paper (A. Arai, *Rev. Math. Phys.* **25** (2013), 1350015,43 pages).

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Contents

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1 Introduction: Some Backgrounds and Motivations

1.1 Partition function in quantum statistical mechanics and an abstract Golden–Thompson inequality

As is well known, a fundamental object in quantum statistical mechanics is the *partition function*

$$Z(\beta) := \operatorname{Tr} e^{-\beta H},$$

where $\beta > 0$ is a parameter denoting the inverse temperature (i.e., $\beta := 1/kT$ with $k > 0$ and $T > 0$ being respectively the Boltzmann constant and the absolute temperature), H is the Hamiltonian of the quantum system under consideration (mathematically a self-adjoint operator on a complex Hilbert space such that $e^{-\beta H}$ is trace class) and Tr denotes trace. One of the important physical quantities derived from the partition function is the *Helmholtz free-energy function*

$$F(\beta) := -\frac{1}{\beta} \log Z(\beta).$$

If there exists a constant $\beta_0 > 0$ such that $e^{-\beta_0 H}$ is trace class, then, for all $\beta \geq \beta_0$, $e^{-\beta H}$ is trace class and

$$\lim_{\beta \to \infty} F(\beta) = E_0(H) := \inf \sigma(H), \tag{1.1}$$

where $\sigma(H)$ denotes the spectrum of H. The number $E_0(H)$ is called the *ground state energy* of H. Hence the Helmholtz free energy function approaches to the ground state energy of the quantum system under consideration as the absolute temperature tends to zero.

If there exists a constant $C_\beta > 0$ depending on β such that

$$Z(\beta) \leq C_\beta,$$

then

$$F(\beta) \geq -\frac{1}{\beta} \log C_\beta.$$

Hence a lower bound for the Helmholtz free-energy function is obtained from an upper bound for the partition function. Similarly one can obtain an upper bound for the Helmholtz free-energy function from a lower bound for the partition function. Therefore to estimate the partition function from both above and below has some physical importance. This leads one to consider inequalities for $\operatorname{Tr} e^{-\beta H}$. Historically one of such inequalities from above was discovered independently by G. Golden [10] and C. J. Thompson [18] (cf. also [17]) in the case where H is of the form $H = H_0 + H_1$ with H_0 and H_1 being Hermitian matrices. Since then, the inequality is called the *Golden–Thompson (GT) inequality*. Nowadays a general form of it is established:

Theorem 1.1 *Let H_0 and H_1 be bounded below self-adjoint operators on a Hilbert space such that $H := H_0 + H_1$ is essentially self-adjoint and $e^{-\beta H_1/2} e^{-\beta H_0} e^{-\beta H_1/2}$ is trace class for some $\beta > 0$. Then $e^{-\beta \overline{H}}$ is trace class, where $\overline{H}$ denotes the closure of H, and*

$$\operatorname{Tr} e^{-\beta \overline{H}} \leq \operatorname{Tr}\left(e^{-\beta H_0} e^{-\beta H_1}\right) \tag{1.2}$$

For a proof of this theorem, see, e.g., [13, p.320] and [15].

Remark 1.2 Under the assumption of Theorem 1.1, $e^{-\beta H_0} e^{-\beta H_1}$ is trace class and

$$\operatorname{Tr}\left(e^{-\beta H_1/2} e^{-\beta H_0} e^{-\beta H_1/2}\right) = \operatorname{Tr}\left(e^{-\beta H_0} e^{-\beta H_1}\right).$$

Remark 1.3 If H_0 and H_1 are strongly commuting (i.e., the spectral measure of H_0 commutes with that of H_1), then the equality in (1.2) holds, because, in this case, $e^{-\beta \overline{H}} = e^{-\beta H_0} e^{-\beta H_1}$ for all $\beta > 0$.

Remark 1.4 (An upper bound for $F(\beta)$) It is obvious that $Z(\beta) \geq d_0 e^{-\beta E_0(H)}$, where $d_0 = \dim \ker(H - E_0(H))$, the multiplicity of the eigenvalue $E_0(H)$. Hence

$$F(\beta) \leq E_0(H) - \frac{1}{\beta} \log d_0.$$

1.2 A GT inequality for a Schrödinger operator

As a simple application of Theorem 1.1, we briefly discuss a Schrödinger operator and point out some "defects" of the GT inequality in this case.

Let us consider the quantum system of a non-relativistic quantum particle with mass $m > 0$ and without spin in the n-dimensional Euclidean vector space $\mathbb{R}^n$ ($n \in \mathbb{N}$) under the influence of a Borel measurable scalar potential $V : \mathbb{R}^n \to \mathbb{R}$. Then the Hamiltonian of the system is given by the Schrödinger operator

$$H_V := -\frac{\hbar^2}{2m}\Delta + V \tag{1.3}$$

acting in $L^2(\mathbb{R}^n)$, where $\hbar > 0$ is a parameter denoting the Planck constant h divided by 2π ($\hbar := h/2\pi$) and Δ is the generalized Laplacian acting in $L^2(\mathbb{R}^n)$.

Suppose that V is in $L^2_{\rm loc}(\mathbb{R}^n)$,[1] bounded below and $\int_{\mathbb{R}^n} e^{-\beta V(x)}dx < \infty$ for some $\beta > 0$. Then H_V is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ [12, Theorem X.28] and bounded below. Let

$$T := e^{-\frac{\beta V}{2}} e^{-\frac{\beta\hbar^2}{2m}\Delta} e^{-\frac{\beta V}{2}}.$$

Then $T = S^*S$ with

$$S = e^{-\frac{\beta\hbar^2}{4m}\Delta} e^{-\frac{\beta V}{2}}.$$

We recall that, for all $t > 0$, the bounded self-adjoint operator $e^{t\Delta}$ is an integral operator on $L^2(\mathbb{R}^n)$ with the integral kernel

$$e^{t\Delta}(x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-|x-y|^2/4t}, \quad x,y \in \mathbb{R}^n. \tag{1.4}$$

For a proof of this fact, see, e.g., [12, p.59, Example 3].

It follows from (1.4) that S is an integral operator on $L^2(\mathbb{R}^n)$ with the integral kernel

$$k(x,y) = \left(\frac{m}{\pi\hbar^2\beta}\right)^{d/2} \exp\left(-\frac{m|x-y|^2}{\hbar^2\beta}\right) e^{-\beta V(y)/2}, \quad x,y \in \mathbb{R}^n,$$

i.e.,

$$Sf(x) = \int_{\mathbb{R}^n} k(x,y) f(y) \mathrm{d}y, \quad f \in L^2(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

Hence

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |k(x,y)|^2 \mathrm{d}x\mathrm{d}y = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{d/2} \int_{\mathbb{R}^n} e^{-\beta V(y)} \mathrm{d}y < \infty.$$

Hence S is Hilbert–Schmidt. Therefore T is trace class and

$$\operatorname{Tr} T = \|S\|_2^2 = \int_{\mathbb{R}^n\times\mathbb{R}^n} |k(x,y)|^2 \mathrm{d}x\mathrm{d}y = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{d/2} \int_{\mathbb{R}^n} e^{-\beta V(y)} \mathrm{d}y,$$

where $\|\cdot\|_2$ denotes Hilbert–Schmidt norm. Thus one can apply Theorem 1.1 to the case where $H_0 = -\hbar^2\Delta/2m$ and $H_1 = V$ to obtain

$$\operatorname{Tr} e^{-\beta \overline{H}_V} \le \left(\frac{m}{2\pi\beta\hbar^2}\right)^{d/2} \int_{\mathbb{R}^n} e^{-\beta V(y)} \mathrm{d}y. \tag{1.5}$$

Note that the right hand side is written as follows:

$$\left(\frac{m}{2\pi\beta\hbar^2}\right)^{d/2} \int_{\mathbb{R}^n} e^{-\beta V(y)} \mathrm{d}y = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\beta H_V^{\rm cl}(x,p)} \mathrm{d}x\mathrm{d}p,$$

where

$$H_V^{\rm cl}(x,p) := \frac{p^2}{2m} + V(x), \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n,$$

[1] $L^p_{\rm loc}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \cup \{\pm\infty\}, \text{Borel measurable} \mid \forall R > 0, \int_{|x|\le R} |f(x)|^p \mathrm{d}x < \infty\}$ $(p > 0)$.

is the corresponding classical Hamiltonian. The classical partition function $Z_V^{\rm cl}(\beta)$ is defined by

$$Z_V^{\rm cl}(\beta) := \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n\times\mathbb{R}^n} e^{-\beta H_V^{\rm cl}(x,p)}\mathrm{d}x\mathrm{d}p.$$

Thus we arrive at

$$\operatorname{Tr} e^{-\beta\overline{H}_V} \le Z_V^{\rm cl}(\beta). \tag{1.6}$$

This is sometimes called the *GT inequality of the Schrödinger operator H_V*.

Remark 1.5 Inequality (1.6) can be derived also by using functional integral methods and extended to a more general class of V (see, e.g., [6, Chapter 4] and [14, Theorem 9.2]).

Now it would be natural to ask when the equality holds in (1.6) or equivalently in (1.5). In the case $V = 0$, the equality in (1.5) holds with the both sides being infinite, but this is meaningless.

From a quantum mechanical point of view, the case where

$$V(x) = V_{\rm os}(x) := \sum_{j=1}^n \frac{m\omega_j^2}{2}x_j^2, \quad x = (x_1,\dots,x_n)\in\mathbb{R}^n,$$

an n-dimensional harmonic oscillator potential with $\omega_j > 0$ $(j = 1,\dots,n)$ being a constant, should be examined if it gives the equality in (1.6).

Example 1.6 Let

$$H_{\rm os} := H_{V_{\rm os}} = -\frac{\hbar^2}{2m}\Delta + \sum_{j=1}^n \frac{m\omega_j^2}{2}x_j^2. \tag{1.7}$$

It is well known that $H_{\rm os}$ is self-adjoint and

$$\sigma(H_{\rm os}) = \sigma_{\rm p}(H_{\rm os}) = \left\{\sum_{j=1}^n\left(k_j+\frac{1}{2}\right)\hbar\omega_j \middle| k_1,\dots,k_n\in\{0\}\cup\mathbb{N}\right\},$$

counting multiplicities, where, for a linear operator A on a Hilbert space, $\sigma_{\rm p}(A)$ denotes the point spectrum (the set of eigenvalues) of A. Hence it follows that, for all $\beta > 0$, $e^{-\beta H_{\rm os}}$ is trace class and

$$\operatorname{Tr} e^{-\beta H_{\rm os}} = \prod_{j=1}^n \frac{e^{-\beta\hbar\omega_j/2}}{1-e^{-\beta\hbar\omega_j}} = \prod_{j=1}^n \frac{1}{2\sinh\frac{\beta\hbar\omega_j}{2}}.$$

On the other hand,

$$Z_{V_{\rm os}}^{\rm cl}(\beta) = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{d/2}\prod_{j=1}^n\int_{\mathbb{R}} e^{-\beta m\omega_j^2x_j^2/2}\mathrm{d}x_j = \prod_{j=1}^n\frac{1}{\beta\hbar\omega_j} = \prod_{j=1}^n\frac{1}{2\,\frac{\beta\hbar\omega_j}{2}}.$$

Since $\sinh\chi > \chi$ for all $\chi > 0$, it follows that

$$\operatorname{Tr} e^{-\beta H_{\rm os}} < Z_{V_{\rm os}}^{\rm cl}(\beta).$$

Thus *the equality in (1.6) does not hold in the case $V = V_{\rm os}$*.

It would be desirable to have a GT type inequality which attains the equality in the case where $V = V_{\mathrm{os}}$. Indeed, this can be done if we take the unperturbed Hamiltonian to be the Hamiltonian of a quantum harmonic oscillator:

$$H_{\mathrm{os}}(V) = H_{\mathrm{os}} + V = H_{V_{\mathrm{os}}+V}. \tag{1.8}$$

We will come back to this point later (see Section 2).

Another "defect" in (1.5) or (1.6) is that it is not of a form which indicates an infinite dimensional version (heuristically the case $n = \infty$), since there is no infinite dimensional Lebesgue measure.

We remark, however, that, as for Schrödinger operator cases, a unified general formulation including both finite and infinite dimensional cases and overcoming the "defects" mentioned above was given in [3], where functional integral representations are established for the trace of objects related to $e^{-\beta H}$ with H being a self-adjoint operator on the boson Fock space over a Hilbert space $\mathcal{H}$, which, in the case $\dim \mathcal{H} = \infty$, may be regarded as an infinite dimensional Schrödinger operator, and GT type inequalities are derived. In these GT type inequalities, the equality is attained in the case where H is a free field Hamiltonian (a harmonic oscillator Hamiltonian in the case $\dim \mathcal{H} < \infty$) as desired.

1.3 Supersymmetric GT inequalities

In a paper [11], Klimek and Lesniewski considered a model in supersymmetric quantum mechanics (SQM) and, using a functional integral representation for the partition function of the model, derived a GT type inequality. This is an extension of (1.5) to the case where H_V is replaced by a supersymmetric Hamiltonian. For the reader's convenience, we briefly review the supersymmetric GT inequality by Klimek and Lesniewski.[2]

Let $n, r \in \mathbb{N}$. The Hilbert space of a boson–fermion system is given by

$$\mathcal{F}_{n,r} := L^2(\mathbb{R}^n) \otimes \wedge(\mathbb{C}^r), \tag{1.9}$$

with $\wedge(\mathbb{C}^r)$ being the fermion Fock space over $\mathbb{C}^r$:

$$\wedge(\mathbb{C}^r) := \oplus_{p=0}^r \wedge^p(\mathbb{C}^r) = \{\psi = (\psi^{(p)})_{p=0}^r | \psi^{(p)} \in \wedge^p(\mathbb{C}^r), p = 0, 1, \ldots, r\}, \tag{1.10}$$

where $\wedge^p(\mathbb{C}^r)$ is the p-fold anti-symmetric tensor product of $\mathbb{C}^r$.

Note that $L^2(\mathbb{R}^n) \cong \otimes^n L^2(\mathbb{R})$. Moreover,

$$L^2(\mathbb{R}) \cong \mathcal{F}_{\mathrm{b}}(\mathbb{C}) = \left\{ \phi = \{\phi^{(k)}\}_{k=0}^\infty | \phi^{(k)} \in \mathbb{C},\ k \geq 0, \sum_{k=0}^\infty |\phi^{(k)}|^2 < \infty \right\},$$

the boson Fock space over $\mathbb{C}$. Hence $L^2(\mathbb{R}^n) \cong \otimes^n \mathcal{F}_{\mathrm{b}}(\mathbb{C})$. In this sense, $L^2(\mathbb{R}^n)$ can be interpreted as a Hilbert space of a quantum system consisting of bosons of n kind without space degrees. In the present paper, we take this point of view, keeping in mind possible infinite dimensional extensions.

[2] In Section 6 in the present paper, we briefly describe a general mathematical framework of SQM. For physical aspects of SQM, see, e.g., [8].

One has the following natural isomorphism:

$$\mathcal{F}_{n,r} \cong L^2(\mathbb{R}^n; \wedge(\mathbb{C}^r)) \cong \int_{\mathbb{R}^n}^{\oplus} \wedge(\mathbb{C}^r)\mathrm{d}x, \tag{1.11}$$

where $L^2(\mathbb{R}^n; \wedge(\mathbb{C}^r))$ is the Hilbert space of $\wedge(\mathbb{C}^r)$-valued square integrable functions on $\mathbb{R}^n$ and $\int_{\mathbb{R}^n}^{\oplus} \wedge(\mathbb{C}^r)\mathrm{d}x$ is the constant fiber direct integral over $\mathbb{R}^n$ with fiber $\wedge(\mathbb{C}^r)$.

Let b_j $(j = 1, \ldots, r)$ be the linear operator on $\wedge(\mathbb{C}^r)$ such that its adjoint b_j^* is of the following form:

$$(b_j^*\psi)^{(0)} = 0, \quad (b_j^*\psi)^{(p)} = \sqrt{p}A_p(e_j \otimes \psi^{(p-1)}), \quad \psi \in \wedge(\mathbb{C}^r),\ 1 \le p \le r,\ j = 1, \ldots, r, \tag{1.12}$$

where $\{e_j\}_{j=1}^r$ is the standard orthonormal basis of $\mathbb{C}^r$. The operator b_j (resp. b_j^*) is called the j-th fermion annihilation (resp. creation) operator on $\wedge(\mathbb{C}^r)$. It follows that

$$\{b_j, b_k^*\} = \delta_{jk}, \quad ,\{b_j, b_k\} = 0, \quad \{b_j^*, b_k^*\} = 0, \quad j, k = 1, \ldots, r,$$

where $\{A, B\} := AB + BA$, the anti-commutator of A and B.

The Hilbert space of a supersymmetric quantum system is given by

$$\mathcal{H}_n := \mathcal{F}_{n,n}, \tag{1.13}$$

$\mathcal{F}_{n,r}$ with the case $r = n$. In this case, Klimek and Lesniewski [11] consider the following supersymmetric Hamiltonian:

$$H_{\mathrm{KL}} = -\frac{\hbar^2}{2}\Delta - \frac{\hbar}{2}\Delta P + \frac{1}{2}|\nabla P|^2 + \sum_{j,k=1}^{n} \hbar(\partial_j\partial_k P)b_j^* b_k$$

acting in $\mathcal{H}_n$, where P is a polynomial of $x_1, \ldots, x_n$, $(x_1, \ldots, x_n) \in \mathbb{R}^n$. They derived the following GT type inequality:

$$\mathrm{Tr}\, e^{-\beta H_{\mathrm{KL}}} \le \frac{1}{(2\pi\beta)^{n/2}\hbar^n} \int_{\mathbb{R}^n} \det(I + e^{-\beta\hbar\nabla\otimes\nabla P(x)}) e^{-\frac{\beta}{2}\left(|\nabla P(x)|^2 - \hbar\Delta P(x)\right)}\mathrm{d}x \tag{1.14}$$

for all $\beta > 0$, where $\nabla \otimes \nabla P(x)$ $(x \in \mathbb{R}^n)$ is the $n \times n$ matrix whose (j, k) component is equal to $\partial_j\partial_k P(x)$ $(j, k = 1, \ldots, n)$ and, for an $n \times n$ matrix M, $\det M$ denotes the determinant of M.

In this case too, it is interesting to ask when the equality is attained in (1.14). But, as shown in the next example, the equality in (1.14) is not attained in the case of a supersymmetric quantum harmonic oscillator, one of the simplest models in SQM and a finite dimensional version of a free supersymmetric quantum field model. In this sense, the inequality (1.14) is somewhat unsatisfactory.

Example 1.7 (A supersymmetric quantum harmonic oscillator) Consider the case where

$$P(x) = \frac{1}{2}\sum_{i=1}^{n} \omega_i x_i^2, \quad x \in \mathbb{R}^n$$

with constants $\omega_i > 0, i = 1, \cdots, n$. Then H_{KL} takes the form

$$H_\omega := \widehat{H}_{\mathrm{os}} + H_{\mathrm{f}}, \tag{1.15}$$

where

$$\widehat{H}_{\mathrm{os}} := -\frac{\hbar^2}{2}\Delta - \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j + \frac{1}{2}\sum_{j=1}^{n}\omega_j^2 x_j^2,$$

$$H_{\mathrm{f}} := \sum_{j=1}^{n} \hbar\omega_j b_j^* b_j.$$

Note that the operator $\widehat{H}_{\mathrm{os}}$ is the Hamiltonian $H_{\mathrm{os}} - \frac{\hbar}{2}\sum_{j=1}^{n}\omega_j$ with $m = 1$ (see (1.7)). Hence

$$\sigma(\widehat{H}_{\mathrm{os}}) = \sigma_{\mathrm{p}}(\widehat{H}_{\mathrm{os}}) = \left\{ \sum_{j=1}^{n} k_j\hbar\omega_j \middle| k_j \in \{0\} \cup \mathbb{N}, j = 1, \dots, n \right\},$$

counting multiplicities. Therefore, as in Example 1.6, we have

$$\operatorname{Tr} e^{-\beta \widehat{H}_{\mathrm{os}}} = \frac{1}{\prod_{j=1}^{n}\left(1 - e^{-\beta\hbar\omega_j}\right)}. \tag{1.16}$$

It is well known or easy to see that

$$\sigma(H_{\mathrm{f}}) = \sigma_{\mathrm{p}}(H_{\mathrm{f}}) = \left\{ \sum_{j=1}^{n} k_j\hbar\omega \middle| k_1, \dots, k_n \in \{0, 1\} \right\},$$

counting multiplicities. Hence

$$\operatorname{Tr} e^{-\beta H_{\mathrm{f}}} = \prod_{j=1}^{n}\left(1 + e^{-\beta\hbar\omega_j}\right).$$

Therefore, for all $\beta > 0$, $e^{-\beta H_\omega}$ is trace class and

$$\operatorname{Tr} e^{-\beta H_\omega} = \left(\operatorname{Tr} e^{-\beta \widehat{H}_{\mathrm{os}}}\right)\left(\operatorname{Tr} e^{-\beta H_{\mathrm{f}}}\right) = \prod_{j=1}^{n} \frac{1 + e^{-\beta\hbar\omega_j}}{1 - e^{-\beta\hbar\omega_j}} = \prod_{j=1}^{n} \coth\frac{\beta\hbar\omega_j}{2}.$$

Let

$$I_P(\beta) := \frac{1}{(2\pi\beta)^{n/2}\hbar^n}\int_{\mathbb{R}^n} \det(I + e^{-\beta\hbar\nabla\otimes\nabla P(x)})e^{-\frac{\beta}{2}\left(|\nabla P(x)|^2 - \hbar\Delta P(x)\right)}\mathrm{d}x$$

Then (1.14) takes the form

$$\operatorname{Tr} e^{-\beta H_{\mathrm{KL}}} \le I_P(\beta).$$

In the present example, we have

$$\nabla P = (\omega_j x_j)_{j=1}^n,\ \nabla \otimes \nabla P = (\omega_j \delta_{jk}),\ \Delta P = \sum_{j=1}^n \omega_j.$$

Hence

$$\begin{aligned} I_P(\beta) &= \frac{1}{(2\pi\beta)^{n/2}\hbar^n} \int_{\mathbb{R}^n} \left\{ \prod_{j=1}^n (I + e^{-\beta\hbar\omega_j}) \right\} e^{-\beta \sum_{j=1}^n \omega_j^2 x_j^2/2 + \beta\hbar \sum_{j=1}^n \omega_j/2} \mathrm{d}x \\ &= \prod_{j=1}^n \frac{\cosh \dfrac{\beta\hbar\omega_j}{2}}{\dfrac{\beta\hbar\omega_j}{2}}. \end{aligned}$$

But, since $\sinh\chi > \chi$ for all $\chi > 0$,

$$\coth \frac{\beta\hbar\omega_j}{2} < \frac{\cosh \dfrac{\beta\hbar\omega_j}{2}}{\dfrac{\beta\hbar\omega_j}{2}}.$$

Hence

$$\operatorname{Tr} e^{-\beta H_\omega} < I_P(\beta).$$

Thus the equality in (1.14) does not hold.

From a quantum field theoretical point of view, it would be desirable to find a GT type inequality which has the following properties:

(i) It attains the equality in the case of supersymmetric quantum harmonic oscillators.

(ii) It can be extended in natural way to an GT type inequality in infinite dimensions.

This is one of the motivations for this work.

2 A Unification of GT Type Inequalities for a Boson System

Before discussing boson-fermion systems in general, we first present a unification of GT type inequalities for a boson system whose Hilbert space of state vectors is $L^2(\mathbb{R}^n)$.

A new idea here is to take, as an unperturbed operator, a *self-adjoint operator* H_{b} *on* $L^2(\mathbb{R}^n)$ *bounded from below such that* $e^{-\beta H_{\mathrm{b}}}$ *(*$\beta > 0$*) is an integral operator with an integral kernel* $K_\beta(x,y)$ *(*$x, y \in \mathbb{R}^n$*) which is strictly positive, continuous in* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$K_\beta \in C(\mathbb{R}^n \times \mathbb{R}^n), \quad K_\beta(x,y) > 0, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{2.1}$$

$$e^{-\beta H_{\mathrm{b}}} f(x) = \int_{\mathbb{R}^n} K_\beta(x,y) f(y) \mathrm{d}y, \quad f \in L^2(\mathbb{R}^n),\ \beta > 0, \text{ a.e.} x \in \mathbb{R}^n. \tag{2.2}$$

Let V be a real-valued Borel measurable function on $\mathbb{R}^n$, *bounded below*, and

$$H_{\mathrm{b}}(V) := H_{\mathrm{b}} + V, \tag{2.3}$$

acting in $L^2(\mathbb{R}^n)$.

The following conditions (A.1)–(A.2) will be needed:

(A.1) The operator $H_{\mathrm{b}}(V)$ is essentially self-adjoint.

(A.2)

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} K_{\beta/2}(x,y)^2 e^{-\beta V(y)}\mathrm{d}x\mathrm{d}y < \infty, \tag{2.4}$$

where $\beta > 0$ is a constant parameter.

We denote the set of trace class operators on a Hilbert space $\mathcal{K}$ by $\mathcal{J}_1(\mathcal{K})$.

A basic fact on H_{b} and V is given in the following lemma:

Lemma 2.1 *Under condition (A.2),* $e^{-\beta V/2}e^{-\beta H_{\mathrm{b}}}e^{-\beta V/2} \in \mathcal{J}_1(L^2(\mathbb{R}^n))$ *and*

$$\mathrm{Tr}\, e^{-\beta V/2}e^{-\beta H_{\mathrm{b}}}e^{-\beta V/2} = \int_{\mathbb{R}^n} K_\beta(x,x)e^{-\beta V(x)}\mathrm{d}x. \tag{2.5}$$

Proof. Let

$$A := e^{-\beta V/2}e^{-\beta H_{\mathrm{b}}}e^{-\beta V/2}.$$

Then $A = B^*B$ with

$$B = e^{-\beta H_{\mathrm{b}}/2}e^{-\beta V/2}.$$

It is easy to see that B is an integral operator on $L^2(\mathbb{R}^n)$ with the integral kernel

$$k_B(x,y) := K_{\beta/2}(x,y)e^{-\beta V(y)/2}, \quad x,y\in\mathbb{R}^n.$$

Hence

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} |k_B(x,y)|^2\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n\times\mathbb{R}^n} K_{\beta/2}(x,y)^2e^{-\beta V(y)}\mathrm{d}y < \infty.$$

Hence B is Hilbert–Schmidt. Therefore A is trace class and

$$\mathrm{Tr}\, A = \|B\|_2^2 = \int_{\mathbb{R}^n\times\mathbb{R}^n} |k_B(x,y)|^2\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n\times\mathbb{R}^n} K_{\beta/2}(x,y)^2e^{-\beta V(y)}\mathrm{d}x\mathrm{d}y$$

We note the following facts:

$$\text{(Hermiticity) } K_t(x,y) = K_t(y,x), \quad t>0,\ (x,y)\in\mathbb{R}^n\times\mathbb{R}^n, \tag{2.6}$$

$$\text{(chain rule) } \int_{\mathbb{R}^n} K_t(x,y)K_s(y,z)\mathrm{d}y = K_{t+s}(x,z), \quad s,t>0,\ x,z\in\mathbb{R}^n. \tag{2.7}$$

Using these facts, we have

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} K_{\beta/2}(x,y)^2e^{-\beta V(y)}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n} K_\beta(y,y)e^{-\beta V(y)}\mathrm{d}y.$$

Hence (2.5) follows. ∎

Theorem 2.2 *Under conditions (A.1) and (A.2), $e^{-\beta\overline{H_{\mathrm{b}}(V)}} \in \mathfrak{J}_1(L^2(\mathbb{R}^n))$ and*

$$\operatorname{Tr} e^{-\beta\overline{H_{\mathrm{b}}(V)}} \leq \int_{\mathbb{R}^n} K_\beta(x,x)e^{-\beta V(x)}\mathrm{d}x. \tag{2.8}$$

Proof. By Lemma 2.1 and Theorem 1.1, $e^{-\beta\overline{H_{\mathrm{b}}(V)}}$ is trace class and

$$\operatorname{Tr} e^{-\beta\overline{H_{\mathrm{b}}(V)}} \leq \operatorname{Tr} e^{-\beta V/2}e^{-\beta H_{\mathrm{b}}}e^{-\beta V/2}.$$

By this inequality and (2.5), we obtain (2.8). ∎

Remark 2.3 By a limiting argument, one can extend (2.8) for a more general class of V. But, here, we omit the details. The same applies to statements below.

If $e^{-\beta H_{\mathrm{b}}} \in \mathfrak{J}_1(L^2(\mathbb{R}^n))$, then $\int_{\mathbb{R}^n} K_\beta(x,x)\mathrm{d}x < \infty$ and

$$\operatorname{Tr} e^{-\beta H_{\mathrm{b}}} = \int_{\mathbb{R}^n} K_\beta(x,x)\mathrm{d}x. \tag{2.9}$$

Hence, if $e^{-\beta H_{\mathrm{b}}} \in \mathfrak{J}_1(L^2(\mathbb{R}^n))$, the equality in (2.8) with a finite value is attained in the case $V = 0$. Moreover, if we take $H_{\mathrm{b}} = -\hbar^2\Delta/2m$ (in this case, for each $\beta > 0$, $e^{-\beta H_{\mathrm{b}}} \notin \mathfrak{J}_1(L^2(\mathbb{R}^n))$), then (2.8) yields (1.5) (see Example 2.4 below). In these senses, (2.8) improves and generalizes (1.5). From a structural point of view, inequality (2.8) gives a unification for known GT type inequalities.

Example 2.4 A simple and elementary example is given by the case where

$$H_{\mathrm{b}} = -\frac{\hbar^2}{2m}\Delta.$$

In this case, we have by (1.4)

$$K_\beta(x,y) = \left(\frac{m}{2\pi\beta\hbar^2}\right)^{n/2} e^{-m|x-y|^2/2\hbar^2\beta}.$$

Hence (2.8) gives (1.5).

Example 2.5 A next example of H_{b} one may have in mind is the Hamiltonian of a quantum harmonic oscillator:

$$H_{\mathrm{b}} = \widehat{H}_{\mathrm{os}}.$$

We already know that $e^{-\beta\widehat{H}_{\mathrm{os}}}$ is trace class and (1.16) holds. Moreover, as is well known (e.g., [9, Theorem 1.5.10], [14, pp.37–38]) $e^{-\beta\widehat{H}_{\mathrm{os}}}$ ($\beta > 0$) is an integral operator with the integral kernel

$$Q_\beta(x,y) := \prod_{j=1}^n Q_\beta^{(j)}(x_j,y_j), \quad x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n) \in \mathbb{R}^n, \tag{2.10}$$

where

$$Q_\beta^{(j)}(x_j,y_j) := \sqrt{\frac{\omega_j e^{\hbar\omega_j\beta}}{2\pi\hbar\sinh\hbar\omega_j\beta}}\exp\Big(-\frac{\omega_j}{2\hbar}(x_j^2+y_j^2)\coth\hbar\omega_j\beta + \frac{\omega_j}{\hbar\sinh\hbar\omega_j\beta}x_jy_j\Big), \quad (x_j,y_j)\in\mathbb{R}\times\mathbb{R}. \tag{2.11}$$

It is easy to see that

$$Q_\beta(x,x) = \prod_{j=1}^n \sqrt{\frac{\omega_j e^{\hbar\omega_j\beta}}{2\pi\hbar\sinh\hbar\omega_j\beta}}\exp\left(-\frac{\omega_j\tanh\frac{\hbar\omega_j\beta}{2}}{\hbar}x_j^2\right).$$

Hence (2.8) gives the following GT type inequality:

$$\mathrm{Tr}\, e^{-\beta\overline{(\widehat{H}_{\mathrm{os}}+V)}} \le \int_{\mathbb{R}^n} e^{-\beta V(x)}\prod_{j=1}^n \sqrt{\frac{\omega_j e^{\hbar\omega_j\beta}}{2\pi\hbar\sinh\hbar\omega_j\beta}}\exp\left(-\frac{\omega_j\tanh\frac{\hbar\omega_j\beta}{2}}{\hbar}x_j^2\right)\mathrm{d}x. \tag{2.12}$$

We also note that taking the limit $\omega_j \downarrow 0$ $(j=1,\dots,n)$ in (2.12) recovers (1.5) with $m=1$. In this sense too, (2.12) is a generalization of (1.5) and a better inequality.

A unification of Examples 2.4 and 2.5 is given in the following example.

Example 2.6 Consider the case where

$$H_{\mathrm{b}} = \overline{H}_U,$$

the Schrödinger operator given by (1.3) with $V=U$. Suppose that U is continuous on $\mathbb{R}^n$ and bounded below. Then, using a functional integral representation with a Brownian bridge, one can show that, for all $\beta>0$, $e^{-\beta\overline{H}_U}$ is an integral operator with a non-negative continuous integral kernel $e^{-\beta\overline{H}_U}(x,y)$ (see, e.g., [6, Theorem 4.43], [14, Theorem 6.6]). Hence, in the present example, (2.8) gives

$$\mathrm{Tr}\, e^{-\beta\overline{(\overline{H}_U+V)}} \le \int_{\mathbb{R}^n} e^{-\beta\overline{H}_U}(x,x)e^{-\beta V(x)}\mathrm{d}x, \tag{2.13}$$

provided that $\overline{H}_U+V$ is essentially self-adjoint and the integral on the right hand side of (2.13) is finite.

3 Applications

GT type inequalities can be applied to study spectral properties of a self-adjoint operator.

Let A be a self-adjoint operator. For each $E\in\mathbb{R}$, we define

$$N_E(A) := \#\{\lambda\in\sigma_{\mathrm{p}}(A)|\lambda\le E\},$$

the number of the eigenvalues of A less than or equal to E, counting multiplicities.

Lemma 3.1 *If $e^{-\beta A}$ is trace class for some $\beta > 0$, then*

$$N_E(A) \leq \operatorname{Tr} e^{-\beta(A-E)}, \tag{3.1}$$

independently of β.

Proof. Let $\{\lambda_n\}_n$ be the set of distinct eigenvalues of A with $\lambda_1 < \lambda_2 < \cdots$ and m_j be the multiplicity of λ_j. Then

$$\operatorname{Tr} e^{-\beta(A-E)} \geq \sum_{\lambda_j \leq E} m_j e^{-\beta(\lambda_j - E)} \geq \sum_{\lambda_j \leq E} m_j = N_E(A).$$

∎

Theorem 3.2 *Under (A.1) and (2.4) for some β, the spectrum of $\overline{H_{\mathrm{b}}(V)}$ is purely discrete and, for each $E \in \mathbb{R}$,*

$$N_E(\overline{H_{\mathrm{b}}(V)}) \leq \int_{\mathbb{R}^n} K_\beta(x,x) e^{-\beta(V(x)-E)} \mathrm{d}x \tag{3.2}$$

Proof. The discreteness of the spectrum of $\overline{H_{\mathrm{b}}(V)}$ follows from that $e^{-\beta \overline{H_{\mathrm{b}}(V)}}$ is trace class and hence compact. Inequality (3.2) follows from Lemma 3.1 with $A = \overline{H_{\mathrm{b}}(V)}$ and (2.8). ∎

Remark 3.3 *Assume (A.1) and that (2.4) holds for all $\beta > 0$. Then (3.2) implies a more refined inequality:*

$$N_E(\overline{H_{\mathrm{b}}(V)}) \leq \inf_{\beta > 0} \int_{\mathbb{R}^n} K_\beta(x,x) e^{-\beta(V(x)-E)} \mathrm{d}x \tag{3.3}$$

Theorem 3.4 *Assume (A.1) and that (2.4) holds for all $\beta \geq \beta_0$ with some $\beta_0 > 0$. In addition, suppose that the following hold:*

(i) *For some $E \in \mathbb{R}$, $V(x) > E$ a.e. $x \in \mathbb{R}^n$ and*

$$\lim_{\beta \to \infty} K_\beta(x,x) e^{-\beta(V(x)-E)} = 0, \quad \text{a.e.} x \in \mathbb{R}^n. \tag{3.4}$$

(ii) *There exists an integrable function $g \geq 0$ on $\mathbb{R}^n$ satisfying*

$$K_\beta(x,x) e^{-\beta(V(x)-E)} \leq g(x), \quad \beta \geq \beta_0, \text{ a.e.} x \in \mathbb{R}^n. \tag{3.5}$$

Then

$$\inf \sigma(\overline{H_{\mathrm{b}}(V)}) > E. \tag{3.6}$$

Proof. For all $\beta \geq \beta_0$, (3.2) holds. By (i) and (ii), we can apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{\beta\to\infty}\int_{\mathbb{R}^n} K_\beta(x,x)e^{-\beta(V(x)-E)}\mathrm{d}x = 0.$$

Hence, by (3.2), $N_E(\overline{H_\mathrm{b}(V)}) = 0$. This implies (3.6). ∎

Remark 3.5 In general, for a self-adjoint operator A bounded below, $E_0(A) := \inf\sigma(A)$ (the infimum of the spectrum $\sigma(A)$ of A) is called the *ground state energy of A*. Hence, under the assumption of Theorem 3.4, (3.6) gives a lower bound for the ground state energy $E_0(\overline{H_\mathrm{b}(V)})$ of $\overline{H_\mathrm{b}(V)}$:

$$E_0(\overline{H_\mathrm{b}(V)}) > E. \tag{3.7}$$

To consider a meaning of (3.7), let $H_\mathrm{b} = -\hbar^2/\Delta/2m$. Then

$$H_\mathrm{b}(V) = H_V$$

with $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ bounded below satisfying

$$\int_{\mathbb{R}^n} e^{-\beta V(x)}\mathrm{d}x < \infty$$

for all $\beta \geq \beta_0$ ($\beta_0 > 0$ is a constant). Then (A.1) and (2.4) with $\beta \geq \beta_0$ hold. In this case we have by Example 2.4

$$K_\beta(x,x) = \left(\frac{m}{2\pi\hbar^2\beta}\right)^{n/2}.$$

Suppose that $V(x) > E$ for a.e. $x \in \mathbb{R}^n$. Then the assumption of Theorem 3.4 is satisfied. Hence (3.7) gives

$$E_0(\overline{H}_V) > E.$$

Suppose that, for some $x_0 \in \mathbb{R}^n$, $V(x_0) = E$. Then the classical ground state energy

$$E_{\mathrm{cl}} := \inf_{x,p\in\mathbb{R}^n}\left(\frac{p^2}{2m} + V(x)\right)$$

is equal to E. Hence

$$E_0(\overline{H}_V) > E_{\mathrm{cl}}.$$

This means that the quantum ground state energy is more than the classical one. This phenomenon is called the *enhancement of the ground state energy due to quantization.* In a previous paper [4], the enhancement of the ground state energy is discussed by a different method which makes it possible to treat a more general class of potentials V.

4 Functional Integral Representations for a Boson System

In this section we consider a generalization of functional integral representations for a boson system derived in [7]. The idea for that is to use a conditional measure associated with the heat semi-group $\{e^{-\beta H_{\rm b}}\}_{\beta\geq 0}$.

For convenience, we define

$$K_0(x,y) := \delta(x-y), \tag{4.1}$$

the n-dimensional Dirac's delta distribution.

Let $\dot{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ be the one-point compactification of $\mathbb{R}$ and

$$\Omega := \{\omega : [0,\infty) \to \dot{\mathbb{R}}^n\}, \tag{4.2}$$

the set of mappings from $[0,\infty)$ to $\dot{\mathbb{R}}^n$. For each $t\in[0,\infty)$ we define a function $q(t) = (q_1(t),\ldots,q_n(t)) : \Omega \to \mathbb{R}^n$ by

$$q_j(t)(\omega) := \begin{cases} 0 & \text{if } \omega_j(t) = \infty \\ \omega_j(t) & \text{if } \omega_j(t) \in \mathbb{R} \end{cases}, \tag{4.3}$$

where $\omega(t) = (\omega_1(t),\ldots,\omega_n(t)) \in \dot{\mathbb{R}}^n, t\geq 0$. Let $\mathcal{B}$ be the Borel field generated by $\{q_j(t) | j=1,\ldots,n,\ t\in[0,\infty)\}$.

Lemma 4.1 *Let $\beta>0$ and $a,c\in\mathbb{R}^n$ be fixed arbitrarily. Let $0\leq t_1\leq t_2\leq\cdots\leq t_n\leq\beta$. Then there exists a probability measure $P_{a,c;\beta}$ on $(\Omega,\mathcal{B})$ such that the joint distribution of $(q(t_1),\cdots,q(t_n))$ is given by*

$$K_\beta(a,c)^{-1}K_{t_1}(a,x_1)K_{t_2-t_1}(x_1,x_2)\cdots K_{t_n-t_{n-1}}(x_{n-1},x_n)K_{\beta-t_n}(x_n,c)\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

Namely, for all Borel sets $B\subset\mathbb{R}^n$,

$$\begin{aligned} &P_{a,c;\beta}(\{\omega\in\Omega | (q(t_1),\ldots,q(t_n))\in B\}) \\ &= \int_B K_\beta(a,c)^{-1}K_{t_1}(a,x_1)K_{t_2-t_1}(x_1,x_2)\cdots K_{t_n-t_{n-1}}(x_{n-1},x_n) \\ &\qquad\qquad \times K_{\beta-t_n}(x_n,c)\mathrm{d}x_1\cdots\mathrm{d}x_n. \end{aligned} \tag{4.4}$$

Proof. This follows from a simple application of Kolmogorov's theorem (e.g., [14, Theorem 2.1]). For a proof, see [6, Lemma 4.40]. ∎

We define a finite measure $\mu_{a,c;\beta}$ on $(\Omega,\mathcal{B})$ by

$$\mathrm{d}\mu_{a,c;\beta} := K_\beta(a,c)\mathrm{d}P_{a,c;\beta}. \tag{4.5}$$

Note that

$$\int 1\,\mathrm{d}\mu_{x,y;t} = K_\beta(x,y), \quad x,y\in\mathbb{R}^n. \tag{4.6}$$

Remark 4.2 In the case where $H_{\mathrm{b}} = \widehat{H}_{\mathrm{os}}$ so that $K_\beta(x,y) = Q_\beta(x,y)$, $\mu_{a,c;\beta}$ is called a *conditional oscillator measure.* This measure is used in [7] to derive functional integral representations for a boson system.

In what follows, we assume the following:

(A.3) For all $\beta > 0$, $e^{-\beta H_{\mathrm{b}}}$ is trace class.

(A.4) For all real-valued functions V on $\mathbb{R}^n$ which are in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$, $H_{\mathrm{b}}(V)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

For a complex Hilbert space $\mathcal{K}$, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product (linear in the second variable) and norm of $\mathcal{K}$ respectively. We denote by $L^\infty(\mathbb{R}^n)$ the set of essentially bounded Borel measurable functions on $\mathbb{R}^n$ and by $\|f\|_\infty$ the essential supremum of f.

We first derive trace formulae concerning the heat semi-group $\{e^{-\beta H_{\mathrm{b}}} | \beta \geq 0\}$:

Lemma 4.3 *Assume (A.3). Let* $0 < t_1 < \cdots < t_m < \beta$ *and* $f_j \in L^\infty(\mathbb{R}^n)$ $(j = 1, \ldots, m)$. *Then* $e^{-t_1 H_{\mathrm{b}}} f_1 e^{-(t_2-t_1)H_{\mathrm{b}}} f_2 \cdots f_m e^{-(\beta-t_m)H_{\mathrm{b}}}$ *is in* $\mathcal{J}_1(L^2(\mathbb{R}^n))$ *and*

$$\begin{aligned} &\mathrm{Tr}\left(e^{-t_1 H_{\mathrm{b}}} f_1 e^{-(t_2-t_1)H_{\mathrm{b}}} f_2 \cdots f_m e^{-(\beta-t_m)H_{\mathrm{b}}}\right) \\ &= \int_{\mathbb{R}^n} \mathrm{d}x \left(\int f_1(q(t_1)) \cdots f_m(q(t_m)) \mathrm{d}\mu_{x,x;\beta} \right). \end{aligned} \tag{4.7}$$

Proof. Similar to the proof of Lemma 3.1 in [7]. ∎

Using this lemma, one can derive a functional integral representation for $\mathrm{Tr}\, e^{-\beta \overline{H_{\mathrm{b}}(V)}}$.

Theorem 4.4 *Assume (A.3) and (A.4). Suppose that, for all* $\beta > 0$,

$$\int_{\mathbb{R}^n} K_\beta(x,x) e^{-\beta V(x)} \mathrm{d}x < \infty. \tag{4.8}$$

Then, for all $\beta > 0$, $e^{-\beta \overline{H_{\mathrm{b}}(V)}} \in \mathcal{J}_1(L^2(\mathbb{R}^n))$ *and the following* (i) *and* (ii) *hold:*

(i) (A Golden–Thompson type inequality)

$$\mathrm{Tr}\, e^{-\beta \overline{H_{\mathrm{b}}(V)}} \leq \int_{\mathbb{R}^n} K_\beta(x,x) e^{-\beta V(x)} \mathrm{d}x. \tag{4.9}$$

(ii) (A functional integral representation for the partition function)

$$\mathrm{Tr}\, e^{-\beta \overline{H_{\mathrm{b}}(V)}} = \int_{\mathbb{R}^n} \mathrm{d}x \int_\Omega e^{-\int_0^\beta V(q(t))dt} d\mu_{x,x;\beta}. \tag{4.10}$$

Proof. Similar to the proof of [7, Theorem 3.5]. ∎

Remark 4.5

(1) In Theorem 4.4, V is not necessarily bounded below. This may be one of the results showing effectiveness of the functional integral approach.

(2) Inequality (4.9) is a generalization of (2.13). If $V=0$, then the equality in (4.9) holds .

The functional integral representation (4.10) can be extended to a more general class of objects.

Theorem 4.6 *Assume (A.3) and (A.4). Let $V_1,\ldots,V_m \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ be such that, for all $\beta>0$ and $j=1,\ldots,m$,*

$$\int_{\mathbb{R}^n} K_\beta(x,x)e^{-\beta V_j(x)}\mathrm{d}x < \infty.$$

Let $0<t_1<\cdots<t_m<\beta$ and $f_j \in L^\infty(\mathbb{R}^n)$ $(j=1,\ldots,m)$. Then

$$e^{-t_1\overline{H_{\mathrm{b}}(V_1)}}f_1e^{-(t_2-t_1)\overline{H_{\mathrm{b}}(V_2)}}f_2\cdots f_me^{-(\beta-t_m)\overline{H_{\mathrm{b}}(V_m)}}$$

is in $\mathfrak{I}_1(L^2(\mathbb{R}^n))$ and

$$\begin{aligned}&\mathrm{Tr}\left(e^{-t_1\overline{H_{\mathrm{b}}(V_1)}}f_1e^{-(t_2-t_1)\overline{H_{\mathrm{b}}(V_2)}}f_2\cdots f_me^{-(\beta-t_m)\overline{H_{\mathrm{b}}(V_m)}}\right)\\&=\int_{\mathbb{R}^n}\mathrm{d}x\left(\int f_1(q(t_1))\cdots f_m(q(t_m))e^{-\sum_{j=1}^{m+1}\int_{t_{j-1}}^{t_j}V_j(q(t))\mathrm{d}t}\mathrm{d}\mu_{x,x;\beta}\right),\end{aligned} \tag{4.11}$$

where $t_0=0, t_{m+1}=\beta$.

Proof. Similar to the proof of [7, Theorem 3.8]. ∎

5 A Boson–Fermion System

We now consider a boson–fermion system. The Hilbert space of state vectors of the system is taken to be $\mathcal{F}_{n,r}$ defined by (1.9).

The purely bosonic part of the total Hamiltonian of the boson–fermion system is taken to be $H_{\mathrm{b}}(V)$ discussed in the preceding section.

To introduce a fermionic part of the total Hamiltonian, including an interaction between the bosons and the fermions, let $\mathbb{U}=(U_{jk})_{j,k=1,\ldots,r}$ be an $r\times r$ Hermitian matrix-valued function on $\mathbb{R}^n$, i.e., the (j,k) component U_{jk} of $\mathbb{U}$ is a Borel measurable function on $\mathbb{R}^n$ such that $U_{kj}(x)^*=U_{jk}(x)$, $j,k=1,\ldots,r$, a.e.$x\in\mathbb{R}^n$ (for a complex number z, z^* denotes the complex conjugate of z). Then we define

$$H_{\mathrm{f},\mathbb{U}} := \sum_{j,k=1}^r U_{jk}b_j^*b_k = \int_{\mathbb{R}^n}^\oplus \sum_{j,k=1}^r U_{jk}(x)b_j^*b_k\mathrm{d}x.$$

This is the fermionic part of the total Hamiltonian. Note that $H_{\mathrm{f},\mathbb{U}}$ describes an interaction between the bosons and the fermions if $\mathbb{U}$ is not a constant matrix.

The total Hamiltonian is defined by

$$H(V,\mathbb{U}) := H_{\mathrm{b}}(V) + H_{\mathrm{f},\mathbb{U}}. \tag{5.1}$$

We need the following conditions:

(A.5) $V \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and $H_{\mathrm{b}}(V)$ is self-adjoint and bounded below. Moreover, for all $\beta > 0$, $e^{-\beta H_{\mathrm{b}}(V)} \in \mathcal{I}_1(L^2(\mathbb{R}^n))$.

(A.6) There exist constants $\alpha \in [0,1)$ and $a, b > 0$ such that

$$|U_{jk}(x)|^2 \le a|V(x)|^{2\alpha} + b, \quad \text{a.e.} x \in \mathbb{R}^n, j,k = 1,\ldots,r$$

Lemma 5.1 *Assume (A.5) and (A.6). Then $H(V,\mathbb{U})$ is self-adjoint and bounded below.*

Proof. Similar to the proof of [7, Lemma 2.3-(i)]. ∎

Let

$$N_{\mathrm{f}} := \sum_{j=1}^{r} b_j^* b_j,$$

the *fermion number operator* on $\wedge(\mathbb{C}^r)$.

The following theorem is a basic result on the boson-fermion Hamiltonian $H(V,\mathbb{U})$.

Theorem 5.2 *Assume (A.3), (A.5), (A.6) and (4.8). Suppose that, for all $\beta > 0$,*

$$\int_{V(x)<0} e^{\beta \sum_{j,k=1}^{r} |U_{jk}(x)|} e^{-\beta V(x)} K_\beta(x,x)\mathrm{d}x < \infty. \tag{5.2}$$

Let $z \in \mathbb{C}\setminus\{0\}$ and $F \in L^\infty(\mathbb{R}^n)$. Then, for all $\beta > 0$, $e^{-\beta H(V,\mathbb{U})}$ is in $\mathcal{I}_1(\mathcal{F}_{n,r})$ and

$$\mathrm{Tr}\left(F z^{N_{\mathrm{f}}} e^{-\beta H(V,\mathbb{U})}\right) = \int_{\mathbb{R}^n} \mathrm{d}x F(x) \int \det\left(I + z e^{-\int_0^\beta \mathbb{U}(q(t))\mathrm{d}t}\right) e^{-\int_0^\beta V(q(t))\mathrm{d}t} \mathrm{d}\mu_{x,x;\beta}. \tag{5.3}$$

Proof. Similar to the proof of [7, Theorem 4.2]. ∎

One can derive Golden–Thompson type inequalities from (5.3).

By using the chain rule (2.7), one can easily show that

$$L_\beta(x,y) := \frac{1}{\beta}\int_0^\beta K_t(x,y)K_{\beta-t}(y,x)\mathrm{d}t. \tag{5.4}$$

is finite for a.e. $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 5.3 *Assume (A.3), (A.5), (A.6), (4.8) and that (5.2) holds for all* $\beta > 0$. *Let* $z \in \mathbb{C} \setminus \{0\}$, $\beta > 0$ *and* $F \in L^\infty(\mathbb{R}^n)$. *Then*

$$\left|\mathrm{Tr}\,(Fz^{N_\mathrm{f}}e^{-\beta H(V,\mathbb{U})})\right| \le \int_{\mathbb{R}^n} \mathrm{d}x \int_{\mathbb{R}^n} \mathrm{d}y |F(x)| L_\beta(x,y) \det(I + |z| e^{-\beta \mathbb{U}(y)}) e^{-\beta V(y)} \tag{5.5}$$

In particular,

$$\mathrm{Tr}\, e^{-\beta H(V,\mathbb{U})} \le \int_{\mathbb{R}^n} \mathrm{d}x K_\beta(x,x) \det(I + e^{-\beta \mathbb{U}(x)}) e^{-\beta V(x)}. \tag{5.6}$$

Proof. Similar to the proof of [7, Theorem 5.1]. ∎

Remark 5.4 In the same manner as in [7, Theorem 4.6], we can extend Theorems 5.2 and 5.3 to a more general class of V.

6 Application to SQM

The boson-fermion system considered in the preceding section includes, as a special case, a class of SQM (see below). Hence the results concerning the boson-fermion system can be applied to such supersymmetric quantum systems.

For the reader's convenience, we recall an abstract mathematical definition of SQM (see, e.g., [16, Chapter 5] and [5, Chapter 9] for more details).

6.1 Definition of SQM and basic properties

A SQM is a quartet $(\mathcal{H}, \Gamma, Q, H)$ consisting of a complex Hilbert space $\mathcal{H}$, a unitary self-adjoint operator $\Gamma \neq \pm 1$ and self-adjoint operators Q, H satisfying the following conditions:

(SQM.1) The operator Γ leaves $D(Q)$ (the domain of Q) invariant (i.e. $\Gamma D(Q) \subset D(Q)$) and $\{\Gamma, Q\}\psi = 0$, $\forall \psi \in D(Q)$.

(SQM.2) $H = Q^2$.

The operator Q (resp. H) is called the *supercharge* (resp. the *supersymmetric Hamiltonian*).

It follows that $\sigma(\Gamma) = \sigma_\mathrm{p}(\Gamma) = \{\pm 1\}$. Hence $\mathcal{H}$ has the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

with $\mathcal{H}_+ := \ker(\Gamma - 1)$ and $\mathcal{H}_- := \ker(\Gamma + 1)$. The subspace $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) is called the *bosonic* (resp. *fermionic*) subspace.

Property (SQM.1) implies that Q maps $D(Q) \cap \mathcal{H}_\pm$ to $\mathcal{H}_\mp$ and hence Q has the operator matrix representation

$$Q = \begin{pmatrix} 0 & Q_+^* \\ Q_+ & 0 \end{pmatrix}$$

with respect to the row vector representation of $\mathcal{H}$

$$\mathcal{H} = \left\{ \left(\begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) \middle| \psi_\pm \in \mathcal{H}_\pm \right\},$$

where Q_+ is a densely defined closed operator from $\mathcal{H}_+$ to $\mathcal{H}_-$. Hence it follows from (SQM.2) that H is reduced by $\mathcal{H}_\pm$ and

$$H = H_+ \oplus H_- = \left(\begin{array}{cc} H_+ & 0 \\ 0 & H_- \end{array} \right)$$

with $H_+ := Q_+^* Q_+$ and $H_- = Q_+ Q_+^*$. The reduced part H_+ (resp. H_-) is called the *bosonic* (resp. *fermionic*) *Hamiltonian.*

If $\ker Q \neq \{0\}$, then each non-zero vector in $\ker Q$ is called a *supersymmetric state.* If $\ker Q = \{0\}$, then the supersymmetry is said to be *spontaneously broken.*

Remark 6.1 In the physical view point which regards supersymmetry as a more fundamental principle in the universe, supersymmetry is expected to be spontaneously broken. In this context too, it is important to investigate $\ker Q$.

The easily proved relation

$$\ker Q = \ker H = \ker H_+ \oplus \ker H_- \tag{6.1}$$

is useful to investigate $\ker Q$.

A standard method to see if spontaneous supersymmetry breaking occurs is to estimate the *analytical index*

$$\mathrm{ind}(Q_+) := \dim\ker Q_+ - \dim\ker Q_+^*$$

of Q_+, which is defined under the condition that at least one of $\dim\ker Q_+$ and $\dim\ker Q_+^*$ is finite. If supersymmetry is spontaneously broken, then $\ker Q_+ = \{0\}$ and $\ker Q_+^* = \{0\}$ and hence $\mathrm{ind}(Q_+) = 0$. Therefore $\mathrm{ind}(Q_+) = 0$ gives a necessary condition for supersymmetry to be spontaneously broken. The following fact is well known (e.g., [16, Theorem 5.19] and [5, Theorem 9.16]):

Lemma 6.2 *Suppose that, for some $\beta > 0$, $e^{-\beta H}$ is trace class on $\mathcal{H}$. Then Q_+ is a Fredholm operator and*

$$\mathrm{ind}(Q_+) = \mathrm{Tr}\,(\Gamma e^{-\beta H}),$$

independently of β.

6.2 A model of SQM

We now discuss a model of SQM which includes the model considered by Klimek and Lesniewski [11].

Let $\mathcal{H}_n$ be the Hilbert space given by (1.13) and

$$\Gamma_n = (-1)^{N_f}. \tag{6.2}$$

Then it is easy to see that Γ_n is a unitary self-adjoint with $\Gamma_n \neq \pm 1$ and

$$\mathcal{H}_n = \mathcal{H}_{n+} \oplus \mathcal{H}_{n-}$$

with

$$\mathcal{H}_{n+} = \ker(\Gamma_n - 1) = \bigoplus_{p:\text{even}} L^2(\mathbb{R}^n) \otimes \wedge^p(\mathbb{C}^n),$$
$$\mathcal{H}_{n-} = \ker(\Gamma_n + 1) = \bigoplus_{p:\text{odd}} L^2(\mathbb{R}^n) \otimes \wedge^p(\mathbb{C}^n).$$

Let

$$a_j := \overline{\frac{i}{\sqrt{\hbar\omega_j}}\left(-i\hbar\frac{\partial}{\partial x_j} - i\omega_j x_j\right) \restriction C_0^\infty(\mathbb{R}^n)}, \; j = 1, \ldots, n,$$

Then, as is well known, the renormalized harmonic oscillator Hamiltonian $\widehat{H}_{\text{os}}$ defined by (1.16) is written as follows:

$$\widehat{H}_{\text{os}} = \sum_{j=1}^n \hbar\omega_j a_j^* a_j$$

The operator a_j (resp. a_j^*) is called the j-th *bosonic annihilation* (resp. *creation*) *operator.* Note that the following commutation relations hold on $C_0^\infty(\mathbb{R}^n)$:

$$[a_j, a_k^*] = \delta_{jk}, \quad [a_j, a_k] = 0, \quad [a_j^*, a_k^*] = 0, \; j, k = 1, \ldots, n,$$

where $[A, B] := AB - BA$, the commutator of A and B.

We introduce a Dirac type operator

$$Q_0 := i \sum_{j=1}^n \sqrt{\hbar\omega_j}(a_j b_j^* - a_j^* b_j).$$

It is not so difficult to show that Q_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and

$$H_\omega = \overline{Q}_0^2, \tag{6.3}$$

where H_ω is the operator defined by (1.15). Moreover, one can show that Γ_n leaves $D(\overline{Q}_0)$ invariant and

$$\{\Gamma_n, \overline{Q}_0\} = 0 \quad \text{on } D(\overline{Q}_0).$$

Thus $(\mathcal{H}_n, \Gamma_n, \overline{Q}_0, \widehat{H}_\omega)$ is a SQM. Using (6.1), one can prove that $\dim\ker Q = 1$. Hence, in this model, supersymmetry is not spontaneously broken.

We consider a perturbation of the Dirac type operator Q_0 to obtain a new supercharge (a perturbed Dirac type operator). Let W be a real distribution on $\mathbb{R}^n$ such that

$$W_j := D_j W \in L^4_{\text{loc}}(\mathbb{R}^n), \quad W_{jk} := D_j D_k W \in L^2_{\text{loc}}(\mathbb{R}^n) \quad (j, k = 1, \ldots, n),$$

where D_j denotes the distributional partial differential operator in the variable x_j, and

$$Q_1 := \frac{i}{\sqrt{2}} \sum_{j=1}^{n} \left(b_j^* W_j - b_j W_j \right).$$

Then a candidate for a new supercharge is defined by

$$Q_W := Q_0 + Q_1. \tag{6.4}$$

At this statge, we only know that Q_W is a symmetric operator on $\mathcal{H}_n$ satisfying

$$\{\Gamma_n, \overline{Q_W}\} = 0 \quad \text{on } D(\overline{Q_W}).$$

The self-adjointness of $\overline{Q_W}$ may depend on properties of W. Here we do not go into discussing the problem when $\overline{Q_W}$ is self-adjoint. Instead, we consider as a substitute for a perturbed supersymmetric Hamiltonian

$$H_{\text{SS}} := \overline{Q_W}^* \overline{Q_W}, \tag{6.5}$$

which, by von Neumann's theorem, is non-negative and self-adjoint. We have

$$\ker H_{\text{SS}} = \ker \overline{Q_W}. \tag{6.6}$$

Hence

$$\dim \ker \overline{Q_W} = \dim \ker H_{\text{SS}}. \tag{6.7}$$

To write down an explicit form of H_{SS} on a restricted subspace, let

$$\Phi_W(x) := \sum_{j=1}^{n} \omega_j x_j D_j W(x) + \frac{1}{2} \sum_{j=1}^{n} |D_j W(x)|^2 - \frac{\hbar}{2} \Delta W(x), \quad x \in \mathbb{R}^n. \tag{6.8}$$

Then we have

$$H_{\text{SS}} = \widehat{H}_{\text{os}} + \Phi_W + H_{\text{f}} + \hbar \sum_{j,k=1}^{n} W_{jk} b_j^* b_k \tag{6.9}$$

on

$$\mathcal{D}_0 := C_0^\infty(\mathbb{R}^n) \hat{\otimes} \wedge (\mathbb{C}^n), \tag{6.10}$$

where $\hat{\otimes}$ means algebraic tensor product. Hence $H_{\text{SS}} \restriction \mathcal{D}_0$ is the operator $H(V, \mathbb{U}) \restriction \mathcal{D}_0$ with

$$H_{\text{b}} = \widehat{H}_{\text{os}}, \; V = \Phi_W, \; \mathbb{U} = \hbar \mathbb{D} + \mathbb{W}, \tag{6.11}$$

where $\mathbb{D} := (\omega_j \delta_{jk})_{j,k=1,\ldots,n}$ and $\mathbb{W} = (W_{jk})_{j,k=1,\ldots,n}$. Therefore, if we impose suitable additional conditions on W, then we may apply the results in Section 5 to H_{SS}. Such conditions are given as follows:

(A.7) There exists a nonnegative continuous function $U \in L^2_{\text{loc}}(\mathbb{R}^n)$ satisfying the following conditions:

(a) For all $\varepsilon \in (0, \delta)$ with a constant $\delta > 0$, $\widehat{H}_{\rm os} + \varepsilon U$ is self-adjoint.

(b) For all $\eta > 0$, there exists a constant $c_\eta > 0$ such that

$$|\Phi_W(x)|^2 \le \eta^2 U(x)^2 + c_\eta^2, \quad \text{a.e.} x \in \mathbb{R}^n.$$

(c) There exist constants $\alpha \in [0,1)$ and $a, b > 0$ such that

$$|W_{jk}(x)|^2 \le aU(x)^{2\alpha} + b, \quad \text{a.e.} x \in \mathbb{R}^n, \quad j,k = 1,\dots,n.$$

(d) $D(\overline{Q_W}) \cap D(U^{1/2})$ is a core of $\overline{Q_W}$.

Let $Q_\beta(x,y)$ $(\beta > 0)$ be the integral kernel of $e^{-\beta \widehat{H}_{\rm os}}$ (see (2.10)) and R_β be the function $L_\beta(x,y)$ with $K_\beta = Q_\beta$ (see (5.4)):

$$R_\beta(x,y) := \frac{1}{\beta}\int_0^\beta Q_t(x,y)Q_{\beta-t}(y,x)\mathrm{d}t. \tag{6.12}$$

We denote by $\nu_{x,y;\beta}$ the conditinal measure $\mu_{x,y;\beta}$ in the case where $K_\beta = Q_\beta$. We call $\nu_{x,y;\beta}$ the *conditional oscillator measure.*

Theorem 6.3 *Assume (A.7). Let $z \in \mathbb{C}\setminus\{0\}$ and $F \in L^\infty(\mathbb{R}^n)$.*

(i) *Suppose that*

$$\int_{\mathbb{R}^n} Q_\beta(x,x) \det\left(I + e^{-\beta\hbar(\mathbb{D}+\mathbb{W}(x))}\right) e^{-\beta\Phi_W(x)}\mathrm{d}x < \infty, \quad \forall\beta > 0. \tag{6.13}$$

Then, for all $\beta > 0$, $e^{-\beta H_{\rm SS}}$ is trace class and the spectrum of $H_{\rm SS}$ is purely discrete. Moreover,

$$\mathrm{Tr}\, e^{-\beta H_{\rm SS}} \le \int_{\mathbb{R}^n} Q_\beta(x,x)\det(1 + e^{-\beta\hbar(\mathbb{D}+\mathbb{W}(x))})e^{-\beta\Phi_W(x)}\mathrm{d}x, \quad \forall\beta > 0. \tag{6.14}$$

(ii) *Suppose that*

$$\int_{\mathbb{R}^n} K_\beta(x,x) \det\left(1 + |z|e^{-\beta\hbar(\mathbb{D}+\mathbb{W}(x))}\right) e^{-\beta\Phi_W(x)}\mathrm{d}x < \infty, \quad \forall\beta > 0. \tag{6.15}$$

Then, for all $\beta > 0$,

$$\begin{aligned} |\mathrm{Tr}\,(Fz^{N_{\rm f}}e^{-\beta H_{\rm SS}})| \;&\le\; \int_{\mathbb{R}^n} \mathrm{d}x|F(x)| \int_{\mathbb{R}^n} \mathrm{d}y R_\beta(x,y) \\ &\qquad \times \det\left(1 + |z|e^{-\beta\hbar(\mathbb{D}+\mathbb{W}(y))}\right) e^{-\beta\Phi_W(y)} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Tr}\,(Fz^{N_{\rm f}}e^{-\beta H_{\rm SS}}) \;&=\; \int_{\mathbb{R}^n} \mathrm{d}xF(x) \int \det\left(1 + ze^{-\beta\hbar\mathbb{D}-\hbar\int_0^\beta \mathbb{W}(q(s))\mathrm{d}s}\right) \\ &\qquad \times e^{-\int_0^\beta \Phi_W(q(s))\mathrm{d}s}\mathrm{d}\nu_{x,x;\beta}. \end{aligned} \tag{6.16}$$

For a proof of the theorem, we refer the reader to [7, Section 6].

Corollary 6.4 *Assume (A.7) and (6.13). Then*

$$\dim\ker\overline{Q_W} \leq \inf_{\beta>0}\int_{\mathbb{R}^n} Q_\beta(x,x)\det(1+e^{-\beta\hbar(\mathbb{D}+\mathrm{W}(x))})e^{-\beta\Phi_W(x)}\mathrm{d}x. \tag{6.17}$$

Proof. By (6.7) and the obvious inequality

$$\dim\ker H_{\mathrm{SS}} \leq \mathrm{Tr}\, e^{-\beta H_{\mathrm{SS}}}$$

(note that $H_{\mathrm{SS}} \geq 0$), we have

$$\dim\ker\overline{Q_W} \leq \mathrm{Tr}\, e^{-\beta H_{\mathrm{SS}}}$$

independently of $\beta > 0$. Hence, using (6.14), we obtain (6.17). ∎

Corollary 6.5 *Assume (A.7) and (6.13). Suppose that there exists a $\beta_0 \in \mathbb{R}$ such that*

$$\int_{\mathbb{R}^n} Q_{\beta_0}(x,x)\det(1+e^{-\beta_0\hbar(\mathbb{D}+\mathrm{W}(x))})e^{-\beta_0\Phi_W(x)}dx < 1.$$

Then $\ker\overline{Q_W} = \{0\}$.

Proof. By (6.17), $\dim\ker\overline{Q_W} < 1$. Hence $\ker\overline{Q_W} = \{0\}$. ∎

The following theorem gives a functional integral representation for the index of $\overline{Q_{W+}}$ under the condition that $\overline{Q_W}$ is self-adjoint:

Theorem 6.6 *Assume (A.7). Suppose that Q_W is essentially self-adjoint on $\mathcal{D}_0$ and, for some $\beta > 0$,*

$$\int_{\mathbb{R}^n} Q_\beta(x,x)\det\left(I+e^{-\beta\hbar(\mathbb{D}+\mathrm{W}(x))}\right)e^{-\beta\Phi_W(x)}\mathrm{d}x < \infty. \tag{6.18}$$

Then $e^{-\beta H_{\mathrm{SS}}}$ is trace class and $\overline{Q_{W+}}$ is Fredholm. Moreover,

$$\mathrm{ind}(\overline{Q_{W+}}) = \int_{\mathbb{R}^n}\mathrm{d}x\int\det\left(1-e^{-\beta\hbar\mathbb{D}-\hbar\int_0^\beta \mathrm{W}(q(t))\mathrm{d}t}\right)e^{-\int_0^\beta\Phi_W(q(t))\mathrm{d}t}d\nu_{x,x;\beta} \tag{6.19}$$

independently of $\beta > 0$.

Proof (Outline). We have

$$\mathrm{ind}(\overline{Q_{W+}}) = \mathrm{Tr}\left(\Gamma_n e^{-\beta H_{\mathrm{SS}}}\right) = \mathrm{Tr}\left((-1)^{N_{\mathrm{f}}}e^{-\beta H_{\mathrm{SS}}}\right),$$

where we have use (6.2). Applying (6.16) with $F = 1$ and $z = -1$ to the right hand side, we obtain (6.19). ∎

7 Concluding Remarks

In the present paper we have considered a class of boson-fermion systems with finite degrees of freedom including supersymmetric quantum ones. This theory can be extended to a class of boson-fermion systems with *infinite degrees of freedom* including *supersymmetric quantum field models.* A mathematical framework for this purpose is given by the abstract boson-fermion Fock space over a pair of two infinite dimensional Hilbert spaces. Basic partial results in this direction have been obtained in [1, 2]. Further studies are under progress from view-points of analysis on infinite dimensional Dirac type operators (recall that Q_W is a finite dimensional Dirac type operator).

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