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Circumscribed and inscribed ellipses induced by Blaschke products of degree 3

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Abstract

A bicentric polygon is a polygon which has both an inscribed circle and a circumscribed one. For given two circles, the necessary and sufficient condition for existence of bicentric triangle for these two circles is known as Chapple's formula or Euler's theorem.

As one of natural extensions of this formula, we characterize the inscribed ellipses of a triangle which is inscribed in the unit circle. We also discuss the condition for the "circumscribed" ellipse of a triangle which is circumscribed about the unit circle.

For the proof of these results, we use some geometrical properties of Blashcke products on the unit disk.

1 Introduction

A bicentric polygon is a polygon which has both an inscribed circle and a circumscribed one. Any triangle is bicentric because every triangle has a unique pair of inscribed circle and circumscribed one. Then, for a triangle, what relation exists between the inscribed circle and the circumscribed one? For this simple and natural question, Chapple gave a following answer (see [1]):

The distance d between the circumcenter and incenter of a triangle is given by $d^2 = R(R-2r)$, where R and r are the circumradius and inradius, respectively. In particular, if circumscribed circle is the unit circle, then the distance is given by $d^2 = 1-2r$.

From now on, we assume that the circumscribed circle is the unit circle. The converse of Chapple's theorem also holds. That is, there exists a bicentric triangles, if the inradius satisfies the Chapple's formula.

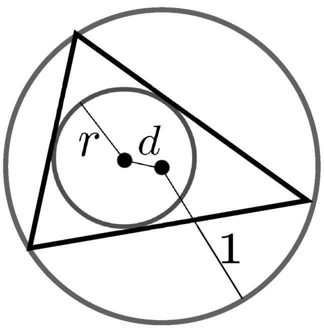

Figure 1: Chapple's formula: $d^2 = 1-2r$.

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In this paper, as one of natural extensions of this formula, we characterize the inscribed ellipses of a triangle which is inscribed in the unit circle. We also discuss the condition for the "circumscribed" ellipse of a triangle which is circumscribed about the unit circle.

For the proof of these results, we use some geometrical properties of Blashcke products on the unit disk.

2 Conics on $\mathbb{C}$

There are almost no documents or textbooks written about conics on the complex plane. Therefore, we summarize that here.

A conic on the complex plane is usually written by using its geometrical characterization.

Circle: Circle is the locus of points such that the distance to center a is constant,

$$|z-a| = r \quad (r > 0).$$

Ellipse: Ellipse is the locus of points such that the sum of the distances to two foci a and b is constant,

$$|z-a| + |z-b| = r \quad (r > 0).$$

Hyperbola: Hyperbola is the locus of points such that the absolute value of the difference of the distances to two foci a and b is constant,

$$|z-a| - |z-b| = \pm r \quad (r > 0).$$

Parabola: Parabola is the locus of points such that the distance to the focus b equals the distance to the directrix $\overline{a}z + a\overline{z} + r = 0$,

$$\frac{|\overline{a}z + a\overline{z} + r|}{4|a|} = |z-b| \quad (r \in \mathbb{R}).$$

Each of the above equation is simple and easy to see geometrical properties such as foci or directrix, but it includes "complex absolutes". It is hard that we deal "complex absolutes" by symbolic and algebraic computation system. If $\alpha \in \mathbb{C}$ is given, then $|\alpha|$ is uniquely determined. But, this procedure is not reversible. The value α is not determined uniquely when $|\alpha|$ is given, because there is no information on "argument" $\arg \alpha$.

Now, we introduce the other expression of conics that does not include absolutes.

Lemma 1
In the complex plane, equation of "generalized conic" is given by

$$\overline{u}z^2 + pz\overline{z} + u\overline{z}^2 + \overline{v}z + v\overline{z} + q = 0, \tag{1}$$

where $u, v \in \mathbb{C}$, $p, q \in \mathbb{R}$.

Proof For the equation of generalized conic on the real xy-plane

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0,$$

substituting $x = \dfrac{z+\overline{z}}{2}$, $y = \dfrac{z-\overline{z}}{2i}$, we have

$$\frac{1}{4}(c_1 - c_3 - c_2 i)z^2 + \frac{1}{2}(c_1 + c_3)z\overline{z} + \frac{1}{4}(c_1 - c_3 + c_2 i)\overline{z}^2 + \frac{1}{2}(c_4 - c_5 i)z + \frac{1}{2}(c_4 + c_5 i)\overline{z} + c_6 = 0.$$

Then, we have the assertion. ∎

Remark 1
A generalized conic is one of a circle, an ellipse, a hyperbola, a parabola, two intersecting lines, two parallel lines, a single double line, a single point, and no points.
In the case of $u=0$, (1) is the equation of generalized circle (either circle or line).

A generalized conic (1) is classified by using data of coefficients.

Lemma 2
The conic on the complex plane

$$\overline{u}z^2+pz\overline{z}+u\overline{z}^2+\overline{v}z+v\overline{z}+q=0, \quad (u,v\in\mathbb{C},\ p,q\in\mathbb{R}) \tag{2}$$

can be classified as follows.

1. *In the case of $p^2-4u\overline{u}<0$;*
   (a) *if $-u\overline{v}^2+pv\overline{v}+4u\overline{u}q-\overline{u}v^2-p^2q\neq 0$, the equation represents a hyperbola,*
   (b) *if $-u\overline{v}^2+pv\overline{v}+4u\overline{u}q-\overline{u}v^2-p^2q=0$, the equation represents two intersecting lines.*
2. *In the case of $p^2-4u\overline{u}>0$;*
   (a) *if $p(-u\overline{v}^2+pv\overline{v}+4u\overline{u}q-\overline{u}v^2-p^2q)>0$, the equation represents an ellipse,*
   (b) *if $p(-u\overline{v}^2+pv\overline{v}+4u\overline{u}q-\overline{u}v^2-p^2q)<0$, the equation represents no points,*
   (c) *if $-u\overline{v}^2+pv\overline{v}+4u\overline{u}q-\overline{u}v^2-p^2q=0$, the equation represents a single point.*
3. *In the case of $p^2-4u\overline{u}=0$;*
   (a) *if $u\overline{v}^2-pv\overline{v}+\overline{u}v^2\neq 0$, the equation represents a parabola,*
   (b) *if $u\overline{v}^2-pv\overline{v}+\overline{u}v^2=0$;*
      i. *if $-v\overline{v}+2pq=0$, the equation represents a single double line,*
      ii. *if $-v\overline{v}+2pq>0$, the equation represents two parallel lines,*
      iii. *if $-v\overline{v}+2pq<0$, the equation represents no points.*

Proof Substituting $z=x+iy$ for the equation (1), we have

$$\big(p+2\Re(u)\big)x^2+4\Im(u)xy+\big(p-2\Re(u)\big)y^2+2\Re(v)x+2\Im(v)y+q=0. \tag{3}$$

This is an equation of a conic on the real xy-plane. Therefore, we can apply classification theorem of conics on the real xy-plane, and obtain the assertion. ∎

3 Geometry of Blaschke product

3.1 Blaschke product

A *Blaschke product* of degree d is a rational function defined by

$$B(z)=e^{i\theta}\prod_{k=1}^{d}\frac{z-a_k}{1-\overline{a_k}z} \qquad (a_k\in\mathbb{D},\ \theta\in\mathbb{R}).$$

In the case that $\theta=0$ and $B(0)=0$, i.e.

$$B(z)=z\prod_{k=1}^{d-1}\frac{z-a_k}{1-\overline{a_k}z} \qquad (a_k\in\mathbb{D}),$$

B is called *canonical.*

It is well known that a Blaschke product is a holomorphic function on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and maps $\mathbb{D}$ onto itself. Moreover, the derivative of a Blaschke product has no zeros on $\partial\mathbb{D}$.

For a Blaschke product

$$B(z) = e^{i\theta} \prod_{k=1}^{d} \frac{z - a_k}{1 - \overline{a_k} z}$$

of degree d, set

$$f_1(z) = e^{-\frac{\theta}{d} i} z, \quad \text{and} \quad f_2(z) = \frac{z - (-1)^d a_1 \cdots a_d e^{i\theta}}{1 - (-1)^d \overline{a_1 \cdots a_d e^{i\theta}} z}.$$

Then, the composition $f_2 \circ B \circ f_1$ is canonical. Although there are many ways to make canonical Blaschke products, the above construction plays an essential role in the argument in the following sections.

3.2 Inscribed ellipses

In this section, we summarize the results in [5], and give an extension of Chapple's theorem. The following result by Daepp, Gorkin, and Mortini is a key tool in this section.

Lemma 3 (Daepp, Gorkin, and Mortini [2])
Let

$$B(z) = z \cdot \frac{z - a}{1 - \overline{a} z} \cdot \frac{z - b}{1 - \overline{b} z} \quad (a, b \neq 0,\ a \neq b,\ a, b \in \mathbb{D}),$$

and z_1, z_2, z_3 the three distinct preimages of $\lambda \in \partial\mathbb{D}$ by B. Then, the lines joining z_k, z_ℓ $(k, \ell = 1, 2, 3,\ k \neq \ell)$ are tangent to the ellipse

$$E: \quad |z - a| + |z - b| = |1 - \overline{a} b|.$$

Conversely, each point of E is the point of tangency of a line that passes through two distinct points ζ_1, ζ_2 on $\partial\mathbb{D}$ for which

$$B(\zeta_1) = B(\zeta_2).$$

Every triangle has a unique inscribed circle. But, there are many ellipses inscribed in a triangle. The following lemma assert that, for each point a in a triangle, there is an inscribed ellipse having a as one of the foci.

Lemma 4
For every mutually distinct points z_1, z_2, z_3 on $\partial\mathbb{D}$, let T be the closed set surrounded by $\triangle z_1 z_2 z_3$. For every $a \in int(T)$, there exists a unique pair of $\lambda \in \partial\mathbb{D}$ and $b \in int(T)$ such that

$$B(z_1) = B(z_2) = B(z_3) = \lambda$$

with

$$B(z) = z \cdot \frac{z - a}{1 - \overline{a} z} \cdot \frac{z - b}{1 - \overline{b} z}.$$

Here, we remark that the inscribed ellipses of a triangle which is inscribed in the $\partial\mathbb{D}$ are characterized as follows.

Lemma 5
For any triangle that is inscribed in $\partial\mathbb{D}$, there exists an ellipse that is inscribed in the triangle if and only if the ellipse is associated with a Blaschke product of degree 3.

Now, we give a natural extension of Chapple's Theorem.

Theorem 6
For an ellipse E, the following two conditions are equivalent.

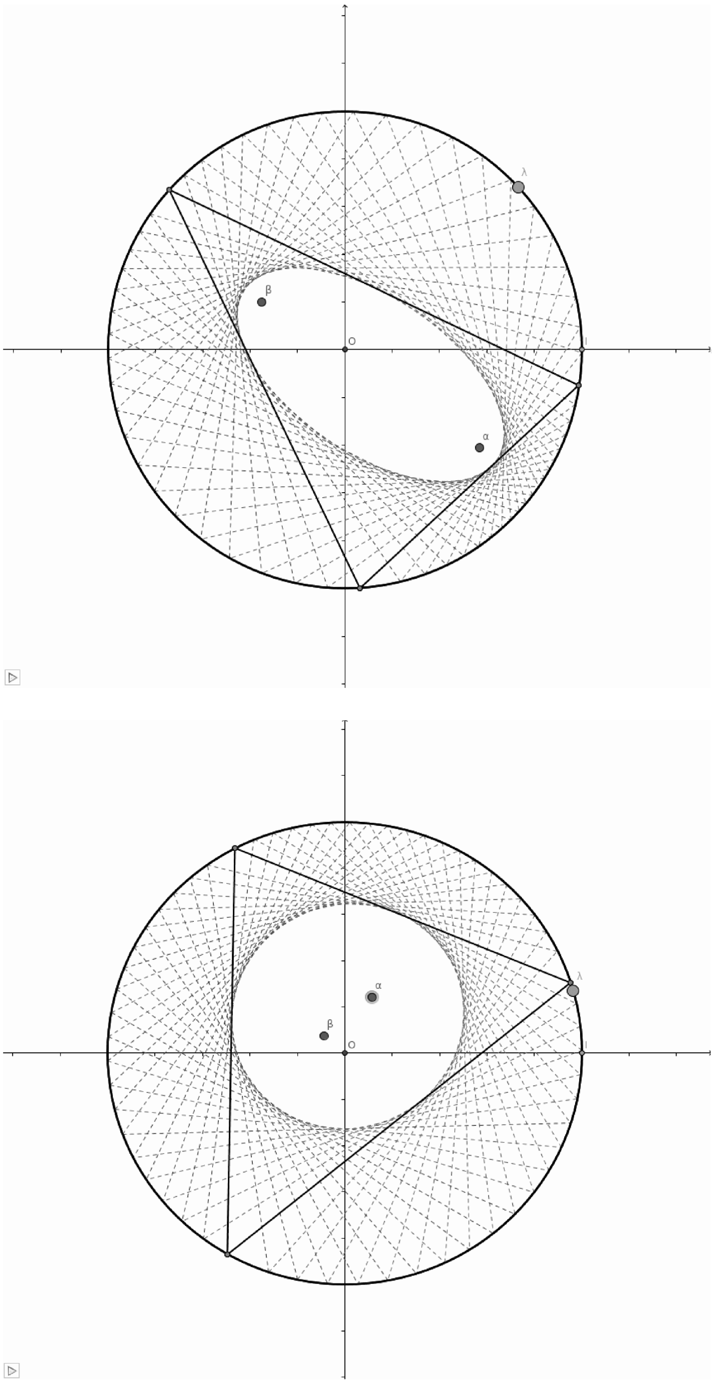

Figure 2: “inscribed ellipse” is appears as the envelope of family of lines joining the preimages.

- *There exists a triangle which E is inscribed in and $\partial\mathbb{D}$ is circumscribed about.*
- *For some $a, b \in \mathbb{D}$, E is defined by the equation*

$$|z-a|+|z-b|=|1-\bar{a}b|.$$

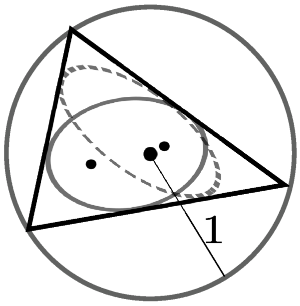

Figure 3: A natural extension of Chapple's formula: "inscribed circle" ⇒ "inscribed ellipse".

If Blashcke product has a double zero point at a $(\neq 0)$, the associated ellipse becomes a circle.

Corollary 7
For a circle C, the following two conditions are equivalent.

- *There exists a triangle which C is inscribed in and $\partial\mathbb{D}$ is circumscribed about.*
- *For some $a \in \mathbb{D}$, C is defined by the equation*

$$|z-a|=\frac{1}{2}(1-|a|^2). \tag{4}$$

The equation (4) coincides with Chapple's formula.

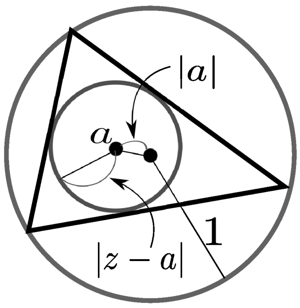

Figure 4: The equation (4) implies Chapple's formula.

3.3 Circumscribed ellipses

For a canonical Blaschke product $B(z)=z\cdot\dfrac{z-a}{1-\bar{a}z}\cdot\dfrac{z-b}{1-\bar{b}z}$ $(a, b \in \mathbb{D})$ and the three preimages z_1, z_2, z_3 of $\lambda \in \partial\mathbb{D}$ by B, let l_k be a tangent line of the unit circle at a point z_k $(k=1,2,3)$. Then, the equation of l_k is given by

$$l_k \ : \quad z+z_k^2\bar{z}-2z_k=0 \qquad (k=1,2,3).$$

Let T_λ be the triangle composed by the three lines l_k $(k=1,2,3)$.

In general, the unit circle is not inscribed in triangle T_λ (see Figure 5 and 7), but we obtain the following.

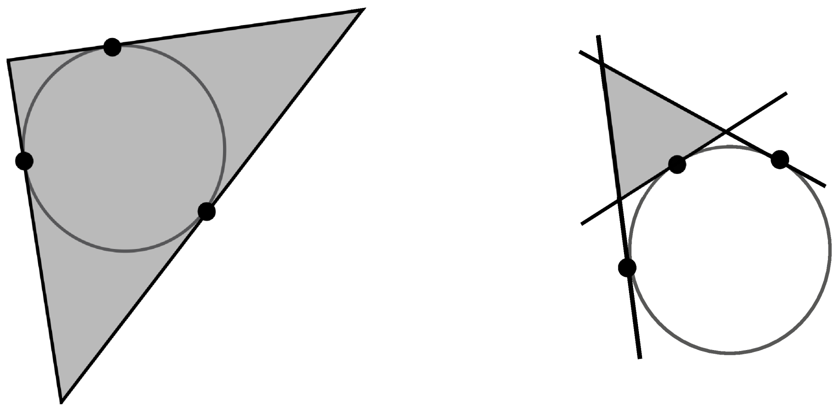

Figure 5: The unit circle is ether the circumscribed circle or the escribed circle, for each triangle T_λ.

Theorem 8
If $(|a+b|-1)^2 > |ab|^2$, for each $\lambda \in \partial\mathbb{D}$, triangle T_λ is circumscribed about the unit circle and is inscribed in an ellipse that depends only on two zeros a, b.

To prove Theorem 8, we need the following lemmas.

Lemma 9
The trace of the vertices of $\{T_\lambda\}_\lambda$ forms an ellipse if and only if $(|a+b|-1)^2 > |ab|^2$.

Proof (Proof of Lemma 9) The intersection point of two tangent lines l_k, l_j to the unit circle is given by

$$z = \frac{2z_k z_j}{z_k + z_j}. \tag{5}$$

(If $z_k + z_j = 0$, the intersection point is the point at infinity.) Recall that the three preimages z_k ($k = 1, 2, 3$) are the solutions of

$$B(z) = z \cdot \frac{z-a}{1-\overline{a}z} \cdot \frac{z-b}{1-\overline{b}z} = \lambda. \tag{6}$$

Eliminating z_k, λ from (5) and (6), we have the equation of z variable,

$$\overline{b}\overline{a}z^2 + (-a\overline{a}b\overline{b} + (a+b)(\overline{a}+\overline{b}) - 1)z\overline{z} + ab\overline{z}^2 - 2(\overline{a}+\overline{b})z - 2(a+b)\overline{z} + 4 = 0. \tag{7}$$

Here, the elimination ideal corresponding to the above equation is obtained by computing a Gröbner basis. We use "risa/asir", a symbolic and algebraic computation system.

From Lemma 1, this equation (7) is an equation of generalized conic on $\mathbb{C}$. Moreover, we can check that (7) represents an ellipse if and only if

$$(|a+b|-1)^2 > |ab|^2,$$

from Lemma 2. We can check also that an ellipse never degenerate to a point or the empty set. ∎

Lemma 10
In the case of $(|a+b|-1)^2 > |ab|^2$, the unit circle is always inscribed in T_λ, for each $\lambda \in \partial\mathbb{D}$.

That is, the unit circle will never become an escribed circle of T_λ.

Proof (Proof of Lemma 10) Suppose that there exists a $\lambda \in \partial\mathbb{D}$ such that the unit circle is an escribed circle of T_λ. Then, corresponding preimages z_1, z_2, z_3 are on the arc of a semicircle.

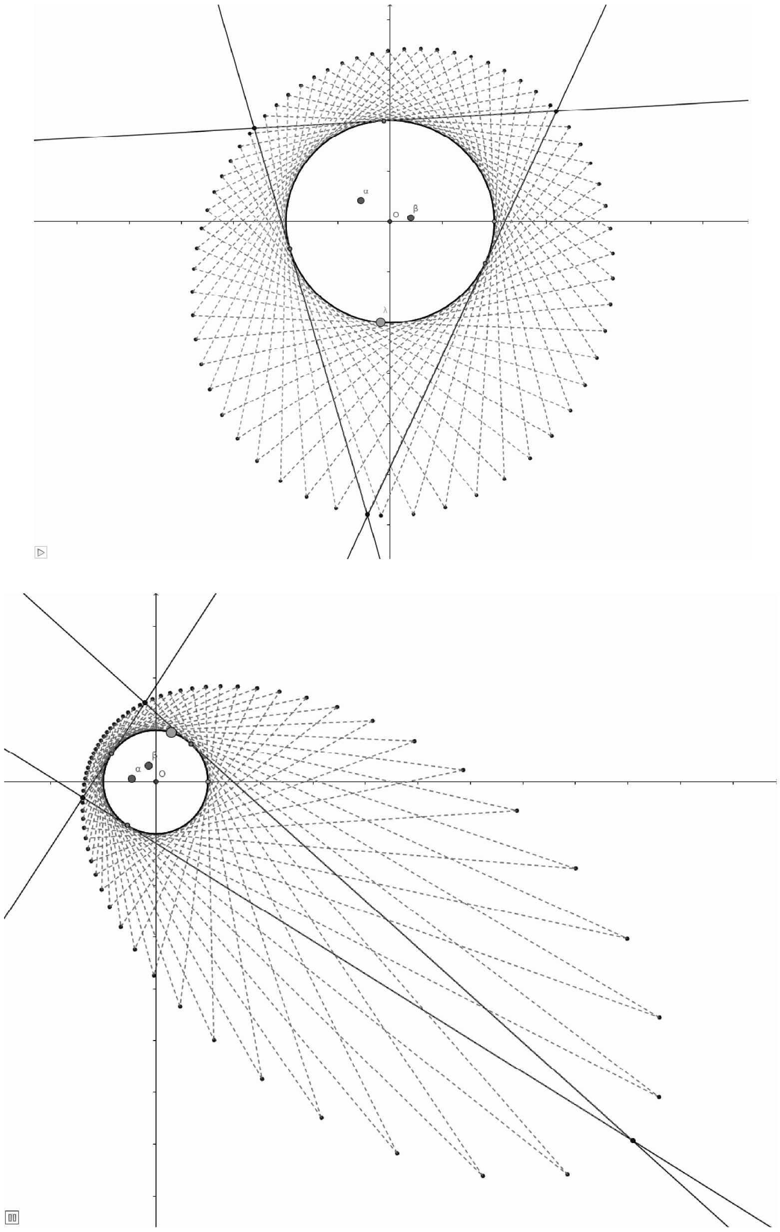

Figure 6: The trace of intersection points of tangent lines forms an ellipse if and only if $(|a+b|-1)^2 > |ab|^2$.

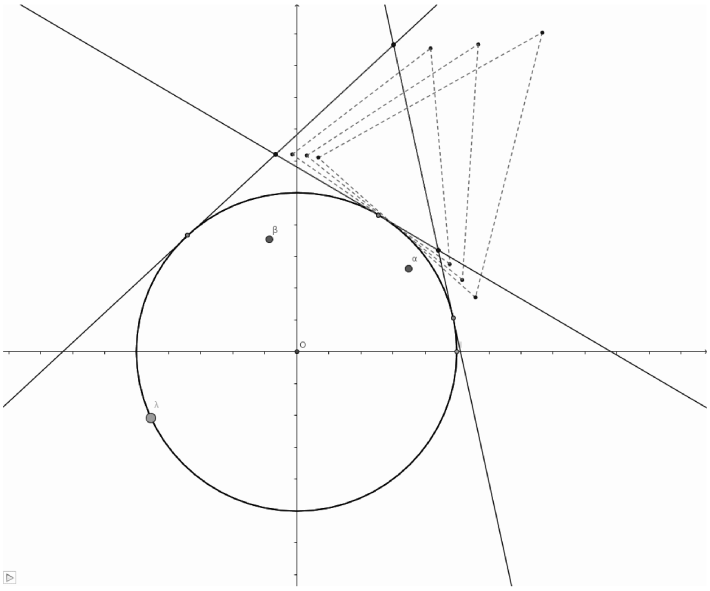

Figure 7: The unit circle is the escribed circle for T_λ. In this case, the trace is not an ellipse.

Now, we may assume that the preimages z_1, z_2, z_3 satisfy

$$0 \leq \operatorname{Arg} z_1 < \operatorname{Arg} z_2 < \operatorname{Arg} z_3 < 2\pi \qquad \text{and} \qquad \operatorname{Arg} z_3 - \operatorname{Arg} z_1 < \pi.$$

Moreover, we assume that the preimages move clockwise when λ moves on the unit circle clockwise. The other case can be treated similarly.

The derivative of a Blaschke product never vanishes on $\partial\mathbb{D}$. Therefore, each preimage z_k moves smoothly and monotonically on $\partial\mathbb{D}$. From the intermediate value theorem, there exist $\tilde{z}_1$ in arc $\widehat{z_1 z_2}$, $\tilde{z}_3$ in arc $\widehat{z_3 z_1}$, and $\tilde{\lambda} \in \partial\mathbb{D}$ such that $\tilde{z}_1, \tilde{z}_3 \in B^{-1}(\tilde{\lambda})$ and $\tilde{z}_3 = -\tilde{z}_1$.

Then, two tangent lines l_1 and l_3 are parallel, and have no intersection point on $\mathbb{C}$. Therefore, the trace of the intersection points includes the point at infinity. This contradicts with the fact of Lemma 9 that the trace of intersection points is an ellipse.

Hence, the unit circle will never become an escribed circle of T_λ, and we have the assertion. ∎

Proof (Proof of Theorem 8) If $(|a+b|-1)^2 > |ab|^2$, each triangle T_λ is inscribed in an ellipse (7) that depends only on two zeros a, b, from Lemma 9. Moreover, the unit circle is inscribed in each triangle T_λ, from Lemma 10. ∎

Remark 2

From the above argument, we also obtain the following.

For any $a, b \in \mathbb{D}$, there exists a $\lambda \in \partial\mathbb{D}$ such that the unit circle is inscribed in T_λ. (It is impossible that the unit circle is always escribed circle of T_λ for all $\lambda \in \partial\mathbb{D}$.)

Moreover, the two foci of the circumscribed ellipse are given as follows.

Proposition 11

The circumscribed ellipse in Theorem 8 is given by

$$|z - f_1| + |z - f_2| = r, \tag{8}$$

where f_1, f_2 are the two solutions of

$$F_{a,b}(t) = ((a\bar{a}b\bar{b} + 1 - (a+b)(\bar{a}+\bar{b}))^2 - 4a\bar{a}b\bar{b})t^2 + 4((\bar{a}\bar{b}b - \bar{b} - \bar{a})a^2 + (\bar{a}\bar{b}b^2 + 1)a + (-\bar{b} - \bar{a})b^2 + b)t + 4(a-b)^2 = 0,$$

and r is the unique positive solution of

$$R_{a,b}(r) = r^2 - \frac{16(b\bar{b}-1)(\bar{a}b-1)(a\bar{b}-1)(a\bar{a}-1)(a\bar{a}b\bar{b} - (a+b)(\bar{a}+\bar{b}) + 2|a||b| + 1)}{((a\bar{a}b\bar{b} + 1 - (a+b)(\bar{a}+\bar{b}))^2 - 4a\bar{a}b\bar{b})^2} = 0.$$

Proof Two equations $F_{a,b}(t) = 0$ and $R_{a,b}(r) = 0$ are obtained by comparing the coefficients of two equations (7) and (8).

Moreover, the last factor of the numerator of the constant term of $R_{a,b}$ is written as

$$(|ab|^2 - |a+b|^2 + 2|a||b| + 1) = (|ab| + 1 + |a+b|)(|ab| + 1 - |a+b|)$$

and, the second factor of above equality satisfies

$$|ab| + 1 - |a+b| > |ab| + 1 - |a| - |b| = (1-|a|)(1-|b|) > 0.$$

Hence, we can check that the constant term of $R_{a,b}(r)$ is non-positive, and $R_{a,b}(r) = 0$ has a unique positive real solution. ∎

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