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Sato Hyperfunctions and Reproducing Kernels
(佐藤超関数と再生核形式)

By

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Abstract

We survey the theory of the microlocal energy method for hyperfunctions and microfunctions developed in [5, 6, 7]; in particular, we introduce the important notions, *positivity* and *quasi-positivity* for hermitian microkernels. The quasi-positivity is based on the good properties of Bergman's reproducing kernels for some conic domains in $T^*\mathbb{C}^n$. Further we introduce some recent result on the Sobolev type 2-forms of order 0 for microfunctions with real analytic parameters obtained by Kaito Yamasaki [11].

§ 1. What is an energy method in the theory of hyperfunctions?

For a Sato hyperfunction $f(x)$, we cannot define $|f(x)|^p$ or L^p-norm $\|f\|_p$ in general. For $p = 2$, however, we can consider a microfunction $f(x)\overline{f(u)}$ as the substitute of $|f(x)|^2$ as introduced in [5, 6]. Further, for a hyperfunction $f(t, x)$ with real analytic parameters $t \in T$, we can consider a hyperfunction

$$E(x,u) := \int_{\overline{T}} f(t,x)\overline{f(t,u)}dt \ \left(= \int_{\mathbb{R}^m} \mathrm{ext}_{\overline{T}}(f(t,x)\overline{f(t,u)})dt\right)$$

as the L^2-energy form of f (the precise definition of real analytic parameters $t \in T$ and the meaning of the integration over $\overline{T}$ will be given in the later section). At the same time, in energy arguments we use some inequality (an order relation)

$$k_1(x,u) \ll k_2(x,u) \quad \text{at } (\mathring{x}, \mathring{x}; i\mathring{\eta}, -i\mathring{\eta})$$

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instead of the equality. In order to illustrate our energy method, we consider the following example: Let $\Omega \subset \mathbb{R}^n_x$ be a bounded domain with real analytic boundary $N = \partial\Omega$, and $f(t,x)$ be a hyperfunction in $\{t \in \mathbb{R}; a < t < b\} \times \Omega$ satisfying the boundary value problem:

$$\text{(1.1)} \qquad \begin{cases} (\partial_t - \Delta_x)f(t,x) = 0 & \text{in } (a,b) \times \Omega, \\ f(t,x) = 0 & \text{on } (a,b) \times \partial\Omega. \end{cases}$$

We remark here that, since $(a,b) \times \partial\Omega$ is non-characteristic to $\partial_t - \Delta_x$, the boundary value $f|_{(a,b)\times\partial\Omega}$ is well-defined for any hyperfunction solution f. Then, our conclusion is that $f \in \mathscr{A}((a,b) \times \overline{\Omega})$ (real analyticity of f up to the boundary). To prove this, we consider the energy form:

$$\text{(1.2)} \qquad E(t,s) := \int_{\overline{\Omega}} f(t,x)\, \overline{f(s,x)}\, dx.$$

Indeed, this is a well-defined hyperfunction in (t,s) because we have not only an estimate

$$\mathrm{SS}(f) \subset \{(t,x; i\tau, i\xi);\ x \in \Omega, \xi = 0\},$$

but also the estimates up to the boundary:

$$\begin{aligned} &\mathrm{SS}(\mathrm{ext}_{\overline{\Omega}}(f(t,x))) \subset \{(t,x;i\tau,i\xi); x \in \overline{\Omega}, \xi = 0\} \cup \{(t,x;i\tau,i\lambda d\varphi(x)); \varphi(x) = 0\}, \\ &\mathrm{SS}(\mathrm{ext}_{\overline{\Omega}}(f(t,x)\overline{f(s,x)})) \subset \\ &\qquad \{(t,s,x;i\tau_1,i\tau_2,i\xi); x \in \overline{\Omega}, \xi = 0\} \cup \{(t,s,x;i\tau_1,i\tau_2,i\lambda d\varphi(x)); \varphi(x) = 0\}. \end{aligned}$$

Here, $\varphi(x) \in C^\omega(\overline{\Omega})$ satisfies $\Omega = \{\varphi(x) > 0\}, d\varphi(x) \neq 0$, and the boundary value theory of hyperfunction solutions permits the extension $\mathrm{ext}_{\overline{\Omega}}(f(t,x)\overline{f(s,x)})$ (roughly, $= \chi_{\overline{\Omega}}(x)f(t,x)\overline{f(s,x)}$) of $f(t,x)\overline{f(s,x)}$ satisfying the above estimates. Therefore (1.2) is a hyperfunction on $(a,b)^2$. Then, we have

$$\begin{aligned} \partial_t E(t,s) &= \int_{\mathbb{R}^n} \mathrm{ext}_{\overline{\Omega}}(\partial_t f(t,x) \cdot \overline{f(s,x)}\,)\, dx = \int_{\mathbb{R}^n} \mathrm{ext}_{\overline{\Omega}}(\Delta_x f(t,x) \cdot \overline{f(s,x)})\, dx \\ &= -\int_{\mathbb{R}^n} \mathrm{ext}_{\overline{\Omega}}(\nabla_x f(t,x) \cdot \overline{\nabla_x f(s,x)}\,)\, dx = \partial_s E(t,s). \end{aligned}$$

Hence, $(\partial_t - \partial_s)E(t,s) = 0$. Since $\partial_t - \partial_s$ is an elliptic operator on

$$\Delta^a(\sqrt{-1}T^*\mathbb{R}) := \{(t,s;i\tau_1,i\tau_2); t = s, \tau_1 = -\tau_2\},$$

we get an estimate:

$$\mathrm{SS}(E(t,s)) \cap \Delta^a(\sqrt{-1}T^*\mathbb{R}) \subset \{\tau_1 = \tau_2 = 0\}.$$

Then, some theorem on the integration of positive hermitian microkernels concludes the real analyticity of the integrand $f(t,x)$ up to the boundary.

The plan of this article is the following: In Section 2, we give a brief introduction to positivity of analytic hermitian kernels. In Section 3 we show the importance of some Bergman reproducing kernels in our theory with respect to the definition of quasi-positivity. Section 4 is devoted to introduce the definition of positivity in microlocal analysis. In Section 5 we give the definition of quasi-positivity of hermitian microkernels by using positive hermitian pseudodifferential operators of infinite order. In Section 6 we introduce K. Yamasaki's result on Sobolev type 2-forms of order 0 for microfunctions with real analytic parameters $t \in T$.

Hereafter we consider hyperfunctions $f(x)$ or $f(t,x)$, where $t = (t_1, \ldots, t_m)$ are real analytic parameters. So the variables t, x change the roles from Section 2.

§ 2. Positive analytic hermitian kernels

Definition 2.1. Let X be a set. A $\mathbb{C}$-valued function $K(x,y)$ on $X \times X$ is said to be a hermitian kernel on X if

$$K(y,x) = \overline{K(x,y)} \quad (\forall x, y \in X).$$

Further a hermitian kernel $K(x,y)$ on X is said to be $K \gg 0$ (positive)$\Longleftrightarrow$
For $\forall N \in \mathbb{N}, \forall x_1, \ldots, \forall x_N \in X$, $\forall \xi_1, \ldots, \forall \xi_N \in \mathbb{C}$, we have

$$\sum_{j,k=1}^{N} K(x_j, x_k)\xi_j \overline{\xi_k} \geq 0.$$

Then, we define an order relation for hermitian kernels K_1, K_2 on X by

$$K_1 \gg K_2 \Longleftrightarrow K_1 - K_2 \gg 0.$$

It is easy to see that

$$K_1 \gg 0, K_2 \gg 0 \Longrightarrow K_1 + K_2 \gg 0, \quad K_1 \cdot K_2 \gg 0,$$

where $K_1 \cdot K_2$ is a product as functions on $X \times X$.

Definition 2.2. Let X be a domain of $\mathbb{C}^n$. Then, a hermitian kernel $K(z,w)$ on X is said to be an analytic hermitian kernel on X if $K(z,w)$ is holomorphic in variables $(z, \overline{w})$ on $X \times X^*$, where

$$X^* := \{z \in \mathbb{C}^n; \overline{z} \in X\}.$$

Example 2.3. Put $X = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$, and $\alpha > -1$. Then we have a positive analytic hermitian kernel on X:

$$\{(z - \overline{w})/i\}^{-1-\alpha} = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{itz} \cdot \overline{e^{itw}} t^\alpha dt \gg 0.$$

The following theorem is the most important for our theory, which is due to several authors; the first statement is due to Krĕin [8], Bremermann [2], Sommer-Mehring [10], Meschkowski [9], Donoghue [3, 4], and the second statement is due to Krĕin [8], Donoghue [4], Meschkowski [9].

Theorem 2.4. *Let X, X' be domains of $\mathbb{C}^n$ such that $\emptyset \neq X' \subset X$. Let K_1, K_2 be analytic hermitian kernels on X. Then we have the following:*

i) $K_1 \ll K_2$ on $X' \times X' \Longrightarrow K_1 \ll K_2$ on $X \times X$.

ii) Let K_3 be a hermitian kernel (not necessarily analytic) on X' satisfying $K_1 \ll K_3 \ll K_2$ on $X' \times X'$. Then there exists a unique analytic hermitian kernel $\widetilde{K_3}$ on X satisfying

$$K_1 \ll \widetilde{K_3} \ll K_2 \text{ on } X \times X, \quad \widetilde{K_3}|_{X' \times X'} = K_3.$$

Let T be a domain of $\mathbb{R}^m$, and X, X' be domains of $\mathbb{C}^n$ such that $\emptyset \neq X' \subset X$.

Theorem 2.5. *(Corollary 1.14 in [6]). Let $K(z, w; t)$ be a C^ω-function (or C^∞-function) on $X' \times X' \times T$ satisfying the following i), ii):*

i) For $\forall t \in T$, $K(z, w; t)$ is a positive analytic hermitian kernel on X'.

ii) For any compact subset $L \subset X'$, $K(z, w; t)$ is integrable on $L \times L \times T$.

If

$$E(z, w) := \int_T K(z, w; t) dt$$

extends to $X \times X^*$ analytically with respect to $(z, \overline{w})$, then $K(z, w; t)$ extends uniquely to $X \times X \times T$ as a positive analytic hermitian kernel on X with C^ω (or C^∞ resp.) parameters $t \in T$.

§ 3. Quasi-positivity and Bergman reproducing kernels

Definition 3.1. For a domain $X \subset \mathbb{C}^n$, we set

$$A^2(X) := \{f(z) \in \mathscr{O}(X); \int_X |f(z)|^2 dxdy < \infty\},$$

where $z = x + iy$. It is well-known that $A^2(X)$ is a Hilbert space (Bergman space). Let $\{\varphi_j(z)\}_{j=1}^{\infty}$ be any completely orthonormal system for $A^2(X)$. Then,

$$K_X(z,w) := \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}$$

is said to be a Bergman kernel of X. Indeed, it is well-known that this series converges locally uniformly on $X \times X$, and that K_X does not depend on the choice of $\{\varphi_j(z)\}_{j=1}^{\infty}$. It is clear that K_X is a positive analytic hermitian kernel on X.

The importance of Bergman kernels is based on the following proposition ([6]). For the reader's convenience, we will give a proof.

Proposition 3.2. *Let $K(z,w)$ be an analytic hermitian kernel on X such that*

$$\|K\|_{X\times X} := \sqrt{\iint_{X\times X} |K(z,w)|^2 dxdydudv} < \infty.$$

Then we have an inequality on $X \times X$: $\quad -\|K\|_{X\times X} \cdot K_X \ll K \ll \|K\|_{X\times X} \cdot K_X$.

Proof. Define a linear operator $T : A^2(X) \ni f \mapsto \int_X K(z,w)f(w)dudv \in A^2(X)$, where $w = u + iv$. Then this is an integral operator of Hilbert-Schmidt type. Therefore there exist a completely orthonormal system $\{\varphi_j\}_{j=1}^{\infty}$ of $A^2(X)$, and real numbers $\lambda_j \in \mathbb{R}$ $(j = 1, 2, \ldots)$ such that

$$K(z,w) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(z)\overline{\varphi_j(w)}.$$

Hence we have

$$\sum_{j=1}^{\infty} \lambda_j^2 = \iint_{X\times X} |K(z,w)|^2 dxdydudv < \infty.$$

In particular, $|\lambda_j| \leq \|K(z,w)\|_{L^2(X\times X)}$ $(\forall j)$. Therefore,

$$-\|K\|_{L^2(X\times X)} K_X \ll K \ll \|K\|_{L^2(X\times X)} K_X.$$

□

Let X, Y be domains of $\mathbb{C}^n$ such that $Y \subset X$, and T be a domain of $\mathbb{R}^m$. Let $p(t,z)$ and $f(t,z)$ be holomorphic functions defined in some neighborhoods of $T \times X$, and $T \times Y$ $(\subset \mathbb{C}^{m+n})$, respectively. Assume that for some constant $M > 0$ we have

$$\mathrm{Re}\, p(t,z) > 0, \text{ and } \|\log\left(p(t,z) + \overline{p(t,w)}\right)\|_{X\times X} \leq M \ (\forall t \in T).$$

Then the analytic hermitian kernel on Y

$$E(z,w) := \int_T (p(t,z) + \overline{p(t,w)}) f(t,z)\overline{f(t,w)}\, dt$$

is not positive, but the energy argument in Theorem 2.5 is applicable to this $E(z,w)$. This is because we can write $p(t,z) + \overline{p(t,w)} = e^{\log\left(p(t,z)+\overline{p(t,w)}\right)}$, and so we have

$$e^{M\cdot K_X(z,w)}(p(t,z) + \overline{p(t,w)}) \gg 0 \ (\forall t \in T).$$

Hence, if $E(z,w)$ extends to an analytic hermitian kernel on X, then $f(t,z)$ extends to a holomorphic function defined in a neighborhood of $T \times X$. Since $e^{M\cdot K_X(z,w)}$ is positive and invertible, we can generalize this argument by introducing a weaker order relation, quasi-positivity "$\gg_q 0$", than hermitian positivity:

$$K_1 \gg_q K_2 \Longleftrightarrow \exists M > 0 \,\text{such that}\, e^{M\cdot K_X}(K_1 - K_2) \gg 0.$$

It is easy to see that "$\gg_q 0$" is an order relation for analytic hermitian kernels on X. However, in the most applications to partial differential equations, we must consider 2-forms of the following type:

$$E(z,w) := \int_T (P(t,z,\partial_z) + \overline{P(t,w,\partial_w)}) f(t,z)\overline{f(t,w)}\, dt, \tag{3.1}$$

where $P(t,z,\partial_z)$ is an elliptic differential operator with parameters t. Therefore we must consider Bergman kernels in the symbol spaces of analytic pseudo-differential operators. A symbol $P(z,\xi)$ of an analytic pseudo-differential operator $P(z,\partial_z)$ at $(\mathring{z};\mathring{\xi}) \in T^*\mathbb{C}^n$ is a holomorphic function defined in some unbounded domain

$$\Omega_\delta(\mathring{z};\mathring{\xi}) := \{(z;\xi) \in T^*\mathbb{C}^n; |z - \mathring{z}| < \delta, |\xi/|\xi| - \mathring{\xi}/|\mathring{\xi}|| < \delta, |\xi| > \delta^{-1}\}$$

satisfying the following condition for some positive numbers δ, C, and some $\mu \in \mathbb{R}$:

$$|P(z,\xi)| \leq C|\xi|^\mu \quad \text{in } \Omega_\delta(\mathring{z};\mathring{\xi}).$$

Since $\mu > 0$ in general, $P(z,\xi)$ is not in $L^2(\Omega; dv(z,\xi))$, where $dv(z,\xi)$ is the Lebesgue measure on $\mathbb{C}^n_z \times \mathbb{C}^n_\xi \simeq \mathbb{R}^{4n}$. So we must consider Bergman kernels with some weight, for example, $(|\xi|+1)^{-\beta}$ $(\beta > 0)$. Though it is difficult to calculate the Bergman kernel for $\Omega_\delta(\mathring{z};\mathring{\xi})$ with such a weight, we can find some Bergman kernel satisfying the equivalent conditions. For that purpose, we have the following procedure:

Step1. Take a real n-dimensional vector subspace L of $\mathbb{C}^n$ such that $\mathring{\xi} \in L \subset L + \sqrt{-1}L = \mathbb{C}^n$.

Step2. Choose $\mathbb{R}$-linearly independent n elements $\xi^1, \dots, \xi^n$ of $L \cap \{\xi \in \mathbb{C}^n; |\xi/|\xi| - \overset{\circ}{\xi}/|\overset{\circ}{\xi}|| < \delta\}$ such that

$$\overset{\circ}{\xi} \in \{s_1\xi^1 + \cdots + s_n\xi^n; s_1, \dots, s_n > 0\}.$$

Step3. Take $\theta^1, \dots, \theta^n \in \mathbb{C}^n$ such that $\langle \xi^k, \theta^\ell \rangle := \sum_{j=1}^n \xi_j^k \theta_j^\ell = \delta_{k\ell}$.

Step4. Choose a large integer $N > 0$ such that for a large $\lambda > 0$ we have

$$\begin{aligned} \lambda\overset{\circ}{\xi} \in U_N &:= \bigcap_{j=1}^n \{\xi \in \mathbb{C}^n; |\arg(\langle \xi, \theta^j \rangle - N)| < \pi/(2N)\} \\ &\subset \{\xi \in \mathbb{C}^n; |\xi/|\xi| - \overset{\circ}{\xi}/|\overset{\circ}{\xi}|| < \delta, |\xi| > \delta^{-1}\}. \end{aligned}$$

Hereafter, we use z^* instead of $\overline{z}$. Put $\rho(\xi) := |1 + \langle \xi, \overset{\circ}{\xi}{}^* \rangle|^{-2n-\sigma}$ $(\sigma < 1)$, and

$$\Omega := \{z \in \mathbb{C}^n; |z - \overset{\circ}{z}| < r\} \times U_N.$$

Then, the Bergman kernel of Ω with respect to $\rho(\xi)dzdz^*d\xi d\xi^*$ is given by

$$\begin{aligned} K^\rho_\Omega(z, \xi, w, \eta) &= C_1 \left(1 - \frac{\langle z - \overset{\circ}{z}, w^* - \overset{\circ}{z}{}^* \rangle}{r^2}\right)^{-n-1} \\ &\times \prod_{j=1}^n \left[\frac{(\xi_j' \eta_j'^*)^{N-1}\{(1 + \langle \xi, \overset{\circ}{\xi}{}^* \rangle)(1 + \langle \eta^*, \overset{\circ}{\xi} \rangle)\}^{1+\sigma/(2n)}}{(\xi_j'^N + \eta_j'^{*N})^2}\right], \end{aligned}$$

where $\xi_j' = \langle \xi, \theta^j \rangle - N$, $\eta_j' = \langle \eta, \theta^j \rangle - N$ (see Theorem 3.11 in [6]). The most important property of this kernel is the following growth order estimate which is a key for our definition of quasi-positivity for micro hermitian kernels:

$$\begin{aligned} &|\partial_{(z,w^*)} K^\rho_\Omega| \le C_2 \min\{|\xi|^\sigma, |\eta|^\sigma\} \quad (0 \le \sigma < 1), \\ &|\partial_{(z,w^*)} K^\rho_\Omega| \le C_2(|\xi| + |\eta|)^\sigma \qquad (\sigma < 0), \\ &|\partial_{(\xi,\eta^*)} K^\rho_\Omega| \le C_3(|\xi| + |\eta|)^{\sigma-1} \quad (\sigma < 1). \end{aligned}$$

Indeed, such a growth order property is indispensable to prove the exponential calculus of pseudo-differential operators with symbol $\exp\left(M \cdot K^\rho_\Omega(z, \xi, w^*, \eta^*)\right)$ (cf. Aoki [1]).

§ 4. Positivity for hermitian microkernels

We consider hermitian kernels $K(x, u)$ as microfunctions on the anti-diagonal set:

$$\Delta^a(\sqrt{-1}T^*\mathbb{R}^n_x) = \{(x, u; i\xi, i\eta) \in \sqrt{-1}T^*(\mathbb{R}^n_x \times \mathbb{R}^n_u); x = u, \xi + \eta = 0\}.$$

Definition 4.1. Let $k(x,u)$ be a germ at $\overset{\circ}{p} = (\overset{\circ}{x}, \overset{\circ}{x}; i\overset{\circ}{\xi}, -i\overset{\circ}{\xi})$ $(\overset{\circ}{\xi} \neq 0)$ of microfunctions in $(x,u) \in \mathbb{R}^n_x \times \mathbb{R}^n_u$. Then, $k(x,u)$ is said to be a hermitian microkernel at $\overset{\circ}{p}$ if

$$k(x,u) = \overline{k(u,x)}.$$

Further, $k(x,u) \gg 0$ at $\overset{\circ}{p}$ (positive as a hermitian microkernel at $\overset{\circ}{p}$) if there exist a small $r > 0$, some open convex cones $\Gamma_1, \ldots, \Gamma_N$ in $\mathbb{R}^n$, and some positive analytic hermitian kernel $K_j(z,w)$ on

$$D_j := \{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r, \operatorname{Im} z \in \Gamma_j\}$$

for $j = 1, \ldots, N$ such that

$$k(x,u) = \sum_{j=1}^{N} [K_j(x + i0\Gamma_j, \overline{u - i0\Gamma_j})] \quad \text{at } \overset{\circ}{p}.$$

Here, we used the notation $K_j(z, \overline{w})$ because $K_j(z, \overline{w})$ is holomorphic in (z,w) on $D_j \times D_j^*$. Further, for two hermitian microkernels $k_1(x,u), k_2(x,u)$ we define an order relation:

$$k_1(x,u) \gg k_2(x,u) \text{ at } \overset{\circ}{p} \iff k_1(x,u) - k_2(x,u) \gg 0 \text{ at } \overset{\circ}{p}.$$

Theorem 4.2. *The relation $k_1 \gg k_2$ at $\overset{\circ}{p}$ satisfies the axioms of order relations; in particular, "$k \gg 0$ at $\overset{\circ}{p}$ and $-k \gg 0$ at $\overset{\circ}{p}$" implies "$k = 0$ at $\overset{\circ}{p}$".*

Example 4.3.

$$\delta(x-u) \gg 0 \quad \text{on} \quad \Delta^a(\sqrt{-1}T^*\mathbb{R}^n_x).$$

Definition 4.4. Let $T \subset \mathbb{R}^m$ be a bounded domain, and $\varphi(t) \in C^\omega(\overline{T})$ be a real-valued function satisfying

$$T = \{\varphi(t) > 0\}, \quad \text{and } \varphi(t) = 0,\ \nabla\varphi(t) \neq 0 \text{ on } \partial T.$$

Let $(\overset{\circ}{t}, \overset{\circ}{x})$ be a point of $\overline{T} \times \mathbb{R}^n$. For a small $r > 0$, a hyperfunction $f(t,x)$ defined on $\{(t,x) \in T \times \mathbb{R}^n; |t - \overset{\circ}{t}| < r, |x - \overset{\circ}{x}| < r\}$ is said to have real analytic parameters $t \in T$ at $(\overset{\circ}{t}, \overset{\circ}{x})$ if the following i) and ii) are satisfied (also see the remark below):

i) When $\overset{\circ}{t} \in \partial T$, f is mild on $\partial T \times \mathbb{R}^n$ from $T \times \mathbb{R}^n$. When $\overset{\circ}{t} \in T$, $\{|t - \overset{\circ}{t}| < r\} \subset T$.

ii) The extension $\mathrm{ext}_{\overline{T}}(f)$ of f to $\{|t - \overset{\circ}{t}| < r, |x - \overset{\circ}{x}| < r\}$ with support in $\overline{T}$ satisfies

$$\begin{aligned} &\mathrm{SS}(\mathrm{ext}_{\overline{T}}(f)) \cap \{(t,x; i\tau, i\xi); \xi = 0, |t - \overset{\circ}{t}| < r, |x - \overset{\circ}{x}| < r\} \\ &\subset \sqrt{-1}T^*_{\partial T \times \mathbb{R}^n}(\mathbb{R}^{m+n}) := \{(t,x; i\lambda\nabla\varphi(t), 0); t \in \partial T, \lambda \in \mathbb{R}\}. \end{aligned}$$

Remark. The above definition for $f(t,x)$ is equivalent to the following condition: There exist some $r' > 0$, some open convex cones $\Gamma_1, \dots, \Gamma_N$ in $\mathbb{R}^n$, and some holomorphic function $F_j(\tilde{t}, z)$ defined in

$$\{(\tilde{t}, z) \in \mathbb{C}^{m+n};\ \mathrm{dis}(\tilde{t}, T) < r', |z - \mathring{x}| < r', \mathrm{Im}\, z \in \Gamma_j, \\ (-\varphi(\mathrm{Re}\, \tilde{t}))_+ + |\mathrm{Im}\, \tilde{t}| < r'|\mathrm{Im}\, z|\}$$

$(j = 1, \dots, N)$ such that

$$f(t,x) = \sum_{j=1}^{N} F_j(t, x + i0\Gamma_j) \quad \text{in } \{t \in T, |x - \mathring{x}| < r'\}.$$

Here, $(s)_+ = s\ (s \geq 0), = 0\ (s < 0)$. Further, let $Y(s)$ be the Heaviside function. Then,

$$\mathrm{ext}_{\overline{T}}(f(t,x)) = \sum_{j=1}^{N} F_j(t,z)\, Y(\varphi(t))|_{\mathrm{Im}\, z \to 0 \cdot \Gamma_j}.$$

§ 5. Positive hermitian pseudodifferential operators and quasi-positivity

We denote by z, w the complexifications of x, u, and by ξ, η the symbols for ∂_z, ∂_w. Further, we use the notation z^*, ξ^* for the complex conjugate of z, ξ; for example, $\mathring{z}^*$. Therefore, the coordinates of $T^*(\mathbb{C}^n_z \times C^n_w)$ are given as $(z, w; \xi, \eta)$, and the hermitian pseudo-differential operators are defined on the hermitian diagonal set of $T^*(\mathbb{C}^n_z \times C^n_w)$:

$$\Delta^h(T^*\mathbb{C}^n) := \{(z, w; \xi, \eta) \in T^*(\mathbb{C}^n_z \times C^n_w); w = z^*, \eta = \xi^*\}.$$

Definition 5.1. Let $\mathring{p} = (\mathring{z}, \mathring{z}^*; \mathring{\xi}, \mathring{\xi}^*)$ be a point of $\Delta^h(T^*\mathbb{C}^n)$ ($|\mathring{\xi}| = 1$). Then,

$$P = \sum_{j,k=0}^{\infty} P_{jk}(z, w, \xi, \eta)$$

is said to be a formal symbol at $\mathring{p}$ of product hermitian pseudo-differential operators if there exist some positive numbers r, d, A $(0 < A < 1)$ such that conditions i)$\sim$iii) hold for any j, k :

i) P_{jk} is holomorphic on $V_j \times V_k^*$, where

$$V_j := \{(z; \xi) \in \mathbb{C}^n; |z - \mathring{z}| < r, |(\xi/|\xi|) - \mathring{\xi}| < r,\ (j+1)d < |\xi|\}.$$

ii) For any $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ (independent of j, k) such that

$$|P_{jk}(z, w, \xi, \eta)| \leq C_\varepsilon A^{j+k} e^{\varepsilon(|\xi| + |\eta|)} \quad \text{on } V_j \times V_k^*.$$

iii) $P_{jk}(z,w,\xi,\eta) = \overline{P_{kj}(w^*,z^*,\eta^*,\xi^*)}$ on $V_j \times V_k^*$.

Concerning 0-equivalence class, we have the following definition: $\sum_{j,k} P_{jk} \sim 0$ if

$$\left|\sum_{j=0}^{s}\sum_{k=0}^{t} P_{jk}(z,w,\xi,\eta)\right| \leq C'_\epsilon \exp\big(-\alpha\min\{s,t\} + \epsilon(|\xi|+|\eta|)\big)$$

holds on $V_s \times V_t^*$ for some $\alpha > 0$, every $s,t \geq 0$, and any $\epsilon > 0$ with some $C'_\epsilon > 0$.

Further, $P = \sum_{j,k=0}^{\infty} P_{jk}(z,w,\xi,\eta) \gg 0$ at $\mathring{p} \Longleftrightarrow$

For $\forall S(\geq 1), \forall J(\geq 0), \forall(z_{s,j},\xi_{s,j};\lambda_{s,j}) \in V_j \times \mathbb{C}$ $(s = 1,\ldots,S,\ j = 0,1,\ldots,J)$, we have

$$\sum_{s,t=1}^{S}\sum_{j,k=0}^{J} P_{jk}(z_{s,j},z^*_{t,k},\xi_{s,j},\xi^*_{t,k})\lambda_{s,j}\lambda^*_{t,k} \geq 0.$$

Proposition 5.2. *Any formal symbol* $\sum_{j,k} P_{jk}$ *of product hermitian pseudo-differential operators is equivalent to some simple symbol* $P' = P'_{00}$ $(P'_{jk} = 0$ *for* $\forall(j,k) \neq (0,0))$ *of product hermitian pseudo-differential operator. In particular,* $\sum_{j,k} P_{jk}$ *is identified with a usual pseudo-differential operator* $P(w,z,\partial_z,\partial_w)$ *at* $\mathring{p} = (\mathring{z},\mathring{z}^*;\mathring{\xi},\mathring{\xi}^*) \in \Delta^h(T^*\mathbb{C}^n)$. *Further, the operator product* $P \circ Q$ *of* $P = \sum_{j,k} P_{jk}$ *and* $Q = \sum_{j,k} Q_{jk}$ *is defined by*

$$(P\circ Q)_{jk} := \sum_{\substack{j=|\alpha|+\ell+\ell'\\ k=|\beta|+m+m'}} \frac{1}{\alpha!\,\beta!}\,\partial_\xi^\alpha\partial_\eta^\beta P_{\ell m}\cdot\partial_z^\alpha\partial_w^\beta Q_{\ell' m'}.$$

In particular, the product of two formal symbols of product hermitian pseudo-differential operators is a formal symbol of product hermitian pseudo-differential operators. Further, if $P = \sum_{j,k=0}^{\infty} P_{jk}(z,w,\xi,\eta) \gg 0$ *at* $\mathring{p}$ *and* $Q = \sum_{j,k=0}^{\infty} Q_{jk}(z,w,\xi,\eta) \gg 0$ *at* $\mathring{p}$, *then* $P\circ Q \gg 0$ *at* $\mathring{p}$.

Example 5.3. Let $P(z,\xi)$ be a symbol at $(\mathring{z};\mathring{\xi}) \in T^*\mathbb{C}^n$ of analytic pseudo-differential operator. Put

$$P^*(w,\eta) := \overline{P(w^*,\eta^*)}$$

Then,

$$P(z,\xi) + P^*(w,\eta), \quad P(z,\xi)P^*(w,\eta)$$

are examples of symbols at $p = (\mathring{z},\mathring{z}^*;\mathring{\xi},\mathring{\xi}^*) \in \Delta^h(T^*\mathbb{C}^n)$ of product hermitian pseudo-differential operators. Further, $P(z,\xi)P^*(w,\eta) \gg 0$ at p. Another non-trivial example of positive product hermitian pseudo-differential operators is

$$Q(z,w,\partial_z,\partial_w) := \big(P(z,\partial_z) + P^*(w,\partial_w)\big)^{-1} \gg 0,$$

where $P(z,\xi)$ is a simple symbol of analytic pseudo-differential operators satisfying the following estimate for some $m>0$ and some $C>0$:

$$C^{-1}|\xi|^m \le \operatorname{Re} P(z,\xi) \le |P(z,\xi)| \le C|\xi|^m \quad (|\xi|\to\infty)$$

in a conic neighborhood of $(\mathring{z};\mathring{\xi}) \in T^*\mathbb{C}^n$.

As we explained in Section 3, in order to treat an operator $P(t,x,\partial_x)+P^*(t,u,\partial_u)$ we must introduce a special type of positive product hermitian pseudo-differential operators; for example, $\exp\big(M\cdot K^\rho_\Omega(z,\xi,w^*,\eta^*)\big)$ in Section 3.

Definition 5.4. Let $\mathring{p}=(\mathring{t},\mathring{z},\mathring{z}^*;\mathring{\xi},\mathring{\xi}^*)$ be a point $\mathbb{R}^m\times\Delta^h(T^*\mathbb{C}^n)$ $(|\mathring{\xi}|=1)$. A simple symbol $P(t,z,w,\xi,\eta)$ is said to be a symbol at $\mathring{p}$ with real analytic parameters t of restricted hermitian pseudo-differential operators with growth order σ if P is holomorphic in

$$\begin{aligned}V(r) :=\{&(t,z,w,\xi,\eta)\in\mathbb{C}^{m+4n}; |t-\mathring{t}|<r, |z-\mathring{z}|<r, |w-\mathring{z}^*|<r,\\ &|(\xi/|\xi|)-\mathring{\xi}|<r, |(\eta/|\eta|)-\mathring{\xi}^*|<r,\ |\xi|>r^{-1}, |\eta|>r^{-1}\}\end{aligned}$$

such that $P(t,z,w,\xi,\eta)=\overline{P(t^*,w^*,z^*,\eta^*,\xi^*)}$ and

$$\begin{cases}|\nabla_{(z,w)}P|\le C\cdot\min\{|\xi|^\sigma,|\eta|^\sigma\},\\ |\nabla_{(\xi,\eta)}P|\le C\cdot(|\xi|+|\eta|)^{\sigma-1}.\end{cases} \tag{5.1}$$

Here, $0<\sigma<1/2$, and C,r are some positive constants. Further, a symbol

$$\exp(P(t,z,w,\xi,\eta))$$

with a restricted hermitian symbol P is said to be a simple symbol at $\mathring{p}$ with real analytic parameters t of exponential restricted hermitian pseudo-differential operators with growth order σ. Indeed, it is easy from (5.1) to obtain an estimate

$$|P(t,z,w,\xi,\eta)|\le M(|\xi|^\sigma+|\eta|^\sigma) \text{ on } V(r)$$

with some $M>0$. Hence, $\exp(P)$ is a symbol of product hermitian pseudo-differential operators.

Example 5.5.

$$P:=A(t,z,w)\cdot\frac{(\xi_1\eta_1)^{1+(\sigma/2)}}{\xi_1^2+\eta_1^2} \text{ at } (\mathring{t},\mathring{z},\mathring{z}^*;dz_1+dw_1),$$

is a symbol of positive restricted hermitian pseudo-differential operators with growth order σ, where $A(t,z,w)$ is a holomorphic function in a neighborhood of $(\mathring{t},\mathring{z},\mathring{z}^*)$ such that for any real fixed t, $A(t,z,w^*)$ is a positive analytic hermitian kernel in z,w.

Definition 5.6. Let $\overset{\circ}{p} = (\overset{\circ}{x}, \overset{\circ}{x}; i\overset{\circ}{\xi}, -i\overset{\circ}{\xi})$ be a point of $\Delta^a(\sqrt{-1}T^*\mathbb{R}^n)$ ($|\overset{\circ}{\xi}| = 1$). Then a hermitian microkernel $k(x,u)$ is said to be quasi-positive, $k(x,u) \gg_q 0$ at $\overset{\circ}{p}$, if there exists a symbol $P(z,w,\xi,\eta)$ at $\overset{\circ}{p}$ of positive restricted hermitian pseudo-differential operators with growth order $\sigma < 1/2$ such that : $\exp(P(z,w,\xi,\eta)) : k(x,u) \gg 0$ at $\overset{\circ}{p}$. Namely,

$$k(x,u) \gg_q 0 \iff \exists P \text{ (restricted, positive) s. t.} : \exp(P(z,w,\xi,\eta)) : k(x,u) \gg 0.$$

Here, : Q : means the pseudo-differential operator defined by the symbol Q. It is known that the quasi-positivity satisfies the axioms of order relations for hermitian microkernels (Theorem 2.7 in [7]).

§ 6. K. Yamasaki's Sobolev type 2-form of order 0

Let $T \subset \mathbb{R}^m$ be a bounded domain with real analytic boundary ∂T. Let $f(t,x)$ be a hyperfunction on $T \times \{|x - \overset{\circ}{x}| < r\}$, which have real analytic parameters $t \in T$ at any point of $\overline{T} \times \{\overset{\circ}{x}\}$. For $\mu > 0$, a hermitian microkernel

$$\int_{\mathbb{R}^m} ((-\Delta_x)^\mu + (-\Delta_u)^\mu) \mathrm{ext}_{\overline{T}}(f(t,x)\overline{f(t,u)})dt,$$

in (x,u) is almost identified with Sobolev type 2-form of order μ with respect to x, u. However, concerning t, it is only an L^2-form. Though we can treat a more general 2-form like (3.1) by using quasi-positivity, we cannot treat the following type 2-form:

$$E(x,u) := \int_T (P(t,x,\partial_t,\partial_x) + \overline{P(s,u,\partial_s,\partial_u)}) f(t,x)\overline{f(s,u)}|_{t=s}\, dt, \tag{6.1}$$

where $P(t,x,\partial_t,\partial_x)$ is a pseudo-differential operator including ∂_t. Even in such a case, if P is of finite order with respect to ∂_t, we treat E by using microlocal energy methods for vector-valued functions developed in [7]. Indeed, in that case, we consider all the derivatives $\partial_t^\alpha f(t,x)$ as independent hyperfunctions. However, such a method cannot be applied to the case :

$$P = 1 + \sum_{\ell=1}^{\infty} \partial_t^\ell \partial_x^{-\ell}$$

at $(0,0;0,i) \in \sqrt{-1}T^*(\mathbb{R}_t \times \mathbb{R}_x)$ because P is not of finite order concerning ∂_t. To overcome this difficulty, K. Yamasaki [11] introduced a Sobolev-type 2-form of order 0.

Definition 6.1. Let s, u be the copies of t, x, respectively. For positive numbers C_1, C_2, and constants $p, q \geq 0$, we define a micro-differential operator Λ (a fundamental microdifferential operator) in the variables $(t,x,s,u) \in \mathbb{R}^{m+n+m+n}$ of order 0 (consider in $\{\xi_n \neq 0, \eta_n \neq 0\}$):

$$\Lambda(\partial_t, \partial_{x_n}, \partial_s, \partial_{u_n}) := \sum_{j\geq 0, I\geq 0} \left(\frac{\Gamma((p+1)j + q|I| + 1)}{\Gamma(pj + q|I| + 1)} \right)^2 C_1^{2j} C_2^{2|I|} \partial_t^I \partial_{x_n}^{-j-|I|} \partial_s^I \partial_{u_n}^{-j-|I|},$$

where $j \in \mathbb{N}_+$, $I = (i_1, \dots, i_m) \in \mathbb{N}_+^m$ $(\mathbb{N}_+ = \{0,1,2,\dots\})$.

Definition 6.2. Let $f(t,x)$ and $g(t,x)$ be hyperfunctions on $T \times \{|x - \mathring{x}| < r\}$, which have real analytic parameters $t \in T$ at any point of $\overline{T} \times \{\mathring{x}\}$. Then, we can introduce an inner product form of f, g as follows:

$$E[f,g](x,u) := \int_{\mathbb{R}^m} \mathrm{ext}_{\overline{T}} \big[\Lambda(\partial_t, \partial_{x_n}, \partial_s, \partial_{u_n}) \big(f(t,x) \overline{g(s,u)} \big) \Big|_{t=s} \big] dt.$$

Hence, our order 0 Sobolev type 2-form over $\overline{T}$ of $f(t,x)$ is defined as

$$E[f,f](x,u).$$

Let $A(t,x,\partial_t,\partial_x)$ be any 0-th order analytic pseudo-differential operator. Then, our aim is to get the following inequality:

$$E[Af,g](x,u) + E[g,Af](x,u) \ll_q C_A \big(E[f,f](x,u) + E[g,g](x,u) \big) \tag{6.2}$$

with a positive constant C_A depending only on A. Kaito Yamasaki's main result [11] is the following:

Theorem 6.3. *Put $C_1 = C_2\, C_3$, and*

$$\begin{cases} p \geq 0, \quad q \geq 1, \quad p - q \geq -1, \\ C_2 > \frac{2^{m+2}}{\lambda}, \quad C_3 > \frac{2^{m+3}}{\lambda}, \quad C_2\, C_3 > \max\{\frac{8}{\lambda}, \frac{16(m+n)}{\lambda^2}\}. \end{cases} \tag{6.3}$$

Then, for

$$C_A = 2^{2m+1} N(A;\lambda)$$

we have an inequality (6.2), *where $N(A;\lambda)$ is the formal norm of A due to Boutet de Monvel. Namely, $N(A;\lambda)$ is a formal power series of λ, and so $\lambda > 0$ should be taken small enough such that $N(A;\lambda)$ is convergent. Further, the condition for p, q is the necessary and sufficient to get* (6.2).

Remark. In the definition of $E[f,g]$, we need some stronger assumption on the singular spectrum of f, g for a given C_2 than the assumption ii) in Definition 4.4. Such a condition is fulfilled if C_2 is sufficiently small.

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