数理解析研究所講究録
第 1980 巻 2016 年 152-158

The invertible Toeplitz operators on the Bergman spaces

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Abstract

In this paper, we study the invertible (and Fredholm) Toeplitz operators T_{φ} on the Bergman spaces with harmonic symbol.

Key Words and Phrases : Bergman spaces, Toeplitz operator, closed range, invertible operator, Fredholm operator.

Let D be the open unit disk in complex plane C. For $z,w\in D$, and $0<r<1$, let $\varphi_{w}(z)=\dfrac{w-z}{1-\overline{w}z}$ and $\rho(z,w)=\left|\dfrac{w-z}{1-\overline{w}z}\right|$ and $D(w,r)=\left\{z\in D,\rho(w,z)<r\right\}$.

Let $H(D)$ be the space of all analytic functions on D.

The space $L^{2}(dA(z))$ is defined to be the space of Lebesgue measurable functions f on D such that

$$\| f\|_{L^{2}(dA(z))}=\left\{\int_{D}|f(z)|^{2}dA(z)\right\}^{\frac{1}{2}}<+\infty,$$

where $dA(z)$ denote the area measure on D. The Bergman space $L_{a}^{2}(dA(z))$ is defined by

$$L_{a}^{2}(dA(z))=H(D)\cap L^{2}(dA(z)).$$

For $\varphi\in L^{2}(dA(z))$, the Toeplitz operator T_{φ} with symbol φ is defined on $L_{a}^{2}(dA(z))$ by

$$T_{\varphi}f=P(\varphi f),$$

where $P(f)(z) = \int_D \frac{f(w)}{(1-\overline{w}z)^2} dA(w)$.

Let X, Y be Banach spaces and let T be a linear operator from X into Y. Then T is called to be bounded below from X to Y if there exists a positive constant $C > 0$ such that $\| Tf \|_Y \geq C \| f \|_X$ for all $f \in X$, where $\| * \|_X$, $\| * \|_Y$ be the norm of X, Y, respectively. The Berezin transform of T_φ is given by $\widetilde{\varphi}(z) = \widetilde{T_\varphi}(z) = < T_\varphi k_z, k_z >$, where $k_z(w) = \frac{1-|z|^2}{(1-z\overline{w})^2}$.

If H is a Hilbert space, then a bounded operator T is a Fredholm operator if and only if the range of T is closed, $\dim\ker T$, and $\dim\ker T^*$ is finite.

For $a, b \in C$, $\varphi, \psi \in L^\infty(D)$, then

(a) $T_{a\varphi+b\psi} = aT_\varphi + bT_\psi$,
(b) $T_{\overline{\varphi}} = T_\varphi^*$,
(c) $T_\varphi \geq 0$ $(\varphi \geq 0)$.

For $\varphi \in H^\infty$, then

(d) $T_\psi T_\varphi = T_{\psi\varphi}$,
(e) $T_{\overline{\varphi}} T_\psi = T_{\overline{\varphi}\psi}$

Let $\tilde{\varphi}$ denote the harmonic extension of the function φ to the open unit disk D. In [8], Douglas posed the following problem :
If φ is a function in L^∞ for which $|\varphi| \geq \delta > 0$, $z \in D$, then is T_φ invertible ?
And V.A. Tolokonnikov gave the following:
If $|\tilde{\varphi}(z)| \geq \delta > \frac{45}{46}$, then T_φ is invertible.
In [18], N.K.Nikolskii gave the following:
If $|\tilde{\varphi}(z)| \geq \delta > \frac{23}{24}$, then T_φ is invertible.
In [20], T.H.Wolff gave the following:
If $\inf_D |\tilde{\varphi}(z)| > 0$ and then T_φ is not invertible.

The study of Toeplitz operators on the Bergman spaces and Hardy space have been studied by many authors. In this paper, we study when the Toeplitz operators T_φ on the Bergman spaces with harmonic symbol is invertible or Fredholm.

In [14], the following theorem are well-known.

Theorem A. *Suppose φ is a bounded and nonnegative function. Then the following conditions are equivalent :*

(1) T_φ *is bounded below.*
(2) *There is a constant* $C > 0$ *such that*

$$\int_D |f(z)|^2 \varphi(z) dA(z) \geq C \int_D |f(z)|^2 dA(z),$$

for all $f \in L_a^2(dA(z))$.

In [14], D.Leucking proved the following results.

Theorem B. *Let* $\alpha > -1$ *and* $p > 0$. *Then the following are equivalent*:

(1) *There is a constant* $C > 0$ *such that*

$$\int_D |f(z)|^p dA(z) \leq C \int_G |f(z)|^p dA(z)$$

for all $f \in L_a^p(dA(z))$

(2) *There is a constant* $C > 0$ *such that*

$$\int_D |f(z)|^p (1 - |z|^2)^\alpha dA(z) \leq C \int_G |f(z)|^p (1 - |z|^2)^\alpha dA(z)$$

for all $f \in L_a^p \left((1 - |z|^2)^\alpha dA(z)\right)$

(3) *For any* $a \in D$ *a subset* G *of* D *satisfy the condition that there exist* $\delta > 0$ *and* $r > 0$ *such that* $\delta |D(a,r)| \leq |D(a,r) \cap G|$, *where* $|D(a,r)|$ *is the (normalized) area of* $D(a,r)$.

Theorem C. *Let* φ *be a bounded measurable function on* D. *Tnen there is a constant* $\epsilon > 0$ *such that*

$$\int_D |\varphi(z) f(z)|^p dA(z) \geq \epsilon \int_D |f(z)|^p dA(z)$$

for all $f \in L_a^p(dA(z))$ *if and only if there exists* $r > 0$ *such that the set* $\{z \in D : |\varphi(z)| > r\}$ *satisfies condition* (3) *of Theorem* 3.

Theorem D. *Let φ be a bounded positive measurable function on D. Tnen T_φ is invertible if and only if there exists $r > 0$ such that the set $\{z \in D : |\varphi(z)| > r\}$ satisfies condition (3) of Theorem3.*

The following theorem is well-known(see [21]).

Theorem E. *Suppose that $\varphi \in \mathcal{C}(\overline{D})$. Then the following conditions are equivalent :*

(1) T_φ *is Fredholm.*

(2) φ *is nonvanishing on the unit circle.*

Theorem 1. *Let $g \in H^\infty$. Then the following are equivalent:*

(1) $T_{\overline{g}}$ *is invertible operator on* $L^2_a(dA(z))$

(2) T_g *is invertible operator on* $L^2_a(dA(z))$

(3) $\inf_{z \in D} \left|\widetilde{T_{\overline{g}}}(z)\right| = \inf_{z \in D} |g(z)| > 0$

The problem which we must consider next is following.

Problem. *Let $g, h \in H^\infty$ and $g, h \in \mathcal{C}(\overline{D})$. Then the following are equivalent:*

(1) $T_{g+\overline{h}}$ *is invertible operator on* $L^2_a(dA(z))$

(2) $\inf_{z \in D} \left|\widetilde{T_{g+\overline{h}}}(z)\right| = \inf_{z \in D} |g(z) + \overline{h(z)}| > 0$

At first, we can prove the following.

Theorem 2. *Let $g \in H^\infty$ and $g \in \mathcal{C}(\overline{D})$. Then the following are equivalent:*

(1) $T_{g+\overline{g}}$ *is bounded below on* $L^2_a(dA(z))$

(2) $T_{g+\overline{g}}$ *is invertible operator on* $L^2_a(dA(z))$

(3) $\inf_{z \in D} \left|\widetilde{T_{g+\overline{g}}}(z)\right| = \inf_{z \in D} |g(z) + \overline{g(z)}| > 0$

Next, we can prove the following.

Theorem 3. *Let $g \in H^\infty$ and a constant $c > 1$. Then the following are equivalent:*

(1) T_g *is bounded below on* $L_a^2(dA(z))$
(2) $T_{cg+\overline{g}}$ *is bounded below on* $L_a^2(dA(z))$

Using Theorem 3, we prove the following main result.

Theorem 4. *Let* $g \in H^\infty$ *and* $g \in \mathcal{C}(\overline{D})$. *Then the following are equivalent:*
(1) $T_{\overline{g}}$ *is invertible operator on* $L_a^2(dA(z))$
(2) T_g *is invertible operator on* $L_a^2(dA(z))$
(3) $T_{cg+\overline{g}}$ *is invertible operator on* $L_a^2(dA(z))(c > 0, c \neq 1)$
(4) $\inf_{z\in D} \left|\widetilde{T_{cg+\overline{g}}}(z)\right| = \inf_{z\in D} |cg(z) + \overline{g(z)}| > 0(c > 0, c \neq 1)$
(5) $\inf_{z\in D} \left|\widetilde{T_g}(z)\right| = \inf_{z\in D} \left|\widetilde{T_{\overline{g}}}(z)\right| = \inf_{z\in D} |g(z)| > 0$

Moreover, we can prove the following.

Theorem 5. *Let* $g, h \in H^\infty$ *with* $\inf_{z\in D} |h(z)| - \sup_{z\in D} |g(z)| > 0$. *If* $\inf_{z\in D} \left|\widetilde{T_{g+\overline{h}}}(z)\right| > 0$, then $T_{g+\overline{h}}$ *is invertible on* $L_a^p(dA(z))$, and $T_{h+\overline{g}}$ *is invertible on* $L_a^p(dA(z))$

Theorem 6. *Suppose that* $g \in H^\infty$ *and* $g \in \mathcal{C}(\overline{D})$. *Then the following conditions are equivalent :*
(1) $T_{\overline{g}}$ *is Fredholm.*
(2) $T_{cg+\overline{g}}$ *is Fredholm*$(c > 0, c \neq 1)$.
(3) g *is nonvanishing on the unit circle.*

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