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The Toeplitz operators on the Bergman spaces with radial symbol

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Abstract

In this paper, we study the invertible (and Fredholm) Toeplitz operators T_{φ} on the Bergman spaces with radial symbol.

Key Words and Phrases : Bergman spaces, Toeplitz operator, closed range, invertible operator, Fredholm operator, radial symbol.

§1. Introduction

Let D be the open unit disk in complex plane C. Let $H(D)$ be the space of all analytic functions on D.

The space $L^{p}(dA(z))$ is defined to be the space of Lebesgue measurable functions f on D such that

$$\| f \|_{L^{2}(dA(z))}=\left\{\int_{D}|f(z)|^{2}dA(z)\right\}^{\frac{1}{2}}<+\infty,$$

where $dA(z)$ denote the area measure on D. The Bergman space $L_{a}^{2}(dA(z))$ is defined by

$$L_{a}^{2}(dA(z))=H(D)\cap L^{2}(dA(z)).$$

For $\varphi\in L^{2}(dA(z))$, the Toeplitz operator T_{φ} with symbol φ is defined on $L_{a}^{2}(dA(z))$ by

$$T_{\varphi}f=P(\varphi f),$$

where $P(f)(z) = \int_D \frac{f(w)}{(1-\overline{w}z)^2} dA(w)$.

Let X, Y be Banach spaces and let T be a linear operator from X into Y. Then T is called to be bounded below from X to Y if there exists a positive constant $C > 0$ such that $\| Tf \|_Y \geq C \| f \|_X$ for all $f \in X$, where $\| * \|_X$, $\| * \|_Y$ be the norm of X, Y, respectively.

Let $C(H)$ be the space of the compact operator on the Hilbert space H. If H is a Hilbert space, then a bounded operator T is a Fredholm operator if and only if there exists a bounded operator B such that $TB - I, BT - I \in C(H)$. And a bounded operator T is a Left (Right) Fredholm operator if and only if there exists a bounded operator B such that $BT - I \in C(H)(TB - I \in C(H))$.

The Berezin transform of the Toeplitz operators T_φ is given by

$$\tilde{\varphi}(z) = \widetilde{T_\varphi}(z) = < T_\varphi k_z, k_z >$$

, where $k_z(w) = \frac{1-|z|^2}{(1-z\overline{w})^2}$.

In [4], B.Korenblum and K.Zhu proved the following result.

Theorem A. *Suppose φ is a bounded and radial, that is $\varphi(re^{i\theta}) = \varphi(r)$. Then the following conditions are equivalent :*

(1) T_φ *is compact.*

(2) $\tilde{\varphi}(z) \to 0$ as $|z| \to 1^-$.

(3) $\lim_{x \to 1^-} \frac{1}{1-x} \int_x^1 \varphi(r) dr = 0$.

In [12], N.Zorboska generalized this theorem.

Theorem B. *Let φ be a radial function in $L^1(D)$, and that T_φ be bounded on L^2_a.* If $f(r) - \frac{1}{1-r} \int_r^1 \varphi(s) s ds$ *is bounded for* $0 \leq r < 1$, *then the following conditions are equivalent :*

(1) T_φ *is compact.*

(2) $\tilde{\varphi}(z) \to 0$ as $|z| \to 1^-$.

In [10], the following theorem is well-known.

Theorem C. *Suppose φ is a bounded and nonnegative function. Then the following conditions are equivalent :*
(1) *T_φ is bounded below.*
(2) *There is a constant $C > 0$ such that*

$$\int_D |f(z)|^2 \varphi(z) dA(z) \geq C \int_D |f(z)|^2 dA(z),$$

for all $f \in L_a^2(dA(z))$.

The following theorem is well-known(see [10]).

Theorem D. *Suppose that $\varphi \in \mathcal{C}(\overline{D})$. Then the following conditions are equivalent :*
(1) *T_φ is Fredholm.*
(2) *φ is nonvanishing on the unit circle.*

The study of Toeplitz operators on the Bergman spaces and Hardy space have been studied by many authors. In this paper, we study when the Toeplitz operators T_φ on the Bergman spaces with radial symbol is invertible or Fredholm.

§2. Statement of main results.

To prove our main theorem, we need the following.

Proposition 1. *Suppose φ is a bounded and radial function. Then the following are equivalent:*
(1) *T_φ is bounded below on $L_a^2(dA(z))$.*
(2) *T_φ is an invertible operator on $L_a^2(dA(z))$.*
(3) *There exists a positive constant $K > 0$ such that*

$$(n+1)\left|\int_0^1 \varphi(t) t^{2n+1} dt\right| \geq K,$$

for $n = 0, 1, 2, \cdots$.

Using the above proposition, we can prove the following result.

Proposition 2. *Suppose φ is a bounded and radial function. If there exists a positive* constant $C > 0$ such that $\frac{1}{1-x}\int_x^1 \varphi(t)dt \geq C (x \in [0,1))$ or $\frac{1}{1-x}\int_x^1 \varphi(t)dt \leq -C$ $(x \in [0,1))$, *then T_φ is an invertible operator on $L_a^2(dA(z))$.*

Moreover, we can also prove the following result.

Proposition 3. *Suppose $\varphi \in C([0,1))$ is a bounded and radial real-valued function. If there exists a positive constant $C > 0$ such that $\inf_{x\in[0,1)} \frac{1}{1-x}\left|\int_x^1 \varphi(t)dt\right| \geq C$, then T_φ is an invertible operator on $L_a^2(dA(z))$.*

The following is our main result.

Theorem 4. *Suppose $\varphi \in C(\overline{D})$ is a bounded and radial, and $\varphi \geq 0$(or $\varphi \leq 0$). Then the following are equivalent:*

(1) *T_φ is bounded below on $L_a^2(dA(z))$*

(2) *T_φ is an invertible operator on $L_a^2(dA(z))$*

(3) *There exists a positive constant $C > 0$ such that*

$$\inf_{z\in D} |\tilde{\varphi}(z)| \geq C.$$

(4) *There exists a positive constant $C > 0$ such that*

$$\inf_{x\in[0,1)} \left|\frac{1}{1-x}\int_x^1 \varphi(t)dt\right| \geq C.$$

(5) *There exists a positive constant $K > 0$ such that*

$$(n+1)\left|\int_0^1 \varphi(t)t^{2n+1}dt\right| \geq K\,,$$

for $n = 0, 1, 2, \cdots$.

Remark 5. There exists an example that T_φ is invertible on $L_a^2(dA(z))$ and that (4) of the above theorem does not hold. For example, let $\varphi(t) = t - \frac{7}{10}$. Since there exists a positive constant $K > 0$ such that

$$(n+1)\left|\int_0^1 \varphi(t)t^{2n+1}dt\right| \geq K\,,$$

for $n = 0, 1, 2, \cdots$, T_φ is invertible on $L_a^2(dA(z))$. But for $x = \frac{2}{5}$, an elementaly calculation implies that $\frac{1}{1-x}\int_x^1 \varphi(t)dt = 0$. □

Using Theorem D and Theorem 4, we can prove the following.

Theorem 6. *Suppose $\varphi \in C(\overline{D})$ is a bounded and radial, and $\varphi \geq 0$(or $\varphi \leq 0$). Then the following are equivalent:*
(1) T_φ *is an invertible operator on* $L_a^2(dA(z))$
(2) T_φ *is a Fredholm operator on* $L_a^2(dA(z))$.
(3) φ *is nonvanishing on the unit circle.*

The following is well-known(see [2]).

Proposition E. *Suppose φ is a bounded function. Then the following are equivalent:*
(1) T_φ *is a Left Fredholm operator on* $L_a^2(dA(z))$
(2) $\liminf_{n\to\infty} \| T_\varphi e_n \|_{L_a^2} > 0$, *where e_n be an orthonormal basis of* L_a^2.

When φ is a bounded and radial function, T_φ is a normal operator. So we see the following.

Proposition F. *Suppose φ is a bounded and radial function. Then the following are equivalent:*
(1) T_φ *is a Fredholm operator on* $L_a^2(dA(z))$
(2) $\liminf_{n\to\infty}(n+1)\left|\int_0^1 \varphi(t)t^{2n+1}dt\right| > 0$.

The problem which we must consider next is following.

Problem 7. *Suppose φ is a bounded and radial function. Then the following are equivalent:*
(1) T_φ *is a Fredholm operator on* $L_a^2(dA(z))$
(2) $\liminf_{n\to\infty} \| T_\varphi e_n \|_{L_a^2} > 0$, *where e_n be an orthonormal basis of* L_a^2.
(3) $\liminf_{n\to\infty}(n+1)\left|\int_0^1 \varphi(t)t^{2n+1}dt\right| > 0$.
(4) $\liminf_{x\to 1^-} \frac{1}{1-x}\left|\int_x^1 \varphi(t)dt\right| > 0$.
(5) $\liminf_{|z|\to 1^-} |\tilde{\varphi}(z)| > 0$.

The following results were obtaind.

Theorem 8. *Suppose φ is a bounded and positive radial function. If* $\liminf_{|z|\to 1^-} \varphi(|z|) > 0$, *then T_φ is a Fredholm operator on $L^2_a(dA(z))$.*

Theorem 9. *Suppose φ is a bounded and radial function and that* $\lim_{x\to 1^-} \frac{1}{1-x}\int_x^1 \varphi(t)dt = A$.
Then T_φ is a Fredholm operator on $L^2_a(dA(z))$ if and only if

$$\liminf_{x\to 1^-} \frac{1}{1-x}\left|\int_x^1 \varphi(t)dt\right| = \lim_{x\to 1^-} \frac{1}{1-x}\left|\int_x^1 \varphi(t)dt\right| > 0 \ .$$

Theorem 10. *Suppose φ is a bounded radial function and that* $\lim_{x\to 1^-} \frac{1}{1-x}\int_x^1 \varphi(t)dt = A$. *Then T_φ is a Fredholm operator on $L^2_a(dA(z))$ if and only if* $\liminf_{|z|\to 1^-} |\tilde{\varphi}(z)| > 0$.

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