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What are reproducing kernels ?

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Abstract: Here, we shall state simply a general meaning for reproducing kernels. We would like to answer the general and essential question that: what are reproducing kernels ? By considering the basic problem, we will be able to obtain a general concept of the generalized delta function as a generalized reproducing kernel and, as a general reproducing kernel Hilbert space, we will be able to consider all separable Hilbert spaces consisting of functions. We shall refer to an open question (a general problem) that was proposed by Professor Kenji Fukumizu in the meeting held on 7th–9th Oct. in RIMS.

1 Introduction

We would like to introduce the concept of general reproducing kernels and at the same time, we seek to answer clearly the general and essential question that: what are reproducing kernels ? By considering this basic problem, we will be able to obtain a general concept of the generalized delta function as a generalized reproducing kernel and, as a general reproducing kernel Hilbert space, we will be able to consider all separable Hilbert spaces comprising functions.

See [5, 6] for the background motivation and ideas from a general initial problem for partial differential equations.

2 What is a reproducing kernel?

We shall consider a family of **any complex valued functions** $\{U_n\}_{n=0}^{\infty}$ defined on a nonempty abstract set E that are linearly independent. Then, we consider a function given by:

$$K_N(x,y) = \sum_{n=0}^{N} U_n(x)\overline{U_n(y)} \quad (x, y \in E). \tag{2.1}$$

Then, we will check that K_N is a **reproducing kernel**.

We shall consider the linear space of all the functions, for arbitrary complex numbers $\{C_n\}_{n=0}^{N}$

$$F = \sum_{n=0}^{N} C_n U_n \tag{2.2}$$

and we introduce the norm

$$\|F\| = \sqrt{\sum_{n=0}^{N} |C_n|^2}. \tag{2.3}$$

The function space forms a Hilbert space $H_{K_N}(E)$ determined by the kernel K_N with the inner product induced from the norm (2.3). Then we note that for any $y \in E$,

$$K_N(\cdot, y) \in H_{K_N}(E) \tag{2.4}$$

and for any $F \in H_{K_N}(E)$ and for any $y \in E$

$$F(y) = (F(\cdot), K_N(\cdot, y))_{H_{K_N}(E)} = \sum_{n=0}^{N} C_n U_n(y). \tag{2.5}$$

The properties (2.4) and (2.5) are called a **reproducing property** of the kernel K_N for the Hilbert space $H_{K_N}(E)$, because the functions F in the inner product (2.5) appeared in the left-hand side. This formula tells us that the functions F may be represented by the kernel K_N and that the Hilbert space $H_{K_N}(E)$ is represented by the kernel K_N.

Of course, conversely the reproducing kernel K_N is uniquely determined by the reproducing properties (2.4) and (2.5) in the space $H_{K_N}(E)$.

3 A general reproducing kernel

We shall introduce a preHilbert space by

$$H_{K_\infty} := \bigcup_{N \geqq 0} H_{K_N}(E).$$

For any $F \in H_{K_\infty}$, there exists a space $H_{K_M}(E)$ containing the function F for some $M \geqq 0$. Then, for any N such that $M < N$,

$$H_{K_M}(E) \subset H_{K_N}(E)$$

in view of linear independence of the functions $\{U_n\}_{n=0}^{\infty}$ and, for the function $F \in H_{K_M}$,

$$\|F\|_{H_{K_M}(E)} = \|F\|_{H_{K_N}(E)}.$$

Therefore, the limit exists:

$$\|F\|_{H_{K_\infty}} := \lim_{N \to \infty} \|F\|_{H_{K_N}(E)}.$$

We denote by H_∞ the completion of H_{K_∞} with respect to this norm.

Note that for any $M < N$, and for any $F_M \in H_{K_M}(E)$, $F_M \in H_{K_N}(E)$ and furthermore in particular, note that

$$\langle f, g \rangle_{H_{K_M(E)}} = \langle f, g \rangle_{H_{K_N(E)}}$$

for all $N > M$ and $f, g \in H_{K_M}(E)$. Then, we obtain the main result:

Theorem. Under the above conditions, for any function $F \in H_\infty$ and for F_N^* defined by

$$F_N^*(x) = \langle F, K_N(\cdot, x) \rangle_{H_\infty} \quad (x \in E),$$

$F_N^* \in H_{K_N}(E)$ for all $N > 0$, and as $N \to \infty$, $F_N^* \to F$ in the topology of H_∞.

Proof. Observe that

$$\begin{aligned} |F_N^*(x)|^2 &\leq \|F\|_{H_\infty}^2 \|K_N(\cdot, x)\|_{H_\infty}^2 \\ &\leq \|F\|_{H_\infty}^2 \|K_N(\cdot, x)\|_{H_{K_N}(E)}^2 \\ &= \|F\|_{H_\infty}^2 K_N(x, x). \end{aligned}$$

Therefore, we see that $F_N^* \in H_{K_N}(E)$ and that $\|F_N^*\|_{H_{K_N}(E)} \leq \|F\|_{H_\infty}$.

Indeed, for these, recall the identity

$$K_N(x, y) = \langle K_N(\cdot, y), K_N(\cdot, x)\rangle_{H_\infty}.$$

The mapping $F \mapsto F_N^*$ being uniformly bounded, and so, we can assume that $F \in H_{K_L}(E)$ for any fixed L. However, in this case, the result is clear, since, $F \in H_{K_N}(E)$ for $L < N$

$$\lim_{N\to\infty} F_N^*(x) = \lim_{N\to\infty} \langle F, K_N(\cdot, x)\rangle_{H_\infty} = \lim_{N\to\infty} \langle F, K_N(\cdot, x)\rangle_{H_{K_N}(E)} = F(x).$$

The theorem may be looked as a reproducing kernel in the natural topology and by the sense of the Theorem, the reproducing property may be written as follows:

$$F(x) = \langle F, K_\infty(\cdot, x)\rangle_{H_\infty},$$

with

$$K_\infty(\cdot, x) \equiv \lim_{N\to\infty} K_N(\cdot, x) = \sum_{n=0}^{\infty} U_n(\cdot)\overline{U_n(x)}. \tag{3.1}$$

Here **the limit does, in general, not need to exist**, however, the series are non-decreasing, in the sense: for any $N > M$, $K_N(y, x) - K_M(y, x)$ is a positive definite quadratic form function.

Of course, any member of the completion space H_∞ may be approximated, naturally, by the members of reproducing kernel Hilbert spaces $H_{K_N}(E)$. This viewpoint will be very important for the approximation of functions, because the functions in a reproducing kernel Hilbert space may be given in a concrete way.

4 Conclusion

Any reproducing kernel (separable case) may be considered as the form (3.1) by arbitrary linear independent functions $\{U_n\}$ on an abstract set E, here, the sum does not need to converge. Furthermore, the property of linear independent is not essential.

Recall the **double helix structure of gene** for the form (3.1).

The completion H_∞ may be found, in concrete cases, from the realization of the spaces $H_{K_N}(E)$.

The typical case is that the family $\{U_n\}_{n=0}^{\infty}$ is a complete orthonormal system in a Hilbert space with the norm;

$$\|F\| = \sqrt{\int_E |F(x)|^2 dm(x)} \tag{4.1}$$

with a dm measurable set E in the usual form $L_2(E, dm)$. Then, the functions (2.2) and the norm (2.3) are realized by this norm and the completion of the space $H_{K_\infty}(E)$ is given by this Hilbert space with the norm (4.1).

For any separable Hilbert space comprising functions, there exists a complete orthonormal system, and so, by our generalized sense, for the Hilbert space there exists an approximating reproducing kernel Hilbert spaces and so, the Hilbert space is the generalized reproducing kernel Hilbert space in the sense of this paper.

This will mean that we were able to extend the classical reproducing kernels ([2, 3, 4]), beautifully and completely.

The fundamental applications of the theorem to initial value problems using eigenfunctions and reproducing kernels, see [5, 6].

The form (3.1) may be regarded as a **generalized Delta function** and we were able to get its representation property in a good way, by the reproducing property.

5 Remarks

The common fundamental definitions and results on reproducing kernels are given as follows [2, 3, 4]:

Definition: Let E be an arbitrary abstract nonempty set. Denote by $\mathcal{F}(E)$ the set of all complex-valued functions on E. A reproducing kernel Hilbert spaces on the set E is a Hilbert space $\mathcal{H} \subset \mathcal{F}(E)$ coming with a function $K : E \times E \to \mathcal{H}$, which is called the reproducing kernel, having **the reproducing property** that

$$K_p \equiv K(\cdot, p) \in \mathcal{H} \text{ for all } p \in E \tag{5.1}$$

and that

$$f(p) = \langle f, K_p \rangle_{\mathcal{H}} \tag{5.2}$$

holds for all $p \in E$ and all $f \in \mathcal{H}$.

Definition:

A complex-valued function $k : E \times E \to \mathbb{C}$ is called a **positive definite quadratic form function** on the set E, or shortly, **positive definite function**, when it satisfies the property that, for an arbitrary function $X : E \to \mathbb{C}$ and for any finite subset F of E,

$$\sum_{p,q \in F} \overline{X(p)} X(q) k(p,q) \geq 0. \tag{5.3}$$

Then, the fundamental result is given by: **a reproducing kernel and a positive definite quadratic form function are the same and are one to one correspondence** with the reproducing kernel Hilbert space.

Now, a generalized reproducing kernel does not need to be any function, it is the form (3.1).

An open question (a general problem) was proposed by Professor Kenji Fukumizu in the meeting:

Professor Fukumizu proposed to **consider some convergence rate in the theorem in the case of concrete examples**. We can expect to get valuable information from the reference [1] and the problem will be important, because it shows the convergence of reproducing kernels to the Delta function.

Meanwhile, as we see from the talks of Professors Y. Sawano and T. Matuura in the research meeting, we will be able to consider some general version of the theorem in the cases of integral forms and some more general situation.

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