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Some starlikeness conditions concerned with the second coefficient

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Abstract

Let $\mathcal{A}$ be the class of analytic functions $f(z)=z+a_2z^2+\cdots$ in the open unit disk $\mathbb{U}$. Some starlikeness conditions for $f(z)\in\mathcal{A}$ missing the second coefficient a_2 were given by V. Singh (Math. Math. Sci. **23** (2000), 855-857). By considering starlikeness of order α for $f(z)\in\mathcal{A}$ with $a_2\neq 0$, some starlikeness conditions concerned with the second coefficient a_2 are discussed.

1 Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. For a positive integer n, let $\mathcal{A}_n$ be the class of functions $f(z)\in\mathcal{H}$ of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k$$

with $\mathcal{A}_1=\mathcal{A}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{U}$ is denoted by $\mathcal{S}$. In 1972, Ozaki and Nunokawa [4] proved a univalence criterion for $f(z)\in\mathcal{A}$ as follows.

Lemma 1.1 *If $f(z)\in\mathcal{A}$ satisfies*

$$\left|\frac{z^2f'(z)}{(f(z))^2}-1\right|<1 \qquad (z\in\mathbb{U}),$$

then $f(z)$ is univalent in $\mathbb{U}$, which means that $f(z)\in\mathcal{S}$.

Moreover, let $\mathcal{T}_n(\mu)$ denote the class of functions $f(z)\in\mathcal{A}_n$ which satisfy the inequality

$$\left|\frac{z^2f'(z)}{(f(z))^2}-1\right|<\mu \qquad (z\in\mathbb{U})$$

for some real number μ with $0<\mu\leqq 1$ and $\mathcal{T}_n(1)=\mathcal{T}_n$. The assertion in Lemma 1.1 gives us that $\mathcal{T}_n(\mu)\subset\mathcal{T}_n\subset\mathcal{S}$.

A function $f(z)\in\mathcal{A}$ is said to be starlike of order α in $\mathbb{U}$ if it satisfies

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$$\text{(1.1)} \qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some real number α with $0 \leqq \alpha < 1$. This class is denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^*(0) = \mathcal{S}^*$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$.

For a positive integer n, we define by $\mathcal{B}_n$ the class of functions $w(z) \in \mathcal{H}$ of the form

$$w(z) = \sum_{k=n}^{\infty} c_k z^k$$

which satisfy the inequality $|w(z)| < 1 \quad (z \in \mathbb{U})$. The following lemma is well-known as Schwarz's lemma (see [1]).

Lemma 1.2 *If $w(z) \in \mathcal{B}_n$, then*

$$\text{(1.2)} \qquad |w(z)| \leqq |z|^n$$

for each point $z \in \mathbb{U}$. The equality in (1.2) is attended for $w(z) = e^{i\varphi} z^n \quad (\varphi \in \mathbb{R})$.

Applying Lemma 1.2 with $n = 2$, Singh [5] discussed starlikeness for $f(z) \in \mathcal{T}_2(\mu)$.

Lemma 1.3 *If $f(z) \in \mathcal{A}_2$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{1}{\sqrt{2}} \qquad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*$. This means that $\mathcal{T}_2(\mu)$ is a subclass of $\mathcal{S}^*$ for $0 < \mu \leqq \dfrac{1}{\sqrt{2}}$.

Furthermore, Kuroki, Hayami, Uyanik and Owa [3] deduced some sufficient condition for $f(z) \in \mathcal{A}_n$ to be starlike of order α in $\mathbb{U}$.

Lemma 1.4 *If $f(z) \in \mathcal{A}_n$ with $n \neq 1$ satisfies*

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| < \frac{(n-1)(1-\alpha)}{\sqrt{(n-1+\alpha)^2 + (1-\alpha)^2}} \qquad (z \in \mathbb{U})$$

for some real number α with $0 \leqq \alpha < 1$, then $f(z) \in \mathcal{S}^*(\alpha)$.

In view of Lemma 1.3, Singh [5] discussed some starlikeness condition for $f(z) \in \mathcal{A}$ missing the second coefficient a_2. In the present paper, we consider starlikeness of order α for $f(z) \in \mathcal{A}$ with $a_2 \neq 0$.

2 Main result 1

By using a certain method of the proof of Lemma 1.4 which was discussed by Kuroki, Hayami, Uyanik and Owa [3], we deduce some sufficient condition for $f(z) \in \mathcal{A}$ to be starlike of order α in $\mathbb{U}$ (see [2]).

Theorem 2.1 *If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$ *satisfies*

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \frac{(1-\alpha)\sqrt{2(1+\alpha^2) - |a_2|^2} - (1-\alpha+2\alpha^2)|a_2|}{2(1+\alpha^2)} \qquad (z \in \mathbb{U})$$

for some real number α *with* $0 \leqq \alpha < 1$, *then* $f(z) \in \mathcal{S}^*(\alpha)$.

Remark 2.1 If we take $a_2 = 0$ in Theorem 2.1, then we obtain the assertion of Lemma 1.4 with $n = 2$.

Letting $\alpha = 0$ in Theorem 2.1, we obtain

Corollary 2.1 *If* $f(z) = z + a_2 z^2 + \cdots \in \mathcal{A}$ *satisfies*

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \frac{\sqrt{2 - |a_2|^2} - |a_2|}{2} \qquad (z \in \mathbb{U}), \tag{2.1}$$

then $f(z) \in \mathcal{S}^*$.

Example 2.1 Noting that

$$\frac{\sqrt{2 - |a_2|^2} - |a_2|}{2} = \frac{1}{2} \qquad \text{when} \quad a_2 = \frac{\sqrt{3}-1}{2},$$

let us consider the function $f(z)$ given by

$$f(z) = \frac{z}{1 - \frac{\sqrt{3}-1}{2}z - \frac{1}{2}z^2} = z + \frac{\sqrt{3}-1}{2}z^2 + \frac{3-\sqrt{3}}{2}z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{2.2}$$

in Corollary 2.1. It follows from (2.2) that

$$\left| \frac{z^2 f(z)}{\left(f(z)\right)^2} - 1 \right| = \left| \frac{1}{2} z^2 \right| < \frac{1}{2} \qquad (z \in \mathbb{U}).$$

Thus, we find that $f(z)$ given by (2.2) satisfies the inequality (2.1) with $a_2 = \dfrac{\sqrt{3}-1}{2}$. On the other hand, a simple check gives us that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{1 + \frac{1}{2}z^2}{1 - \frac{\sqrt{3}-1}{2}z - \frac{1}{2}z^2}\right)$$

$$> \frac{33 + (5\sqrt{3} - 3)\sqrt{12\sqrt{3} - 6}}{198} = 0.2765\cdots > 0 \qquad (z \in \mathbb{U}).$$

This leads that $f(z)$ given by (2.2) belongs to the class $\mathcal{S}^*$.

Furthermore, putting $\alpha = \frac{1}{2}$ in Theorem 2.1, we have

Corollary 2.2 *If* $f(z) = z + a_2 z^2 + \cdots \in \mathcal{A}$ *satisfies*

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \frac{\sqrt{\frac{5}{2} - |a_2|^2} - 2|a_2|}{5} \qquad (z \in \mathbb{U}), \tag{2.3}$$

then $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

Example 2.2 Noting that

$$\frac{\sqrt{\frac{5}{2} - |a_2|^2} - 2|a_2|}{5} = \frac{1}{10} \qquad \text{when} \quad a_2 = \frac{1}{2},$$

let us consider the function $f(z)$ given by

$$f(z) = \frac{z}{1 - \frac{1}{2}z - \frac{1}{10}z^2} = z + \frac{1}{2}z^2 + \frac{1}{20}z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{2.4}$$

in Corollary 2.2. It is easy to check that

$$\left| \frac{z^2 f(z)}{\left(f(z)\right)^2} - 1 \right| = \left| \frac{1}{10} z^2 \right| < \frac{1}{10} \qquad (z \in \mathbb{U}).$$

Then, we see that $f(z)$ given by (2.4) satisfies the inequality (2.3) with $a_2 = \frac{1}{2}$. Moreover, we can observe that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\left(\frac{1 + \frac{1}{10}z^2}{1 - \frac{1}{2}z - \frac{1}{10}z^2}\right)$$

$$> \frac{10153 + 792\sqrt{71}}{24534} = 0.6858 \cdots > \frac{1}{2} \qquad (z \in \mathbb{U}),$$

which implies that $f(z)$ given by (2.4) belongs to the class $\mathcal{S}^*\left(\frac{1}{2}\right)$.

3 Main result 2

Suppose that $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{T}_1(\mu)$. It is easy to see that

$$\frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 = \left(a_3 - a_2{}^2\right) z^2 + 2\left(a_4 - 2a_2 a_3 + a_2{}^3\right) z^3 + \cdots \qquad (z \in \mathbb{U}).$$

If we define the function $w(z)$ by

$$(3.1)\qquad w(z)=\frac{1}{\mu}\left(\frac{z^2f'(z)}{(f(z))^2}-1\right)\qquad (z\in\mathbb{U}),$$

then since $f(z)\in\mathcal{T}_1(\mu)$, we see that $w(z)\in\mathcal{B}_2$. On the other hand, let us consider the function $f(z)$ given by

$$(3.2)\qquad f(z)=z+\sum_{k=2}^{n}{a_2}^{k-1}z^k+\sum_{k=n+1}^{\infty}a_kz^k\qquad (z\in\mathbb{U}),$$

where n is positive integer with $n\neq 1$. Noting that

$$\frac{f(z)}{z}=\frac{1-(a_2z)^n}{1-a_2z}+\sum_{k=n}^{\infty}a_{k+1}z^k\qquad (z\in\mathbb{U}),$$

we have

$$\frac{z}{f(z)}=\frac{1-a_2z}{1-(a_2z)^n+(1-a_2z)\sum\limits_{k=n}^{\infty}a_{k+1}z^k}$$

$$=\frac{1-a_2z}{1-({a_2}^n-a_{n+1})z^n-\sum\limits_{k=n+1}^{\infty}(a_2a_k-a_{k+1})z^k}$$

$$=\frac{1-a_2z}{1-\sum\limits_{k=n}^{\infty}(a_2a_k-a_{k+1})z^k}\qquad (a_n={a_2}^{n-1})$$

$$=(1-a_2z)+(1-a_2z)\left\{\sum_{k=n}^{\infty}(a_2a_k-a_{k+1})z^k\right\}+(1-a_2z)\left\{\sum_{k=n}^{\infty}(a_2a_k-a_{k+1})z^k\right\}^2+\cdots$$

$$=1-a_2z+({a_2}^n-a_{n+1})z^n+(2a_2a_{n+1}-{a_2}^{n+1}-a_{n+2})z^{n+1}+(2a_2a_{n+2}-{a_2}^2a_{n+1}-a_{a+3})z^{n+2}+\cdots$$

for $z\in\mathbb{U}$. Therefore, we obtain that

$$\frac{z^2f'(z)}{(f(z))^2}-1=\frac{z}{f(z)}-z\left(\frac{z}{f(z)}\right)'-1$$

$$=(n-1)(a_{n+1}-{a_2}^n)z^n+n(a_{n+2}-2a_2a_{n+1}+{a_2}^{n+1})z^{n+1}+\cdots$$

for $z \in \mathbb{U}$. This gives that $w(z)$ defined by (3.1) belongs to the class $\mathcal{B}_n$ if $f(z) \in \mathcal{T}_1(\mu)$. Hence by applying Lemma 1.2, we deduce some sufficient condition for $f(z)$ given by (3.2) to be starlike of order α in $\mathbb{U}$.

Theorem 3.1 *Let n be a positive integer with $n \neq 1$. If $f(z) = z + \sum_{k=2}^{n} {a_2}^{k-1} z^k + \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \mu_n(\alpha) \qquad (z \in \mathbb{U})$$

for some real number α with $0 \leqq \alpha < 1$, where

$$\mu_n(\alpha) = \frac{(n-1)\left[(1-\alpha)\sqrt{A-(n-1)^2|a_2|^2} - \left\{A-(n-1)(n-1+\alpha)\right\}|a_2|\right]}{A}$$

$$\left(A = (1-\alpha)^2 + (n-1+\alpha)^2\right),$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Remark 3.1 If we take $a_2 = 0$ in Theorem 3.1, then we obtain Lemma 1.4 proven by Kuroki, Hayami, Uyanik and Owa [3].

Remark 3.2 Setting $n = 2$ in Theorem 3.1, we find the assertion of Theorem 2.1.

Letting $n = 3$ in Theorem 3.1, we obtain

Corollary 3.1 *If $f(z) = z + a_2 z^2 + {a_2}^2 z^3 + \sum_{k=4}^{\infty} a_k z^k \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \mu_3(\alpha) \qquad (z \in \mathbb{U})$$

for some real number α with $0 \leqq \alpha < 1$, where

$$\mu_3(\alpha) = \frac{2\left[(1-\alpha)\sqrt{(1-\alpha)^2+(2+\alpha)^2-4|a_2|^2} - \left\{(1-\alpha)^2+\alpha(2+\alpha)\right\}|a_2|\right]}{(1-\alpha)^2+(2+\alpha)^2},$$

then $f(z) \in \mathcal{S}^*(\alpha)$.

Furthermore, taking $\alpha = 0$ in Corollary 3.1, we get

Corollary 3.2 *If $f(z) = z + a_2 z^2 + {a_2}^2 z^3 + \sum_{k=4}^{\infty} a_k z^k \in \mathcal{A}$ satisfies*

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \frac{2\sqrt{5-4|a_2|^2} - 2|a_2|}{5} \qquad (z \in \mathbb{U}), \tag{3.3}$$

then $f(z) \in \mathcal{S}^*$.

Example 3.1 Noting that

$$\frac{2\sqrt{5-4|a_2|^2}-2|a_2|}{5} = \frac{3}{5} \qquad \text{when} \quad a_2 = \frac{1}{2},$$

let us consider the function $f(z)$ given by

$$f(z) = \frac{z}{1-\frac{1}{2}z-\frac{3}{10}z^3} = z + \frac{1}{2}z^2 + \left(\frac{1}{2}\right)^2 z^3 + \frac{17}{40}z^4 + \cdots \qquad (z \in \mathbb{U}) \tag{3.4}$$

in Corollary 3.2. It follows from (3.4) that

$$\left|\frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1\right| = \left|\frac{3}{5}z^3\right| < \frac{3}{5} \qquad (z \in \mathbb{U}).$$

Thus, we find that $f(z)$ given by (3.4) satisfies the inequality (3.3) with $a_2 = \dfrac{1}{2}$. On the other hand, a simple check gives us that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) = \mathrm{Re}\left(\frac{1+\frac{3}{5}z^3}{1-\frac{1}{2}z-\frac{3}{10}z^3}\right) > \frac{2}{9} > 0 \qquad (z \in \mathbb{U}).$$

This leads that $f(z)$ given by (3.4) belongs to the class $\mathcal{S}^*$.

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