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A NOTE ON MULTIVALENT FUNCTIONS

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ABSTRACT. The Noshiro-Warschawski Theorem [1], [5] provides a simple and useful sufficient condition: $\mathfrak{Re}\{f'(z)) > 0$ in D, for the univalence of analytic function $f(z)$ in a convex domain D. In this paper we prove some related results to this theorem. The applications of main results are also presented.

1. INTRODUCTION

The Noshiro-Warschawski Theorem [1], [5] can be used to give a simple and useful condition that is sufficient for the univalence of function $f(z)$ which is analytic in a convex domain D and satisfies the condition $\mathfrak{Re}\{f'(z)\} > 0$ in D. Ozaki [6] extended the above result to the following: if $f(z)$ is analytic in a convex domain D and

$$\mathfrak{Re}\left\{f^{(p)}(z)\right\} > 0 \quad \text{in} \quad D,$$

then $f(z)$ is at most p-valent in D.
Furthermore, Nunokawa [2] has shown the following result.

Theorem 1.1. *Let $p \geq 2$. If*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

is analytic in $|z| < 1$ and

$$\left|\arg\left\{f^{(p)}(z)\right\}\right| < \frac{3\pi}{4}, \tag{1.2}$$

then $f(z)$ is p-valent in $|z| < 1$.

It is the purpose of the present paper to improve Theorem 1.1.

2. MAIN RESULTS

Theorem 2.1. *If $p \geq 2$ and*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad in \quad |z| < 1$$

satisfies the condition

$$\left|\arg\left\{f^{(p)}(z)\right\}\right| < \frac{\alpha\pi}{2} \quad in \quad |z| < 1, \tag{2.1}$$

where $\alpha = 1/\beta_0 = 1.7897771\ldots$,

$$\beta_0 = 1 - \frac{\log 4}{\pi} = \frac{2}{\pi}\int_0^1 \sin^{-1}\frac{2\varrho}{1+\varrho^2}\,\mathrm{d}\varrho,$$

then $f(z)$ is $p-$valent in $|z| < 1$.

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Proof. Let us put

$$q(z) = \frac{1}{p!\,z} f^{(p-1)}(z), \quad q(0) = 1.$$

Then it follows that

$$q(z) + zq'(z) = \frac{f^{(p)}(z)}{p!}$$

and then, from hypothesis (2.1), we have

$$|\arg\{q(z) + zq'(z)\}| < \frac{\alpha\pi}{2}. \tag{2.2}$$

Following the same idea as in the proof of the main theorem in [4, p. 1292–1293] we obtain

$$\begin{aligned} q(z) &= \frac{zq(z)}{z} \\ &= \frac{zf^{(p-1)}(z)}{p!\ z} \\ &= \frac{1}{p!\ re^{i\theta}} \int_0^z (q(t) + tq'(t))\ \mathrm{d}t \\ &= \frac{1}{p!\ re^{i\theta}} \int_0^r \left(q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\right) e^{i\theta}\ \mathrm{d}\varrho. \end{aligned}$$

Applying (2.2) and the principle of subordination, we get that $\left[q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\right]^{1/\alpha}$ lies in the disc $Q(|z| < \varrho)$ for all $0 < \varrho < 1$, $0 \le \theta < 2\pi$, where $Q(z) = (1+z)/(1-z)$. Because $Q(|z| < \varrho)$ is the disc with the center $(1+\varrho^2)/(1-\varrho^2)$ and the radius $2\varrho/(1-\varrho^2)$, by some geometric observation, we can see that $\left[q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\right]^{1/\alpha}$ lies in the sector $|\arg\{w\}| < \gamma$, where $\gamma = \sin^{-1}\{(2\varrho)/(1+\varrho^2)\}$. Thus, we have

$$\begin{aligned} &|\arg\{q(z)\}| \\ &= \left|\arg\left\{\frac{1}{p!\ re^{i\theta}} \int_0^r \left(e^{i\theta} q(\varrho e^{i\theta}) + \varrho e^{2i\theta} q'(\varrho e^{i\theta})\right)\ \mathrm{d}\varrho\right\}\right| \\ &\le \int_0^r \left|\arg\left\{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\right\}\right|\ \mathrm{d}\varrho \\ &\le \alpha \int_0^r \left|\arg\left\{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\right\}^{1/\alpha}\right|\ \mathrm{d}\varrho \\ &\le \alpha \int_0^1 \sin^{-1} \frac{2\varrho}{1+\varrho^2}\ \mathrm{d}\varrho \\ &= \alpha(\sin^{-1} 1 - \log 2) \\ &= \alpha(\pi/2 - \log 2) \\ &= \frac{\pi}{2}\alpha\beta_0 = \frac{\pi}{2}. \end{aligned}$$

Hence, we have

$$\mathfrak{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} = \mathfrak{Re}\left\{\frac{zf^{(p-1)}(z)}{z^2}\right\} > 0$$

in $|z| < 1$. For the same reason as in the proof of the main theorem in [2, p. 454], we obtain that $f(z)$ is p-valent in $|z| < 1$. □

Theorem 2.2. *If $q(z)$ is analytic in $|z| < 1$, with $q(0) = 1$ and satisfies there the condition*

$$|\arg\{q(z) + zq'(z) - \beta\}| < \frac{\alpha\pi}{2}, \tag{2.3}$$

where $0 < \alpha$ *and* $0 < \beta < 1$, *then we have*

$$|\arg\{q(z)\}| < \alpha\left(\frac{\pi}{2} - \log 2\right) \quad in \quad |z| < 1.$$

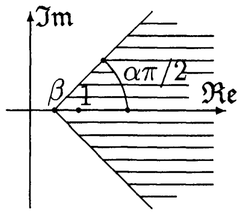

Fig.1.

Proof. We have

$$zq(z) = \int_0^z (tq(t))' \, \mathrm{d}t$$

and so

$$\begin{aligned} q(z) &= \frac{1}{re^{i\theta}} \int_0^r \left(\varrho e^{i\theta} q(\varrho e^{i\theta})\right)' e^{i\theta} \mathrm{d}\varrho \\ &= \frac{1}{r} \int_0^r \left(\varrho e^{i\theta} q(\varrho e^{i\theta})\right)' \mathrm{d}\varrho. \end{aligned}$$

Following the same idea as in the proof of the main theorem in [4, p. 1292–1293] we have

$$\begin{aligned} &|\arg\{q(z)\}| \\ &= \left|\arg\left\{\frac{1}{r}\int_0^r \left(q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\right) \, \mathrm{d}\varrho\right\}\right| \\ &\le \int_0^r \left|\arg\{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\}\right| \, \mathrm{d}\varrho \\ &\le \int_0^r \left|\arg\{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\} - \beta + \beta\right| \, \mathrm{d}\varrho \\ &\le \int_0^r \left|\arg\{q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})\} - \beta\right| \, \mathrm{d}\varrho \\ &+ \int_0^r |\arg \beta| \, \mathrm{d}\varrho. \end{aligned}$$

Applying (2.3) and the principle of subordination we get that $q(\varrho e^{i\theta}) + \varrho e^{i\theta} q'(\varrho e^{i\theta})$ lies in the sector $\beta + Q^{\alpha}(|z| < \varrho)$ for all $0 < \varrho < 1$, $0 \le \theta < 2\pi$, where $Q(z) = (1+z)/(1-z)$, Fig. 1. Therefore, and because $Q(|z| < \varrho)$ is the disc with the center $(1+\varrho^2)/(1-\varrho^2)$

and the radius $2\varrho/(1-\varrho^2)$, by some geometric observation we obtain

$$\begin{aligned}
&|\arg\{q(z)\}|\\
&\leq \int_0^r \left|\arg\left\{\left(\frac{1+\varrho e^{i\theta}}{1-\varrho e^{i\theta}}\right)^{\alpha}\right\}\right| \, \mathrm{d}\varrho\\
&\leq \alpha\int_0^r \left|\arg\left\{\left(\frac{1+\varrho e^{i\theta}}{1-\varrho e^{i\theta}}\right)\right\}\right| \, \mathrm{d}\varrho\\
&\leq \alpha\int_0^r \sin^{-1}\frac{2\varrho}{1+\varrho^2}\, \mathrm{d}\varrho\\
&\leq \alpha\int_0^1 \sin^{-1}\frac{2\varrho}{1+\varrho^2}\, \mathrm{d}\varrho\\
&= \alpha(\pi/2-\log 2).
\end{aligned}$$

This completes the proof. □

Remark 2.3.

$$\begin{aligned}
&\int_0^1 \sin^{-1}\frac{2\varrho}{1+\varrho^2}\, \mathrm{d}\varrho\\
&= \left[\varrho\sin^{-1}\frac{2\varrho}{1+\varrho^2} - \log(1+\varrho^2)\right]_{\varrho=0}^{\varrho=1}\\
&= \pi/2-\log 2\\
&= 0.877649147\ldots.
\end{aligned}$$

Applying the same method as in the proof of Theorem 2.2, we can get the following corollaries.

Corollary 2.4. *If $q(z)$ is analytic in $|z|<1$, with $q(0)=1$ and satisfies the condition*

$$|\arg\{q(z)+zq'(z)-\beta\}| < \frac{\alpha\pi}{2} \quad in \quad |z|<1,$$

where $0<\alpha$ and $0<\beta<1$, then we have

$$|\arg\{q(z)-\beta\}| < \alpha\left(\frac{\pi}{2}-\log 2\right) \quad in \quad |z|<1.$$

Corollary 2.5. *If $q(z)$ is analytic in $|z|<1$, with $q(0)=1$ and satisfies the condition*

$$|\arg\{q(z)+zq'(z)-\beta\}| < \frac{\alpha\pi}{2} \quad in \quad |z|<1,$$

where $0<\alpha$ and $\beta\leq 0$, then we have

$$|\arg\{q(z)-\beta\}| < \alpha\left(\frac{\pi}{2}-\log 2\right) \quad in \quad |z|<1,$$

Corollary 2.6. *If $q(z)$ is analytic in $|z|<1$, with $q(0)=1$ and satisfies the condition*

$$|\arg\{q(z)+zq'(z)-\beta\}| < \frac{\pi^2}{2(\pi-\log 4)}$$

for $|z|<1$, then

$$\mathfrak{Re}\{q(z)\} > \beta \quad in \quad |z|<1.$$

From Corollary 2.5 we can get the following improvement of the main theorem in [3].

Theorem 2.7. *If $p\geq 2$ and*

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad in \quad |z|<1$$

satisfies the condition

$$\left|\arg\left\{\frac{f^{(p)}(z)}{p!}+\frac{\log 4-1}{2-\log 2}\right\}\right| < \frac{\pi^2}{2(\pi-\log 4)} \quad in \quad |z|<1,$$

then $f(z)$ *is p-valent in* $|z|<1$.

Proof. Let us put

$$q(z)=\frac{f^{(p-1)}(z)}{p!\ z}, \quad q(0)=1.$$

Then it follows that

$$q(z)+zq'(z)=\frac{f^{(p)}(z)}{p!}$$

and therefore, we have

$$\left|\arg\left\{q(z)+zq'(z)+\frac{\log 4-1}{2-\log 2}\right\}\right| = \left|\arg\left\{\frac{f^{(p)}(z)}{p!}+\frac{\log 4-1}{2-\log 2}\right\}\right| < \frac{\pi^2}{2(\pi-\log 4)} \quad \text{in} \quad |z|<1.$$

Taking into account Corollary 2.5, we have

$$\Re\left\{\frac{f^{(p-1)}(z)}{z}\right\}>0 \quad \text{in} \quad |z|<1$$
$$\Leftrightarrow \Re\left\{\frac{zf^{(p-1)}(z)}{z^2}\right\}>0 \quad \text{in} \quad |z|<1.$$

For the same reason as in the proof of the main theorem in [3], we get that $f(z)$ is p-valent in $|z|<1$. □

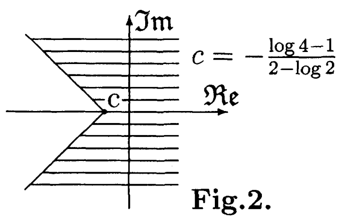

Fig.2.

Remark 2.8. *We have*

$$\frac{\log 4-1}{2-\log 2}=0.29\ldots, \quad \frac{\pi^2}{2(\pi-\log 4)}=\pi\cdot 0.89\ldots.$$

Theorem 2.7 shows that if the image of $|z|<1$ *under the mapping* $w=f^{(p)}(z)/p!$ *is contained in the indicated domain on the Fig.2, then* $f(z)$ *is p-valent in* $|z|<1$*, whenever* $p\geq 2$.

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