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Asymptotic Behavior of Solutions for Semilinear Volterra Diffusion Equations with Spatial Inhomogeneity

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1 Introduction

In this paper we consider the following logistic diffusion equation with spatially inhomogeneous coefficients and continuously delay term:

$$\text{(P)} \quad \begin{cases} u_t = \mathrm{div}\{d(x)\nabla u\} + u\{a(x) - b(x)u - c(x)k * u(t)\} & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, a, b and c are functions of class $L^\infty(\Omega)$ with $b \geq 0$ and $c \geq 0$ in Ω and

$$k * u(t) := \int_0^t k(t-s)u(s)ds.$$

A diffusion coefficient d is a positive function of class $C^{1+\alpha}(\overline{\Omega})$ with $\alpha \in (0,1)$. Boundary operator B represents the following boundary condition

$$Bu = u \quad \text{or} \quad Bu = \partial u/\partial n + \beta(x)u,$$

where $\partial/\partial n$ denotes the exterior normal derivative to $\partial\Omega$ and β is a nonnegative function of class $C^{1+\alpha}(\partial\Omega)$. Moreover, k is assumed to be a nonnegative function of class $C^1(0,\infty)\cap L^1(0,\infty)$ satisfying

$$\int_0^\infty k(t)dt = 1. \tag{1.1}$$

Our problem (P) appears in ecology and u denotes the population density of a biological species. Throughout this paper, we always assume

(A.1) u_0 is a nonnegative (not identically zero) function of class $L^\infty(\Omega)$,

(A.2) $\inf_{x\in\Omega}\{b(x)+c(x)\} > 0$.

If $c \equiv 0$, then (P) is an initial boundary value problem for a spatially inhomogeneous logistic diffusion equation. In this case, the dynamics of solutions of (P) is well known (see Cantrell-Cosner [3, 4, 5]). However, it is more realistic to take account of the past information in the study of population biology. The term $k * u(t)$ is sometimes called a hereditary term and describes effects from the past to the present. Naturally, the following two functions k are typical delay kernels in mathematical biology:

(K.1) $\qquad k(t) = (1/T)e^{-t/T},$

(K.2) $\qquad k(t) = (t/T^2)e^{-t/T}.$

Here, (K.1) and (K.2) are called a weak delay kernel and a strong delay kernel, respectively. For instance, they appear in the bacteria model (for details, see Iida [8]).

Our main purpose is to study

(P.1) Existence and uniqueness of global solutions of (P),

(P.2) Asymptotic behavior of solutions as $t \to \infty$,

(P.3) Existence, uniqueness and stability of positive stationary solutions.

When a, b, c and d are constants, (P.1)-(P.3) are studied by many authors and lots of results are obtained (see e.g. [8, 10, 14, 17, 18] with homogeneous Neumann boundary condition and [15, 20] with homogeneous Dirichlet boundary condition). In addition, some systems of Volterra diffusion equations have also been studied in [1, 19]. However, there are few results for (P) under inhomogeneous environment. So our main purpose is to study Volterra diffusion equations with spatial inhomogeneity. We have to develop some devices and tools to discuss the spatial inhomogeneity. Details for our arguments will be found in the paper of Yoshida-Yamada [21].

The plan of this paper is as follows. In Section 2, we will introduce our results. They are concerned with (P.1)-(P.3) and main results are Theorems 2.4, 2.5 and 2.7. Consider the following eigenvalue problem:

$$\text{(EP)} \qquad \begin{cases} -\operatorname{div}\{d(x)\nabla\psi\} - a(x)\psi = \lambda\psi & \text{in } \Omega, \\ B\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $\lambda_1 \equiv \lambda_1(a, d)$ denote the principal eigenvalue of (EP). For the proof of some theorems, the sign of λ_1 is important. So in Section 3, we will discuss some sufficient conditions for $\lambda_1 < 0$. In Sections 4, 5 and 6, we will prove Theorems 2.4, 2.5 and 2.7, respectively.

Notation

For $p \in [1, \infty]$, $L^p(\Omega)$ denotes the Banach space of measurable functions u in Ω with norm

$$\|u\|_{p,\Omega} := \left\{ \int_\Omega |u(x)|^p dx \right\}^{1/p} < \infty \quad \text{if } p \in [1, \infty),$$
$$\|u\|_{\infty,\Omega} := \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < \infty \qquad \text{if } p = \infty.$$

If there is no confusion, then we will omit the subscript Ω. For each $p \in [1, \infty)$ and integer $k \in [1, \infty)$, $W^{k,p}(\Omega)$ denotes the usual Sobolev space of measurable functions u in Ω such that u and its distributional derivatives up to order k belong to $L^p(\Omega)$. Its norm is defined by

$$\|u\|_{k,p,\Omega} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_p^p \right)^{1/p},$$

where α denotes a multi-index for derivatives. If there is no confusion, then we will also omit the subscript Ω. We sometimes write $H^k(\Omega)$ instead of $W^{k,2}(\Omega)$. Moreover, $W_0^{k,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable functions in Ω with compact support in Ω. In the same way as $H^k(\Omega)$, we sometimes write $H_0^k(\Omega)$ instead of $W_0^{k,2}(\Omega)$.

Let I be any subinterval of $[0,\infty)$ and let X be any Banach space. Denote by $C(I;X)$ the space of X-valued strongly continuous functions in I. For any positive integer j, $C^j(I;X)$ denotes the space of functions u of class $C(I;X)$ such that u is j-times strongly continuously differentiable in I.

2 Main results

Let $p > 1$ be fixed. Define a closed, linear and elliptic operator A with dense domain $D(A)$ by

$$Au = -\operatorname{div}\{d(x)\nabla u\}$$

and

$$D(A) = \begin{cases} W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega) & \text{if } Bu = u, \\ \{v \in W^{2,p}(\Omega) \mid Bu = 0 \text{ on } \partial\Omega\} & \text{if } Bu = \partial u/\partial n + \beta(x)u. \end{cases}$$

For each $\mu \in [0,1]$, we introduce the fractional power spaces $D(A^\mu)$ equipped with the graph norm of A^μ in the standard manner. If $p > \max\{1, N/2\}$, then

$$D(A^\mu) \subset C^\nu(\overline{\Omega}) \quad \text{with } \nu \in [0, 2\mu - (N/p)). \tag{2.1}$$

For the proof of (2.1), see Henry [6] or Pazy [9]. It is well known that $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ in $L^p(\Omega)$. Then we can establish the global existence theorem.

Theorem 2.1. *Let $p > \max\{1, N/2\}$. Then (P) has a unique solution u in the class*

$$u \in C([0,\infty); L^p(\Omega)) \cap C^1((0,\infty); L^p(\Omega)) \cap C((0,\infty); D(A));$$

which satisfies

$$u > 0 \quad in \quad \Omega \times (0,\infty) \quad and \quad \partial u/\partial n < 0 \quad on \quad \partial\Omega \times (0,\infty)$$

if $Bu = u$, and

$$u > 0 \quad in \quad \overline{\Omega} \times (0,\infty)$$

if $Bu = \partial u/\partial n + \beta(x)u$. Moreover, if $\inf_{x\in\Omega} b(x) > 0$, then

$$u \leq m \quad in \ \Omega \times (0,\infty), \tag{2.2}$$

where $m = \max\{\|u_0\|_\infty, \sup_{x\in\Omega}\{a(x)/b(x)\}\}$.

This theorem can be proved in the standard manner. For details, see for instance [7] and [18].

By a stationary solution of (P) we mean any solution of

$$\text{(SP)} \qquad \begin{cases} \operatorname{div}\{d(x)\nabla\varphi\} + \varphi\,[a(x) - \{b(x)+c(x)\}\varphi] = 0 & \text{in } \Omega, \\ B\varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

(note (1.1)). We will look for positive solutions of (SP).

Recall (EP). Then λ_1 is given by the following variational characterization (see [5, Chapter 2]):

$$\lambda_1 = \inf_{\substack{\psi \in H^1(\Omega) \\ \|\psi\|_2 = 1}} \left\{ \int_\Omega d(x)|\nabla\psi|^2 dx + \int_{\partial\Omega} d(x)\beta(x)\psi^2 d\sigma - \int_\Omega a(x)\psi^2 dx \right\}$$

if $B\psi = \partial\psi/\partial n + \beta(x)\psi$, where σ denotes a surface element, while

$$\lambda_1 = \inf_{\substack{\psi \in H_0^1(\Omega) \\ \|\psi\|_2 = 1}} \left\{ \int_\Omega d(x)|\nabla\psi|^2 dx - \int_\Omega a(x)\psi^2 dx \right\}$$

if $B\psi = \psi$.

Then we can obtain the existence and uniqueness of a positive solution of (SP).

Theorem 2.2. *Problem* (SP) *has a positive solution φ if and only if $\lambda_1 < 0$, where λ_1 is the principal eigenvalue of* (EP). *Moreover, when φ exists, it is uniquely determined and it satisfies*

$$0 < \varphi \le M \quad \text{in } \Omega \tag{2.3}$$

and

$$\begin{cases} \partial\varphi/\partial n < 0 & \text{on } \partial\Omega \quad \text{if } B\varphi = \varphi, \\ 0 < \varphi \le M & \text{on } \partial\Omega \quad \text{if } B\varphi = \partial\varphi/\partial n + \beta(x)\varphi, \end{cases}$$

where $M = \sup_{x\in\Omega}\{a(x)/\{b(x) + c(x)\}\}$.

Remark 2.1. Since $\lambda_1 < 0$ requires $\sup_{x\in\Omega} a(x) > 0$, M is a positive number.

Theorem 2.2 can be proved as an application of the monotone method (see Sattinger [13, Theorem 2.1]).

We can show the following result on the asymptotic behavior of solutions for (P) in the case of $\lambda_1 \ge 0$, namely the case of nonexistence of positive stationary solution:

Theorem 2.3. *Assume*

$$\inf_{x\in\Omega} b(x) > 0 \quad \text{and} \quad \lambda_1 \ge 0 \quad \text{or} \quad \inf_{x\in\Omega} b(x) = 0 \quad \text{and} \quad \lambda_1 > 0. \tag{2.4}$$

Then every solution u of (P) *satisfies*

$$\lim_{t\to\infty} u(t) = 0 \quad \text{uniformly in } \overline{\Omega}.$$

Proof. Since c and k are nonnegative, the positivity of u implies

$$u_t \le \operatorname{div}\{d(x)\nabla u\} + u\{a(x) - b(x)u\}.$$

Consider the following problem:

$$\begin{cases} v_t = \operatorname{div}\{d(x)\nabla v\} + v\{a(x) - b(x)v\} & \text{in } \Omega \times (0,\infty), \\ Bv = 0 & \text{on } \partial\Omega \times (0,\infty), \\ v(\cdot,0) = \|u_0\|_\infty & \text{in } \Omega. \end{cases}$$

Owing to (2.4), the theory of dynamical systems shows

$$\lim_{t\to\infty} v(t) = 0 \quad \text{uniformly in } \overline{\Omega}.$$

Since the comparison theorem (see e.g. Smoller [16]) shows $u \le v$, the conclusion easily follows. □

In what follows, we will discuss the case $\lambda_1 < 0$, which assesses that there exists a unique positive stationary solution φ of (SP). First, we will consider the case $\inf_{x\in\Omega} b(x) > 0$. Denote by $\hat{k}$ the Laplace transform of k:

$$\hat{k}(\lambda) = \int_0^\infty e^{-\lambda t} k(t)dt.$$

Then we can prove the global attractivity of φ of (SP).

Theorem 2.4. *Assume* $\inf_{x\in\Omega} b(x) > 0$, $\lambda_1 < 0$ *and* $tk \in L^1(0,\infty)$. *Furthermore, assume that there exists a positive constant* k_0 *such that*

$$b(x) + \mathrm{Re}\,\hat{k}(i\eta)c(x) \geq k_0 \quad \text{for } x \in \Omega \text{ and } \eta \in \mathbb{R}. \tag{2.5}$$

Then every solution u *of* (P) *satisfies*

$$\lim_{t\to\infty} u(t) = \varphi \quad \text{uniformly in } \overline{\Omega}. \tag{2.6}$$

Recall special kernels (K.1) and (K.2). Then both kernels satisfy $tk \in L^1(0,\infty)$. Moreover, for (K.1),

$$\begin{aligned} \inf_{\eta\in\mathbb{R}} \mathrm{Re}\,\hat{k}(i\eta) &= \inf_{\eta\in\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+i\eta T}\right) \\ &= 0, \end{aligned} \tag{2.7}$$

and for (K.2),

$$\begin{aligned} \inf_{\eta\in\mathbb{R}} \mathrm{Re}\,\hat{k}(i\eta) &= \inf_{\eta\in\mathbb{R}} \mathrm{Re}\left(\frac{1}{1+i\eta T}\right)^2 \\ &= -\frac{1}{8}. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), φ is always globally attractive for (K.1), while for (K.2), it is globally attractive if

$$\inf_{x\in\Omega}\left\{b(x) - \frac{c(x)}{8}\right\} > 0. \tag{2.9}$$

When we consider (P) with spatially homogeneous coefficients and homogeneous Neumann boundary condition, it follows from [18] that, if k is given by (K.2), then φ loses its stability and that the Hopf bifurcation occurs. However, there is no result on the Hopf bifurcation in other cases.

We can also consider (P) for the case $\inf_{x\in\Omega} b(x) = 0$. One of the difficulties of this case is to derive a priori estimate of u.

Theorem 2.5. *Let* $\inf_{x\in\Omega} b(x) = 0$, $\lambda_1 < 0$ *and* $tk \in L^1(0,\infty)$. *Assume* $\mathrm{Re}\,\hat{k}(i\eta) \geq 0$ *for* $\eta \in \mathbb{R}$. *Then every solution* u *of* (P) *satisfies*

$$\sup_{t\in[0,\infty)} \|u(t)\|_\infty < \infty.$$

Repeating the proof of Theorem 2.4, we can also obtain the following result:

Theorem 2.6. *In addition to the assumptions of Theorem 2.5, assume* (2.5). *Then every solution u of* (P) *satisfies*

$$\lim_{t\to\infty} u(t) = \varphi \quad \textit{uniformly in } \overline{\Omega}.$$

Recall that if k is defined by (K.1) (resp. (K.2)), it satisfies (2.7) (resp. (2.8)). Then both of (K.1) and (K.2) cannot satisfy (2.5). This implies that Theorem 2.6 is inconvenient from the viewpoint of the application. By putting additional assumptions, we can improve Theorem 2.6 as follows.

Theorem 2.7. *In addition to the assumptions of Theorem 2.5, assume $k(0) \neq 0$ and $k'(= dk/dt) \in L^1(0,\infty)$. Furthermore, assume that there exist positive constants c_0 and k_1 such that $c(x) \geq c_0$ for $x \in \Omega$ and*

$$\mathrm{Re}\left\{\hat{k}(i\eta)\right\}^{-1} \geq k_1 \quad \textit{for } \eta \in \mathbb{R}. \tag{2.10}$$

Then every solution u of (P) *satisfies*

$$\lim_{t\to\infty} v(t) = \varphi \quad \textit{uniformly in } \overline{\Omega}.$$

3 Sufficient conditions for $\lambda_1 < 0$

In this section, we will search some sufficient conditions for $\lambda_1 < 0$. Set

$$\Omega_0 = \{x \in \Omega \mid a(x) > 0\}, \tag{3.1}$$

and always assume $\Omega_0 \neq \emptyset$ in this section. Consider the following eigenvalue problem:

$$\begin{cases} -\operatorname{div}\{d(x)\nabla\rho\} = \mu a(x)\rho & \text{in } \Omega, \\ B\rho = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $\mu_1^+ \equiv \mu_1^+(a,d)$ denote the positive principal eigenvalue of (3.2). It is given by the following variational characterization (see e.g. [5]):

$$\frac{1}{\mu_1^+} = \sup_{\substack{\rho\in H^1(\Omega)\\ \rho\neq 0}} \frac{\displaystyle\int_\Omega a(x)\rho^2 dx}{\displaystyle\int_\Omega d(x)|\nabla\rho|^2 dx + \int_{\partial\Omega} d(x)\beta(x)\rho^2 d\sigma}$$

if $B\psi = \partial\psi/\partial n + \beta(x)\psi$, while

$$\frac{1}{\mu_1^+} = \sup_{\substack{\rho\in H_0^1(\Omega)\\ \rho\neq 0}} \frac{\displaystyle\int_\Omega a(x)\rho^2 dx}{\displaystyle\int_\Omega d(x)|\nabla\rho|^2 dx} \tag{3.3}$$

if $B\psi = \psi$. Note that if $B\rho = \partial\rho/\partial n$, then μ_1^+ exists if and only if

$$\int_\Omega a(x)dx < 0. \tag{3.4}$$

For the proof of (3.4), see, e.g., [5]. The relation between λ_1 and μ_1^+ is given by the following proposition (see [5, Theorem 2.6]):

Proposition 3.1. *Let* λ_1 *be the principal eigenvalue of* (EP) *and let* μ_1^+ *be the positive principal eigenvalue of* (3.2).

(i) *If* $B\psi = \psi$ *or* $B\psi = \partial\psi/\partial n + \beta(x)\psi$ $(\beta \not\equiv 0)$, *then* $\lambda_1 < 0$ *if and only if* $\mu_1^+ < 1$.

(ii) *If* $B\psi = \partial\psi/\partial n$ *with* (3.4), *then* $\lambda_1 < 0$ *if and only if* $\mu_1^+ < 1$. *If* $B\psi = \partial\psi/\partial n$ *with*

$$\int_\Omega a(x)dx > 0, \tag{3.5}$$

then $\lambda_1 < 0$.

Then the following result is obtained:

Proposition 3.2. *Define* Ω_0 *by* (3.1). *Let* $B\psi = \psi$ *or* $B\psi = \partial\psi/\partial n$ *with* (3.4) *or* $B\psi = \partial\psi/\partial n + \beta(x)\psi$ $(\beta \not\equiv 0)$. *Then there exists a positive constant* d^* *such that* $\lambda_1 < 0$ *for any* d *satisfying* $\|d\|_{\infty,\Omega_0} < d^*$.

Proof. We will only discuss the case $B\psi = \psi$. The other cases can be handled similarly. Take any connected set $\Omega_0^* \subset \Omega_0$. Take any function $\rho \in H_0^1(\Omega_0^*)$ and let $\tilde{\rho} : \Omega_0^* \to \mathbb{R}$ be the natural extension of ρ. Observing (3.3), we can estimate

$$\begin{aligned} \frac{1}{\mu_1^+} &\geq \sup_{\substack{\rho\in H_0^1(\Omega_0^*)\\ \rho\neq 0}} \frac{\int_\Omega a(x)\tilde{\rho}^2 dx}{\int_\Omega d(x)|\nabla\tilde{\rho}|^2 dx} \\ &\geq \|d\|_{\infty,\Omega_0^*}^{-1} \sup_{\substack{\rho\in H_0^1(\Omega_0^*)\\ \rho\neq 0}} \frac{\int_{\Omega_0^*} a(x)\rho^2 dx}{\int_{\Omega_0^*} |\nabla\rho|^2 dx}. \end{aligned}$$

Choose d^* as

$$d^* = \sup_{\substack{\rho\in H_0^1(\Omega_0^*)\\ \rho\neq 0}} \frac{\int_{\Omega_0^*} a(x)\rho^2 dx}{\int_{\Omega_0^*} |\nabla\rho|^2 dx}.$$

Then $\mu_1^+ < 1$ for any d satisfying $\|d\|_{\infty,\Omega_0^*} < d^*$. Therefore, Proposition 3.1 yields the conclusion. □

Propositions 3.1 and 3.2 imply that a positive stationary solution exists if a diffusion coefficients in a favorable habitat Ω_0 is sufficiently small or (3.5) is achieved with homogeneous Neumann boundary condition. In ecology, this fact asserts that there is a chance for a species to survive if the species stays in a favorable habitat Ω_0.

4 Proof of Theorem 2.4

For the proof of Theorem 2.4, we will follow the argument used by Yamada [20]. Let φ be a positive solution of (SP). We introduce the following nonnegative functional:

$$\begin{aligned} E(u) &= \int_\Omega \varphi^2(x) g(u(x)/\varphi(x))dx \\ &= \int_\Omega \varphi(x)\left\{u(x) - \varphi(x) - \varphi(x)\log\frac{u(x)}{\varphi(x)}\right\}dx, \end{aligned} \tag{4.1}$$

where

$$g(u) = u - 1 - \log u. \tag{4.2}$$

This functional has also been used in [1].

Lemma 4.1 (cf. [20, Lemma 3.1]). *Define* $E(u)$ *by* (4.1). *Then any solution* u *of* (P) *satisfies*

$$\begin{aligned} \frac{d}{dt}E(u(t)) = &- \int_\Omega d(x)\varphi^2\left|\nabla\left\{\log\frac{u(t)}{\varphi}\right\}\right|^2 dx - \int_\Omega b(x)\varphi\{u(t)-\varphi\}^2 dx \\ &- \int_\Omega c(x)\varphi\{u(t)-\varphi\}k * (u-\varphi)(t)dx \\ &+ \int_t^\infty k(s)ds \int_\Omega c(x)\varphi^2\{u(t)-\varphi\}dx. \end{aligned} \tag{4.3}$$

Proof. We will only prove the case $Bu = u$. The other cases can be proved similarly. Differentiation of (4.1) with respect to t yields

$$\begin{aligned} \frac{d}{dt}E(u(t)) &= \int_\Omega \left\{1 - \frac{\varphi}{u(t)}\right\} u_t(t)\varphi dx \\ &= \int_\Omega \varphi\left\{1 - \frac{\varphi}{u(t)}\right\} \operatorname{div}\{d(x)\nabla u(t)\}dx \\ &\quad + \int_\Omega \varphi\{u(t)-\varphi\}\{a(x) - b(x)u(t) - c(x)k * u(t)\}dx. \end{aligned}$$

In view of (1.1),

$$\begin{aligned} a(x) - b(x)u - c(x)k * u(t) = a(x) &- \{b(x) + c(x)\}\varphi - b(x)(u-\varphi) \\ &- c(x)k * (u-\varphi)(t) + c(x)\varphi\int_t^\infty k(s)ds. \end{aligned}$$

Since φ is a solution of (SP), it follows that

$$\begin{aligned} \frac{d}{dt}E(u(t)) = &\int_\Omega \varphi\left\{1 - \frac{\varphi}{u(t)}\right\}\operatorname{div}\{d(x)\nabla u(t)\}dx - \int_\Omega \{u(t)-\varphi\}\operatorname{div}\{d(x)\nabla\varphi\}dx \\ &- \int_\Omega b(x)\varphi\{u(t)-\varphi\}^2 dx - \int_\Omega c(x)\varphi\{u(t)-\varphi\}k * (u-\varphi)(t)dx \\ &+ \int_t^\infty k(s)ds\int_\Omega c(x)\varphi^2\{u(t)-\varphi\}dx. \end{aligned}$$

Moreover, we also obtain from the integration by parts

$$\int_\Omega \varphi\left\{1-\frac{\varphi}{u(t)}\right\}\operatorname{div}\{d(x)\nabla u(t)\}\,dx-\int_\Omega\{u(t)-\varphi\}\operatorname{div}\{d(x)\nabla\varphi\}\,dx$$
$$=-\int_\Omega d(x)\left\{|\nabla\varphi|^2-\frac{2\varphi}{u(t)}\nabla\varphi\cdot\nabla u(t)-\frac{\varphi^2}{u^2(t)}|\nabla u(t)|^2\right\}dx$$
$$=-\int_\Omega d(x)\varphi^2\left|\nabla\left\{\log\frac{u(t)}{\varphi}\right\}\right|^2dx.$$

Therefore, (4.3) follows. □

Then we are ready to follow the argument in [20]. We will also prepare some regularity results.

Lemma 4.2. *Let u be a bounded solution of* (P) *and let δ be any positive number. Then there exist positive constants K_1, K_2 and K_3, independent of t, such that for $p>1$, $t\in[\delta,\infty)$ and $\mu\in[0,1)$,*

$$\|A^\mu u(t)\|_p\le K_1,$$

and for $h>0$,

$$\|A^\mu\{u(t+h)-u(t)\}\|_p\le K_2h^\vartheta+K_3h^{1-\mu} \tag{4.4}$$

with $\vartheta\in(0,1-\mu)$.

For the proof of Lemma 4.2, see [18, Theorem 3.1] and Rothe [12, Lemma 21].

Proof of Theorem 2.4. We may assume $u_0>0$. Indeed, we can retake $u_0=u(T)>0$ for $T>0$ and prove this theorem with slight modification. Integrating (4.3) over [0,T] with arbitrary number $T>0$, we have

$$\begin{aligned}E(u(T))&+\int_0^T\int_\Omega d(x)\varphi^2\left|\nabla\left\{\log\frac{u(t)}{\varphi}\right\}\right|^2dxdt+\int_0^T\int_\Omega b(x)\varphi\{u(t)-\varphi\}^2dxdt\\&+\int_0^T\int_\Omega c(x)\varphi\{u(t)-\varphi\}k*(u-\varphi)(t)dxdt\\=&E(u_0)+\int_0^T\int_t^\infty k(s)ds\int_\Omega c(x)\varphi^2\{u(t)-\varphi\}dxdt.\end{aligned} \tag{4.5}$$

By virtue of (2.2) and (2.3), the second term in the right hand side of (4.5) is estimated as

$$\int_0^T\int_t^\infty k(s)ds\int_\Omega c(x)\varphi^2\{u(t)-\varphi\}dxdt\le\|c\|_\infty M^2(m+M)|\Omega|\int_0^\infty sk(s)ds.$$

Since $tk\in L^1(0,\infty)$, it follows from (4.5) and the above inequality that

$$\int_0^T\int_\Omega b(x)\varphi\{u(t)-\varphi\}^2dxdt+\int_0^T\int_\Omega c(x)\varphi\{u(t)-\varphi\}k*(u-\varphi)(t)dxdt\le K_4, \tag{4.6}$$

where

$$K_4 = E(u_0) + \|c\|_\infty M^2(m+M)|\Omega| \int_0^\infty tk(t)dt.$$

For $v : [0,T] \to \mathbb{R}$, define v_T by

$$v_T(t) = \begin{cases} v(t) & \text{if } t \in [0,T], \\ 0 & \text{if } t \in (-\infty,\infty)\backslash[0,T], \end{cases}$$

and for $k : [0,\infty) \to \mathbb{R}$, define $\tilde{k}$ by

$$\tilde{k}(t) = \begin{cases} k(t) & \text{if } t \in [0,\infty), \\ 0 & \text{if } t \in (-\infty,0). \end{cases}$$

Then we can derive the following relation (cf. [18, Lemma 2.2]):

$$\mathcal{F}\left(\tilde{k} * v_T\right)(\eta) = \hat{k}(i\eta)\mathcal{F}v_T(\eta), \tag{4.7}$$

where $\mathcal{F}v$ denotes the Fourier transform of v:

$$\mathcal{F}v(\eta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\eta t} v(t)dt.$$

Therefore, making use of (2.5), (4.7), Fubini's theorem and Parseval-Plancherel's equality, we can obtain

$$\begin{aligned} &\int_0^T \int_\Omega b(x)\varphi\{u(t)-\varphi\}^2 dxdt + \int_0^T \int_\Omega c(x)\varphi\{u(t)-\varphi\}k*(u-\varphi)(t)dxdt \\ \geq &k_0 \int_0^T \int_\Omega |u(t)-\varphi|^2 \varphi dxdt, \end{aligned} \tag{4.8}$$

(for details, see [18] and [20]).

Since T is arbitrary and K_4 is independent of T, (4.6) and (4.8) yield

$$\varphi^{1/2}(u-\varphi) \in L^1((0,\infty); L^2(\Omega)). \tag{4.9}$$

On the other hand, (4.4) shows that $\varphi^{1/2}(u-\varphi)$ is uniformly continuous in $(0,\infty)$ with respect to $L^2(\Omega)$ norm. The fact, together with (4.9) implies

$$\lim_{t\to\infty} \int_\Omega |u(t)-\varphi|^2 \varphi dx = 0. \tag{4.10}$$

Then we can prove (2.6) from (4.10). Its proof is exactly the same as in [20] with A replaced by $A+1$. □

Remark 4.1. Take $p > N$ and $\mu \in ((p+N)/(2p), 1)$. Then (2.1) implies

$$\lim_{t\to\infty} u(t) = \varphi \quad \text{in } C^1(\overline{\Omega}).$$

5 Proof of Theorem 2.5

We will prove this theorem along the arguments used in the work of Yamada [20, Proposition 3.3]. We can assume $u_0 > 0$. Since $\lambda_1 < 0$, there exists an unique positive stationary solution φ of (SP). Then integrating (4.3) over $[0, T]$ with $T > 0$, we see

$$E(u(T)) \leq E(u_0) + \|c\|_\infty M \int_0^T \int_t^\infty k(s)ds \int_\Omega \varphi u(t)dxdt \tag{5.1}$$

as in the proof of Theorem 2.4. Since g is a convex function (see (4.2)), it is possible to apply Jensen's inequality (see e.g. [2, pp. 120]) to $E(u(T))$ to get

$$\begin{aligned} g\left(\|\varphi\|_2^{-2} \int_\Omega \varphi u(T)dx\right) &\leq \|\varphi\|_2^{-2} \int_\Omega \varphi^2 g\left(\frac{u(T)}{\varphi}\right) dx \\ &= \|\varphi\|_2^{-2} E(u(T)). \end{aligned} \tag{5.2}$$

Put $V(t) := \|\varphi\|_2^{-2} \int_\Omega \varphi u(t)dx$. Then we obtain from (5.1) and (5.2)

$$\|\varphi\|_2^2 g(V(T)) \leq E(v_0) + \|c\|_\infty \cdot \|\varphi\|_2^2 M \int_0^T \int_t^\infty k(s)dsV(t)dt. \tag{5.3}$$

By using the idea in [20], it can be shown that for sufficiently large T_0,

$$K_6 := \|c\|_\infty M \int_{T_0}^\infty \int_t^\infty k(s)dsdt < 1.$$

Then (5.3) implies that for every $T \geq T_0$,

$$g(V(T)) \leq \|\varphi\|_2^{-2} E(u_0) + \|c\|_\infty M \int_0^{T_0} \int_t^\infty k(s)dsV(t)dt + K_6 \sup_{t \geq T_0} V(t).$$

Recall that g is given by (4.2). Since $K_6 < 1$, it follows from the above inequality that

$$\sup_{t \geq T_0} V(t) \leq K_7 \tag{5.4}$$

with some K_7. Then it follows from (5.4) that

$$\sup_{t \geq 0} \int_\Omega \varphi u(t)dx \leq K_8 \tag{5.5}$$

with some K_8.

Let $r \in (0, 1/2)$. Then we can obtain from (5.5) that

$$\sup_{t \geq 0} \int_\Omega u^r(t)dx \leq K_9, \tag{5.6}$$

where K_9 is a suitable positive constant, independent of t (for details, see [20]). Therefore, the result of Rothe [11, Proposition 2] implies the conclusion. □

6 Proof of Theorem 2.7

We will prove this theorem by the same idea as in [20, Theorem 3.5]. We may assume $u_0 > 0$. By $\lambda_1 < 0$, there exists a unique positive stationary solution φ of (SP). Define a new function v as

$$v(x,t) = k * (u - \varphi)(x,t).$$

Similarly to the proof of Theorem 2.4, integrate (4.3) over $[0,T]$ with an arbitrary $T > 0$; then there exists a positive constant K_{10}, independent of T, such that

$$\begin{aligned} E(u(T)) &+ \int_0^T \int_\Omega d(x)\varphi^2 \left| \nabla \left\{ \log \frac{u(t)}{\varphi} \right\} \right|^2 dxdt \\ &+ \int_0^T \int_\Omega b(x)\varphi\{u(t) - \varphi\}^2 dxdt + \int_0^T \int_\Omega c(x)\varphi\{u(t) - \varphi\}v(t)dxdt \le K_{10}. \end{aligned} \tag{6.1}$$

Since v satisfies

$$v_t(t) = k(0)\{u(t) - \varphi\} + k' * (u - \varphi)(t), \tag{6.2}$$

it follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_\Omega c(x)\varphi v^2(t)dx &= \int_\Omega c(x)\varphi v(t)v_t(t)dx \\ &= \int_\Omega c(x)\varphi v(t) \left[k(0)\{u(t) - \varphi\} + k' * (u - \varphi)(t)\right] dx. \end{aligned}$$

Integrate this identity over $[0,T]$:

$$\begin{aligned} &\frac{1}{2} \int_0^T \frac{d}{dt} \int_\Omega c(x)\varphi v^2(t)dxdt \\ &= \int_0^T \int_\Omega c(x)\varphi v(t) \left[k(0)\{u(t) - \varphi\} + k' * (u - \varphi)(t)\right] dxdt. \end{aligned} \tag{6.3}$$

In view of $v(0) = 0$,

$$\begin{aligned} \int_0^T \frac{d}{dt} \int_\Omega c(x)\varphi v^2(t)dxdt &= \int_\Omega c(x)\varphi v^2(T)dx \\ &\ge 0. \end{aligned}$$

Therefore, we can see from (6.3)

$$\begin{aligned} -\int_0^T \int_\Omega c(x)\varphi v(t)k' * (u - \varphi)(t)dxdt &\le k(0) \int_0^T \int_\Omega c(x)\varphi\{u(t) - \varphi\}v(t)dxdt \\ &\le k(0)K_{10}, \end{aligned} \tag{6.4}$$

where we have used (6.1).

Note

$$\begin{aligned} \hat{k}'(i\eta) &= \int_0^\infty e^{-i\eta t}k'(t)dt \\ &= -k(0) + i\eta\hat{k}(i\eta) \end{aligned}$$

and $\mathcal{F}(v_T)(\eta) = \hat{k}(i\eta)\mathcal{F}((u-\varphi)_T)(\eta)$. Then

$$\mathcal{F}\left(\tilde{k}' * (u-\varphi)_T\right)(\eta) = \left\{-k(0) + i\eta\hat{k}(i\eta)\right\}\left\{\hat{k}(i\eta)\right\}^{-1}\mathcal{F}(v_T)(\eta). \tag{6.5}$$

By virtue of (6.5), Fubini's theorem and Parseval-Plancherel's equality, similarly in the proof of Theorem 2.4, we can show

$$\begin{aligned}
&-\int_0^T\int_\Omega c(x)\varphi v(t)k' * (u-\varphi)(t)dxdt \\
&= -\int_\Omega c(x)\varphi\int_{-\infty}^{\infty} v_T(t)\tilde{k}' * (u-\varphi)_T(t)dtdx \\
&= -\int_\Omega c(x)\varphi\int_{-\infty}^{\infty} \mathrm{Re}\,\mathcal{F}(v_T)(\eta)\mathcal{F}\left(\tilde{k}' * (u-\varphi)_T\right)(\eta)d\eta dx \\
&= k(0)\int_\Omega c(x)\varphi\int_{-\infty}^{\infty} \mathrm{Re}\left\{\hat{k}(i\eta)\right\}^{-1}|\mathcal{F}(v_T)(\eta)|^2 d\eta dx.
\end{aligned}$$

Therefore, (6.4) implies

$$c_0 k_1\int_0^T\int_\Omega \varphi v^2(t)dxdt \le K_{10},$$

(note (2.10)). Since T is arbitrary and K_{10} is independent of T, this fact implies

$$\varphi^{1/2}v \in L^1((0,\infty); L^2(\Omega)). \tag{6.6}$$

One can prove the uniformly continuity of $v(t)$ with respect to t from (6.2) (see [20]). Hence, it follows from (6.6) that

$$\lim_{t\to\infty} \varphi^{1/2}v(t) = 0 \quad \text{in } L^2(\Omega).$$

In the same manner as [20] (replace A by $A+1$),

$$\lim_{t\to\infty} v(t) = 0 \quad \text{uniformly in } \overline{\Omega}. \tag{6.7}$$

The rest of the proof is essentially the same as Yamada [20]. So we omit the details. □

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