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Effect of higher order derivatives of initial data on the blow-up set for a semilinear heat equation

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1 Introduction

This paper deals with the blow-up problem for a semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + u^p, & x \in \Omega, \ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $p > 1$, $N \geq 1$, Ω is a domain in $\mathbf{R}^N$ and $u_0 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ is a nonnegative initial function. We denote by $T(u_0)$ the maximal existence time of the unique classical solution for problem (1.1). If $T(u_0) < \infty$, the solution satisfies

$$\limsup_{t \nearrow T(u_0)} \|u(t)\|_{L^\infty(\Omega)} = +\infty.$$

Then we say that the solution blows up in finite time and call $T(u_0)$ the blow-up time of the solution for problem (1.1). Furthermore, for the solution of (1.1) with $T(u_0) < \infty$, we define the blow-up set $B(u_0)$ by

$$B(u_0) := \Big\{ x \in \overline{\Omega} : \text{ there exists a sequence } \{(x_n, t_n)\} \subset \overline{\Omega} \times (0, T(u_0)) \\ \text{such that } \lim_{n \to \infty} (x_n, t_n) = (x, T(u_0)) \text{ and } \lim_{n \to \infty} u(x_n, t_n) = \infty \Big\}.$$

The purpose of this paper is to characterize the location of the set $B(u_0)$ for "large" initial data u_0. We explain the meaning of "large" initial data later.

We first focus on the case where $u_0(x) = \lambda\varphi(x)$. Here $\lambda > 0$ is a sufficiently large parameter and $\varphi \in C_0^\infty(\Omega)$ is a nonnegative function on $\overline{\Omega}$. For problem (1.1) with this type initial data, it is well known that, for sufficiently large $\lambda > 0$, the blow-up set $B(u_0)$ is approximated by the one for corresponding ordinary differential equation

$$\partial_t u = u^p, \quad x \in \Omega, \ t > 0, \qquad u(x,0) = \lambda\varphi(x), \quad x \in \Omega. \tag{1.2}$$

In fact, the author of this paper and Ishige in [4] proved the following result: Assume that the solution of (1.1) satisfies

$$\sup_{\lambda > \lambda_0} \sup_{0 < t < T(\lambda\varphi)} (T(\lambda\varphi) - t)^{1/(p-1)} \|u(t)\|_{L^\infty(\Omega)} < \infty \tag{1.3}$$

for some $\lambda_0 > 0$, then, for any $\delta > 0$, there exists a constant $\lambda_\delta > 0$ such that

$$B(\lambda\varphi) \subset \{x \in \overline{\Omega} : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - \delta\}$$

for any $\lambda > \lambda_\delta$. Since $\delta > 0$ is arbitrary, this results implies that the solution blows up only near the maximum points of φ if λ is sufficiently large. Furthermore, the set of maximum points of φ corresponds the blow-up set for ODE (1.2), so the blow-up set for (1.1) is approximated by the one for ODE if λ is sufficiently large. Here we give one remark. We define

$$v(x,t) = \lambda^{-1}u(x, \lambda^{-(p-1)}t).$$

Then v satisfies

$$\begin{cases} \partial_t v = \lambda^{-(p-1)}\Delta v + v^p & \text{in} \quad \Omega \times (0, \lambda^{p-1}T(\lambda\varphi)), \\ v(x,t) = 0 & \text{on} \quad \partial\Omega \times (0, \lambda^{p-1}T(\lambda\varphi)), \\ v(x,0) = \varphi(x) & \text{in} \quad \Omega. \end{cases}$$

Therefore, a semilinear heat equation with large initial data is equivalent to a semilinear heat equation with small diffusion, and the above results hold true for

$$\partial_t u = \epsilon\Delta u + u^p, \ \ x \in \Omega, \ \ t > 0, \quad u(x,t) = 0, \ \ x \in \partial\Omega, \ \ t > 0, \quad u(x,0) = \varphi(x), \ \ x \in \Omega,$$

with sufficiently small $\epsilon > 0$.

It is natural to ask what happens for the case where φ has several maximum points. Concerning this question, it has been proved in [5] that, if α, $\beta \in \Omega$ are maximum points of φ and

$$\Delta\varphi(\alpha) > \Delta\varphi(\beta),$$

then there exists a constant $\delta > 0$ such that $B(\lambda\varphi) \cap B(\beta, \delta) = \emptyset$ for sufficiently large λ under the condition (1.3). Therefore we can characterize the location of the blow-up set by using $\Delta\varphi$ at maximum points, and the solution does not blow-up near the maximum point β. In particular, the solution blows up near the maximum point α if φ has only two maximum points α and β. One might ask the natural question

How can we characterize the location of the blow-up set $B(\lambda\varphi)$ if $\Delta\varphi(\alpha) = \Delta\varphi(\beta)$?

However, unfortunately, it seems difficult to consider this case for problem (1.1) with the initial data of the form $u_0(x) = \lambda\varphi(x)$. In this paper, we consider another type of "large" initial data $u_0(x) = \lambda + \varphi(x)$ with sufficiently large $\lambda > 0$, and characterize the location of the blow-up set for problem (1.1). In particular, we show the relationship between the blow-up set and higher order derivatives of the initial data.

Before stating our main results, we introduce some notation. For any $x \in \mathbf{R}^N$ and $r > 0$, we denote the open ball of radius r and center x by $B(x,r)$. Let $BC_+(\overline{\Omega})$ be the set of nonnegative bounded continuos functions on $\overline{\Omega}$. For any $\phi \in BC_+(\mathbf{R}^N)$, we denote by $e^{t\Delta}\phi$ the unique bounded solution of the heat equation $\partial_t u = \Delta u$ in $\mathbf{R}^N \times (0,\infty)$ with $u(x,0) = \phi(x)$ in $\mathbf{R}^N$, that is,

$$(e^{t\Delta}\phi)(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}}\phi(y)\,dy, \quad (x,t) \in \mathbf{R}^N \times (0,\infty).$$

Furthermore, we denote by $M(\phi)$ the set of maximum points of ϕ. We are ready to state our main results.

Theorem 1.1 *Let $p > 1$, $N \geq 1$, Ω be a domain in $\mathbf{R}^N$ and $\varphi \in C^4(\Omega) \cap BC_+(\overline{\Omega})$ satisfy $\varphi(x) < \|\varphi\|_{L^\infty(\Omega)}$ on $\partial\Omega$ and $\limsup_{|x|\to\infty} \varphi(x) < \|\varphi\|_{L^\infty(\Omega)}$. For any $\lambda > 0$, let T_λ and B_λ be the blow-up time and the blow-up set of the solution for problem* (1.1) *with $u_0(x) = \lambda + \varphi(x)$. Assume that $M(\varphi)$ consists of only two points α and β and that*

$$\Delta\varphi(\alpha) = \Delta\varphi(\beta), \qquad \Delta^2\varphi(\alpha) > \Delta^2\varphi(\beta).$$

Furthermore, assume that there exist positive constant C and λ_0 such that

$$\sup_{0<t<T_\lambda} (T_\lambda - t)^{1/(p-1)} \|u(t)\|_{L^\infty(\Omega)} \leq C \tag{1.4}$$

for all $\lambda > \lambda_0$. Then there exist positive constant δ and λ_* such that

$$B_\lambda \subset B(\alpha, \delta)$$

for all $\lambda > \lambda_*$.

Remark 1.1 (i) *Assumption* (1.4) *can be proved under suitable assumptions on p, Ω and φ. In particular, if $\Omega = \mathbf{R}^N$ and $\lambda > 0$ is sufficiently large, then we can prove* (1.4) *with the aid of the argument of* [1].
(ii) *Let $u_0(x) = \lambda + \varphi(x)$ and assume the same situation as in [5]. Let $M(\varphi)$ consist of only two points α and β, and assume that*

$$\Delta\varphi(\alpha) > \Delta\varphi(\beta).$$

Then the solution blows up only near the maximum points α if λ is sufficiently large. Therefore, the similar result as in [5] also holds for initial data of the type $u_0(x) = \lambda + \varphi(x)$.

Unfortunately, we can not prove further results in general. However, under some restriction on p, we can show the effect of higher order derivatives on the blow-up set for (1.1).

Theorem 1.2 *For any $m \in \mathbf{N}$ with $m \geq 3$, let $1 < p < 1 + 1/2(m-2)$. Let $N \geq 1$, Ω be a domain in $\mathbf{R}^N$ and $\varphi \in C^4(\Omega) \cap BC_+(\overline{\Omega})$ satisfy $\varphi(x) < \|\varphi\|_{L^\infty(\Omega)}$ on $\partial\Omega$ and $\limsup_{|x|\to\infty} \varphi(x) < \|\varphi\|_{L^\infty(\Omega)}$. For any $\lambda > 0$, let T_λ and B_λ be the blow-up time and the blow-up set of the solution for problem* (1.1) *with $u_0(x) = \lambda + \varphi(x)$. Assume that $M(\varphi)$ consists of only two points α and β and that*

$$\Delta^k\varphi(\alpha) = \Delta^k\varphi(\beta) \quad (k = 1, \ldots, m-1), \qquad \Delta^m\varphi(\alpha) > \Delta^m\varphi(\beta).$$

Assume (1.4). *Then there exist positive constant δ and λ_* such that*

$$B_\lambda \subset B(\alpha, \delta)$$

for all $\lambda > \lambda_*$.

Remark 1.2 *Under the same assumptions as in Theorem* 1.2, *we have*

$$T_\lambda = \frac{\lambda_\varphi^{-(p-1)}}{p-1}\left(1+\lambda_\varphi^{-(p-1)-1}|\Delta\varphi(\alpha)| - \sum_{k=2}^{m}\frac{\lambda_\varphi^{-k(p-1)-1}}{k!(p-1)^{k-1}}\Delta^k\varphi(\alpha) + o(\lambda^{-m(p-1)-1})\right)$$

for all sufficiently large $\lambda > 0$, *where* $\lambda_\varphi := \lambda + \|\varphi\|_{L^\infty(\Omega)}$.

2 Outline of the proof of Theorem 1.1

This section is devoted to explain the outline of the proof of Theorem 1.1. In order to prove Theorem 1.1, we study the profile of the solution just before the blow-up time. In fact, we study the profile of the solution at

$$t = S_\lambda - \lambda_\varphi^{-3(p-1)-1},$$

where

$$S_\lambda := \frac{\lambda_\varphi^{-(p-1)}}{p-1}\left(1+\lambda_\varphi^{-(p-1)-1}|\Delta\varphi(\alpha)| - \frac{\lambda_\varphi^{2(p-1)-1}}{2(p-1)}\Delta^2\varphi(\alpha)\right).$$

One of the most important point in the proof of Theorem 1.1 is to get the profile of the solution at

$$t = S_\lambda - \lambda_\varphi^{-2(p-1)-1}.$$

Once we get the profile of the solution at this time, we can easily obtain the profile of the solution at $t = S_\lambda - \lambda_\varphi^{-3(p-1)-1}$ by the argument as in [5].

In order to get the profile of the solution just before the blow-up time, we construct comparison functions. For the construction of subsolutions, let z be the solution of

$$\partial_t z = \Delta z \ \text{ in } \Omega\times(0,\infty), \quad z(x,t)=0 \ \text{ in } \partial\Omega\times(0,\infty), \quad z(x,0)=\lambda+\varphi(x) \ \text{ in } \Omega,$$

and put

$$U_0(x,t) := \left(z(x,t)^{-(p-1)} - (p-1)t\right)^{-1/(p-1)}.$$

Then we can easily check that the function U_0 is a subsolution for problem (1.1), and we can get the profile of the solution from below.

For the construction of supersolutions, we employ the cut-off technique. For a positive parameter ϵ, which will be chosen later, we put

$$\varphi_\lambda(x) := \begin{cases} \max\left\{\varphi(x), \|\varphi\|_{L^\infty(\Omega)} - \lambda_\varphi^{-(p-1)+\epsilon}\right\} & \text{if } x\in\Omega, \\ \|\varphi\|_{L^\infty(\Omega)} - \lambda_\varphi^{-(p-1)+\epsilon} & \text{if } x\notin\Omega. \end{cases}$$

Then we have $u(x,0) \le \lambda + \varphi_\lambda(x)$ in Ω and

$$\|\nabla\varphi_\lambda\|_{L^\infty(\mathbf{R}^N)} \lesssim \lambda_\varphi^{-(p-1)/2+\epsilon/2}$$

for all sufficiently large λ. Consider

$$\begin{cases} \partial_t U = \Delta U + U^p, & x \in \mathbf{R}^N, \ t > 0, \\ U(x,0) = \lambda + \varphi_\lambda(x), & x \in \mathbf{R}^N. \end{cases} \tag{2.1}$$

For the construction of supersolutions for problem (1.1), it is enough to construct supersolutions for problem (2.1). For any $\sigma > 0$, we define the function U_σ by

$$U_\sigma(x,t) := \left([\lambda + (e^{t\Delta}\varphi_\lambda)(x)]^{-(p-1)} - (p-1)(1+\sigma)t \right)^{-1/(p-1)}.$$

Then we see that, if U_σ satisfies

$$p \left(\inf_{x \in \mathbf{R}^N} U(x,0) \right)^{-2p} U_\sigma(x,t)^{p-1} \left| \nabla (e^{t\Delta}\varphi_\lambda)(x) \right|^2 \leq \sigma,$$

then the function U_σ is a supersolution for problem (2.1) as long as it exists, and we can get the profile of the solution from above. In order to get precise profile of the solution, we have to take a parameter $\sigma > 0$ as small as possible. For this purpose, we consider the following partition of time interval. Put

$$\begin{cases} I_0 := [0, S_\lambda - \lambda_\varphi^{-(p-1)-1/2}], \\ I_k := [S_\lambda - \lambda_\varphi^{-(p-1)-k/2}, S_\lambda - \lambda_\varphi^{-(p-1)-(k+1)/2}] \quad (k = 1, \dots, 2[p-1]), \\ I := [S_\lambda - \lambda_\varphi^{-(p-1)-[p-1]-1/2}, S_\lambda - \lambda_\varphi^{-2(p-1)-1}]. \end{cases}$$

Then we have

$$[0, S_\lambda - \lambda_\varphi^{-2(p-1)-1}] = I_0 \cup \left(\bigcup_{k=1}^{2[p-1]} I_k \right) \cup I.$$

We construct supersolutions in each interval I_0, I_k and I by following the above manner, and obtain the profile of the solution at $t = S_\lambda - \lambda_\varphi^{-2(p-1)-1}$. As a result, we finally get the profile of the solution at $t = S_\lambda - \lambda_\varphi^{-3(p-1)-1}$.

We conclude the proof of Theorem 1.1. Put

$$v(x,\tau) := \lambda_\varphi^{-3-\frac{1}{p-1}} u(x, S_\lambda - \lambda_\varphi^{-3(p-1)-1} + \lambda_\varphi^{-3(p-1)-1}\tau).$$

Then v satisfies

$$\begin{cases} \partial_\tau v = \lambda_\varphi^{-3(p-1)-1} \Delta v + v^p, & x \in \Omega, \ \tau > 0, \\ v(x,\tau) = 0, & x \in \partial\Omega, \ \tau > 0, \\ v(x,0) = \lambda_\varphi^{-3-\frac{1}{p-1}} u(x, S_\lambda - \lambda_\varphi^{-3(p-1)-1}), & x \in \Omega. \end{cases} \tag{2.2}$$

Furthermore, $v(\cdot,0)$ satisfies the following properties: there exist positive constants δ_1 and δ_2 such that

$$\sup_{x \in B(\beta,\delta_1)} v(x,0) \leq \|v(0)\|_{L^\infty(\Omega)} - \delta_2 \tag{2.3}$$

for all sufficiently large λ. Furthermore, by (1.4) we have

$$\limsup_{\lambda\to\infty} \|v(0)\|_{L^\infty(\Omega)} < \infty.$$

These imply that the function $v(\cdot, 0)$ can not take its maximum near β. On the other hand, the diffusion coefficient of problem (2.2) is sufficiently small, so the solution blows up only near the maximum points of $v(\cdot, 0)$ by the results of [4] and we conclude that the solution blows only near the maximum point α. $\square$

References

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