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Signal-dependent sensitivity preventing blow-up in a fully parabolic chemotaxis system

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1. Introduction

This report summarizes two recent works [**2**] and [**8**] (joint work with Tomomi Yokota).

In various biological contexts, a biological phenomenon called *chemotaxis* plays an important role. *Chemotaxis* is the directed movement of cells towards increasing concentrations of a chemical substance which is produced by cells. Keller and Segel first proposed a mathematical model describing *chemotaxis* in 1970 ([**13**]). After that, this model has attracted considerable attention in mathematical studies. In this report we especially focus on a signal-dependent sensitivity which describes that the cell movement towards higher signal concentration is inhibited at points where these concentrations are high. We consider the Neumann initial-boundary value problem for a fully parabolic chemotaxis system with signal-dependent sensitivity function

$$(1.1)\qquad \begin{cases} u_t = \Delta u - \nabla\cdot(u\chi(v)\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$, $n\geq 2$ with smooth boundary and assume that

$$(1.2)\qquad \begin{cases} u_0\in C^0(\bar\Omega), \quad u_0\geq 0 \quad \text{in } \bar\Omega, \quad u_0\not\equiv 0,\\ v_0\in W^{1,\infty}(\Omega), \quad v_0>0 \quad \text{in } \bar\Omega. \end{cases}$$

The prototypical choice of the sensitivity function is the case $\chi(v)=\frac{\chi_0}{v}$ $(\chi_0>0)$ which was proposed in an original model by Keller and Segel [**13**] building on the so-called Weber-Fechner law.

The diffusive term vs. the cross-diffusive term. Considerable attention has been devoted to analyze the competition between the spreading effect of the diffusive term and the concentrating effect of the cross-diffusive term in (1.1). As to the simplified chemotaxis system ($\chi(v)\equiv 1$), it is well known that the size of initial data determines whether the solution is global and bounded or not as follows:

- $n=1$, or $n\geq 2$ and the initial data is suitably small
  $\Longrightarrow$ (1.1) has a global and bounded solution ([**17, 16, 20**]).
- $n\geq 2$ and the initial data is suitably large $\Longrightarrow$ (1.1) has a blow-up solution ([**9, 22**]).

Many references to earlier works on some variants of chemotaxis system can be found in [**10, 12**].

Weakening the cross-diffusive term by a decaying function. In the past few years, the study of chemotaxis system has developed by having a different point of view. By introducing a decaying function $\chi(v)$ into the cross-diffusive term, the concentrating effect of the cross-diffusive term is weaken and then it is expected that (1.1) has global and bounded solution independently of the size of initial data. Here, we recall some results about (1.1) with $\chi(v) = \frac{\chi_0}{v}$ $(\chi_0 > 0)$. Winkler [**21**] proved that if $\chi_0 < \sqrt{\frac{2}{n}}$, then (1.1) possesses a global classical solution independently of the size of initial data. As pointed out in [**21**], the result did not rule out the possibility that the solution may become unbounded as $t \to \infty$. The question of boundedness of the solution to (1.1) has been posted as an open problem. Moreover as to the present problem, global existence of weak solutions was established when $\chi_0 < \sqrt{\frac{n+2}{3n-4}}$ ([**21**]). In the radially symmetric setting, Stinner and Winkler [**18**] constructed certain weak solutions under the condition $\chi_0 < \sqrt{\frac{n}{n-2}}$. As compared to the above, the parabolic-elliptic case has been studied more precisely ([**1, 15, 5, 7, 6**]).

In the first half of the present report we focus on the case $\chi(v) = \frac{\chi_0}{v}$ $(\chi_0 > 0)$. We improve the approach in [**21**] and establish uniform-in-time boundedness of solutions to (1.1). The first main result reads as follows.

Theorem 1.1 (F. [**2**]). *Let $n \geq 2$. Assume that $\chi(v) = \frac{\chi_0}{v}$ with $0 < \chi_0 < \sqrt{\frac{2}{n}}$ and suppose that u_0 and v_0 satisfy* (1.2). *Then the global solution of* (1.1) *is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0.$$

The above theorem states *uniform-in-time* boundedness of solutions under the same condition as in [**21**]. There are two difficulties in deriving boundedness. The first difficulty stems from the singularity of $\frac{1}{v}$. To overcome this difficulty we shall establish a *time-independent* pointwise lower bound for v (Lemma 2.2). Note that the strong maximum principle easily implies

$$v(\cdot,t) \geq \eta(t) := \min_{x\in\bar{\Omega}} v_0(x) \cdot e^{-t} \qquad \text{for all } t > 0.$$

However, this is useless in proving uniform-in-time boundedness of solutions, since $\eta(t) \to 0$ as $t \to \infty$. The second difficulty lies in deducing *time-independent* L^p-boundedness of solutions. Although the L^p-estimate in [**21**] depends on time, we shall reconstruct the method in [**21**] and remove the dependence. Invoking the above two *time-independent* estimates, we establish boundedness.

In the latter half of the present report we consider the strongly singular sensitivity case: the sensitivity function χ satisfies

$$\chi \in C^{1+\delta}_{\mathrm{loc}}((0,\infty)) \quad \text{for some } \delta > 0 \tag{1.3}$$

and

$$0 < \chi(v) \leq \frac{\chi_0}{v^k} \quad \text{for some } \chi_0 > 0 \text{ and } k > 1. \tag{1.4}$$

In the regular case $0 < \chi(v) \le \frac{\chi_0}{(1+\alpha v)^k}$ ($\alpha > 0$, $\chi_0 > 0$, $k > 1$), global existence and boundedness were shown for all $\chi_0 > 0$ by Winkler [**19**]. Using the *time-independent* pointwise lower bound for v (Lemma 2.2), the boundedness result in [**19**] shall be extended to the strongly singular case $\frac{\chi_0}{v^k}$ ($\chi_0 > 0$, $k > 1$). The second main results reads as follows.

Theorem 1.2 (F.-Yokota [**8**]). *Suppose that χ satisfy* (1.3) *and* (1.4), *and assume that* (u_0, v_0) *fulfils* (1.2). *Then the problem* (1.1) *has a global classical solution* (u, v) *and moreover the solution is bounded in the sense that there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0.$$

This report is organized as follows. Section 2 will be concerned with preliminaries, including the announced pointwise lower bound for v. In Section 3 we focus on the case $\chi(v) = \frac{\chi_0}{v}$. We firstly establish time-independent L^p-boundedness of solutions and give the proof of Theorem 1.1. We consider the strongly singular case ($\chi(v) = \frac{\chi_0}{v^k}$, $k > 1$) in Section 4. The uniform-in-time lower bound for v builds a bridge between the regular case and the singular one.

2. Preliminaries

We first recall the global existence result established in [**21**].

Lemma 2.1. *Assume that $\chi(v) = \frac{\chi_0}{v}$ with $0 < \chi_0 < \sqrt{\frac{2}{n}}$. If the initial data (u_0, v_0) satisfies* (1.2), *then* (1.1) *has a global classical positive solution*

$$u \in C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\bar{\Omega})),$$
$$v \in C^{2,0}(\bar{\Omega} \times (0,\infty)) \cap C^0([0,\infty); C^0(\bar{\Omega})).$$

Moreover, the first component of the solution satisfies the mass identity

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \qquad \textit{for all } t > 0. \tag{2.1}$$

The following lemma is a cornerstone of our work. The mass identity (2.1) plays a key role in the proof of this lemma. We shall denote by (u, v) the classical solution of (1.1) in the rest of the report.

Lemma 2.2. *There exists $\eta > 0$ such that*

$$\inf_{x\in\Omega} v(x,t) \ge \eta > 0 \qquad \textit{for all } t \ge 0,$$

where η does not depend on t.

Proof. We use a known result for the Neumann heat semigroup $e^{t\Delta}$. In the same way as in the proof of [**11**, Lemma 3.1], we can obtain the pointwise estimate from below

$$e^{t\Delta}w(x) \ge \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{(\operatorname{diam}\Omega)^2}{4t}} \cdot \int_\Omega w > 0 \quad (x \in \Omega, t > 0) \quad \text{for all nonnegative } w \in C^0(\bar{\Omega}),$$

where $\operatorname{diam}\Omega := \max_{x,y\in\bar\Omega}|x-y|$. First by the positivity of $v_0 > 0$ in $\bar\Omega$ and the maximum principle we have

$$v(t) \geq \min_{x\in\bar\Omega} v_0(x)\cdot e^{-t} > 0 \qquad \text{for all } t \geq 0.$$

Now fix $\tau > 0$. Then it follows that

$$v(t) \geq \min_{x\in\bar\Omega} v_0(x)\cdot e^{-\tau} =: \eta_1 > 0 \qquad \text{for all } t \in [0,\tau].$$

Next, the representation formula of v, the maximal principle and (2.1) imply that

$$\begin{aligned}
v(t) &= e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(s)\,ds \\
&\geq \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\left((t-s)+\frac{(\operatorname{diam}\Omega)^2}{4(t-s)}\right)} \cdot \left(\int_\Omega u(x,s)\,dx\right) ds \\
&= \|u_0\|_{L^1(\Omega)} \cdot \int_0^t \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\left(r+\frac{(\operatorname{diam}\Omega)^2}{4r}\right)}\,dr \\
&\geq \|u_0\|_{L^1(\Omega)} \cdot \int_0^\tau \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-\left(r+\frac{(\operatorname{diam}\Omega)^2}{4r}\right)}\,dr =: \eta_2 > 0 \qquad \text{for all } t \in [\tau,\infty).
\end{aligned}$$

Therefore we have $v(t) \geq \min\{\eta_1,\eta_2\} =: \eta$ for all $t \geq 0$. This completes the proof. □

To achieve boundedness of the norm of $u(\cdot,t)$ in $L^p(\Omega)$ we shall use the following lemmas.

Lemma 2.3. *Consider the case $\chi(v) = \frac{\chi_0}{v}$. Let $p \in \mathbb{R}$ and $q \in \mathbb{R}$. Then the following identity holds for all $t > 0$:*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p v^q + q\int_\Omega u^p v^q - q\int_\Omega u^{p+1}v^{q-1} \\
&= -p(p-1)\int_\Omega u^{p-2}v^q|\nabla u|^2 + \int_\Omega u^p v^{q-2}\cdot\big[-q(q-1)+pq\chi_0\big]\cdot|\nabla v|^2 \\
&\quad + \int_\Omega u^{p-1}v^{q-1}\cdot\big[-2pq+p(p-1)\chi_0\big]\nabla u\cdot\nabla v.
\end{aligned}$$

Proof. Proceeding analogously to [**21**, Lemma 2.3], we can prove the desired identity. □

Lemma 2.4. *Let $1 \leq \theta,\mu \leq \infty$.*

(i) *If $\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu}) < 1$, then there exists $C > 0$ such that*

$$\|v(\cdot,t)\|_{L^\mu(\Omega)} \leq C\Big(1 + \sup_{s\in(0,\infty)}\|u(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \textit{for all } t > 0.$$

(ii) *If $\frac{1}{2}+\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu}) < 1$, then there exists $C > 0$ such that*

$$\|\nabla v(\cdot,t)\|_{L^\mu(\Omega)} \leq C\Big(1 + \sup_{s\in(0,\infty)}\|u(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \textit{for all } t > 0.$$

Proof. We can argue similarly as in [**21**, Lemma 2.4] due to the estimate for $e^{t(\Delta-1)}$:

$$\|e^{t(\Delta-1)}\varphi\|_{L^\mu(\Omega)} \leq ct^{-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})}e^{-\delta t}\|\varphi\|_{L^\theta(\Omega)} \qquad \text{for all } t > 0,\ \varphi \in L^\theta(\Omega),$$

with some constants $c,\delta > 0$. □

3. Proof of Theorem 1.1

In this section we focus on the case $\chi(v) = \frac{\chi_0}{v}$ ($\chi_0 > 0$). We follow the same way as in [**21**]. The difference is that our estimates are independent of time.

Lemma 3.1. *Let $n \geq 2$ and $\chi(v) = \frac{\chi_0}{v}$ with $0 < \chi_0 < \sqrt{\frac{2}{n}}$. Assume that $p \in (1, \frac{1}{\chi_0^2})$ and $r \in (r_-(p), r_+(p))$, where $r_\pm(p) := \frac{p-1}{2}(1 \pm \sqrt{1 - p\chi_0^2})$. If there exists a constant $c > 0$ such that*

$$\|v(\cdot,t)\|_{L^{p-r}(\Omega)} \leq c \qquad \text{for all } t > 0, \tag{3.1}$$

then there exists $C > 0$ such that

$$\int_\Omega u^p(x,t)v^{-r}(x,t)\,dx \leq C \qquad \text{for all } t > 0.$$

Proof. Choosing $q := -r$ in Lemma 2.3, we obtain

$$\begin{aligned}
\mathbf{I} :=& \frac{d}{dt}\int_\Omega u^p v^{-r} - r\int_\Omega u^p v^{-r} + r\int_\Omega u^{p+1}v^{-r-1} \\
=& -p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 - \int_\Omega u^p v^{-r-2}\big[r(r+1) + pr\chi_0\big]\cdot|\nabla v|^2 \\
&+ \int_\Omega u^{p-1}v^{-r-1}\big[2pr + p(p-1)\chi_0\big]\nabla u\cdot\nabla v
\end{aligned} \tag{3.2}$$

for $t > 0$. Applying Young's inequality to the last term, we have

$$\begin{aligned}
&\Big|\int_\Omega u^{p-1}v^{-r-1}\big[2pr + p(p-1)\chi_0\big]\nabla u\cdot\nabla v\Big| \\
&\leq p(p-1)\int_\Omega u^{p-2}v^{-r}|\nabla u|^2 + \frac{1}{4p(p-1)}\int_\Omega u^p v^{-r-2}\big[2pr + p(p-1)\chi_0\big]^2\cdot|\nabla v|^2.
\end{aligned}$$

Therefore (3.2) yields

$$\mathbf{I} \leq -\int_\Omega u^p v^{-r-2}h(p,r,\chi_0)|\nabla v|^2, \tag{3.3}$$

where

$$h(p,r,\chi_0) := r(r+1) + pr\chi_0 - \frac{\big[2pr + p(p-1)\chi_0\big]^2}{4p(p-1)}. \tag{3.4}$$

As $p \in (1, \frac{1}{\chi_0^2})$ and $r \in (r_-(p), r_+(p))$, we thus obtain

$$\begin{aligned}
4(p-1)h(p,r,\chi_0) &= -4r^2 + 4(p-1)r - p(p-1)^2\chi_0^2 \\
&= 4(r_+(p) - r)(r - r_-(p)) > 0.
\end{aligned}$$

In view of the positivity $h > 0$, (3.2) and (3.3) imply

$$\frac{d}{dt}\int_\Omega u^p v^{-r} + r\int_\Omega u^{p+1}v^{-r-1} \leq r\int_\Omega u^p v^{-r} \qquad \text{for all } t > 0. \tag{3.5}$$

Now unlike the proof of [**21**, Lemma 4.2] we pay attention to the term $r\int_\Omega u^{p+1}v^{-r-1}$. Hölder's inequality implies that

$$\int_\Omega u^p v^{-r} = \int_\Omega (u^{p+1}v^{-r-1})^{\frac{p}{p+1}} \cdot v^{-r-\frac{p(-r-1)}{p+1}} \le \Big(\int_\Omega u^{p+1}v^{-r-1}\Big)^{\frac{p}{p+1}} \Big(\int_\Omega v^{p-r}\Big)^{\frac{1}{p+1}}.$$

In virtue of the assumption (3.1), we see that

$$\int_\Omega u^p v^{-r} \le c^{\frac{p-r}{p+1}} \Big(\int_\Omega u^{p+1}v^{-r-1}\Big)^{\frac{p}{p+1}}.$$

Hence we have that

$$c^{-\frac{p-r}{p}} \Big(\int_\Omega u^p v^{-r}\Big)^{\frac{p+1}{p}} \le \int_\Omega u^{p+1}v^{-r-1}. \tag{3.6}$$

Combining (3.6) with (3.5), we establish the following inequality:

$$\frac{d}{dt}\int_\Omega u^p v^{-r} \le -rc^{-\frac{p-r}{p}} \Big(\int_\Omega u^p v^{-r}\Big)^{\frac{p+1}{p}} + r\int_\Omega u^p v^{-r}.$$

Since we find $\frac{p+1}{p} > 1$, thus the standard ODE technique completes the proof. □

We are now in a position to prove Theorem 1.1.

PROOF OF THEOREM 1.1 The proof is divided into two steps.

(Step 1) In this step we shall gain L^p-boundedness of solutions. We will prove that there exist some $p > \frac{n}{2}$ and $C_p > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C_p \qquad \text{for all } t > 0. \tag{3.7}$$

We consider an iterative argument. First we pick a pair (p_0, r_0) such that

$$\begin{cases} p_0 \in \Big(1, \min\Big\{\dfrac{1}{{\chi_0}^2},\ n+1,\ \dfrac{n+2}{n-2}\Big\}\Big), \\ r_0 := \dfrac{p_0-1}{2}. \end{cases} \tag{3.8}$$

Then we can confirm that

$$p_0 > r_0, \quad r_0 < \frac{n}{2}, \quad r_0 \in (r_-(p_0), r_+(p_0)) \quad \text{and} \quad p_0 - r_0 = \frac{p_0+1}{2} < \frac{n}{n-2}.$$

Since $\frac{n}{2}(1-\frac{1}{p_0-r_0}) < 1$ due to the inequality $p_0 - r_0 < \frac{n}{n-2}$, Lemma 2.4 (i) together with the mass identity (3) allows us to find a constant $c_0 > 0$ fulfilling

$$\|v(\cdot,t)\|_{L^{p_0-r_0}(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^1(\Omega)}\Big) \le c_0 \qquad \text{for all } t > 0.$$

Therefore Lemma 3.1 yields that there exists a constant $c_0' > 0$ such that

$$\int_\Omega u^{p_0} v^{-r_0} \le c_0' \qquad \text{for all } t > 0.$$

Now we claim that for all $q_0 \in (1, \min\{p_0, \frac{n(p_0-r_0)}{n-2r_0}\})$ there exists a constant $c_0'' > 0$ such that

$$\int_\Omega u^{q_0} \le c_0'' \qquad \text{for all } t > 0. \tag{3.9}$$

Indeed, applying Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega u^{q_0} &= \int_\Omega (u^{p_0}v^{-r_0})^{\frac{q_0}{p_0}} \cdot v^{\frac{r_0q_0}{p_0}} \\ &\le \Big(\int_\Omega u^{p_0}v^{-r_0}\Big)^{\frac{q_0}{p_0}} \cdot \Big(\int_\Omega v^{\frac{q_0r_0}{p_0-q_0}}\Big)^{\frac{p_0-q_0}{p_0}} \\ &\le c_0'^{\frac{q_0}{p_0}} \cdot \Big(\int_\Omega v^{\frac{q_0r_0}{p_0-q_0}}\Big)^{\frac{p_0-q_0}{p_0}}. \end{aligned} \tag{3.10}$$

Since $\frac{n}{2}(\frac{1}{q_0} - \frac{p_0-q_0}{q_0r_0}) < 1$ due to $q_0 < \frac{n(p_0-r_0)}{n-2r_0}$, it follows from Lemma 2.4 (i) that

$$\sup_{t>0} \|v(\cdot,t)\|_{L^{\frac{q_0r_0}{p_0-q_0}}(\Omega)} \le K_0\Big(1 + \sup_{t>0}\|u(\cdot,t)\|_{L^{q_0}(\Omega)}\Big)$$

with $K_0 > 0$. Applying this estimate to (3.10), we have

$$\sup_{t>0} \|u(\cdot,t)\|_{L^{q_0}(\Omega)} \le K_0'\Big(1 + (\sup_{t>0}\|u(\cdot,t)\|_{L^{q_0}(\Omega)})^{\frac{r_0}{p_0}}\Big)$$

with $K_0' > 0$. Since $\frac{r_0}{p_0} < 1$, we can verify (3.9).

In the above argument, if $p_0 > \frac{n}{2}$, then we can pick $q_0 > \frac{n}{2}$ and we establish (3.7). On the other hand, if $p_0 \le \frac{n}{2}$, then we consequently deduce that for all $q_0 \in (1, \frac{n(p_0+1)}{2(n-p_0+1)})$ there exists $c_0'' > 0$ satisfying

$$\int_\Omega u^{q_0} \le c_0'' \qquad \text{for all } t > 0 \tag{3.11}$$

due to $p_0 \ge \frac{n(p_0-r_0)}{n-2r_0} = \frac{n(p_0+1)}{2(n-p_0+1)}$ when $p_0 \le \frac{n}{2}$.

We proceed the second iteration. We fix a pair (p_1, r_1) such that

$$\begin{cases} p_1 \in \Big(p_0, \min\Big\{\dfrac{1}{\chi_0{}^2}, n+1, \dfrac{p_0(n+2)}{n-2p_0}\Big\}\Big), \\ r_1 := \dfrac{p_1 - 1}{2}. \end{cases} \tag{3.12}$$

Then we see that

$$p_1 > r_1, \quad r_1 < \frac{n}{2} \quad \text{and} \quad r_1 \in (r_-(p_1), r_+(p_1)).$$

Moreover, we can calculate that

$$\begin{aligned} p_1 - r_1 = \frac{p_1+1}{2} &< \frac{\frac{p_0(n+2)}{n-2p_0} + 1}{2} \\ &= \frac{n(p_0+1)}{2(n-2p_0)} = \frac{n(p_0+1)}{2\{(n-p_0+1)-(p_0+1)\}} = \frac{n \cdot \frac{n(p_0+1)}{2(n-p_0+1)}}{n - 2\cdot\frac{n(p_0+1)}{2(n-p_0+1)}}. \end{aligned}$$

Hence, we can find some $q_0 \in (1, \frac{n(p_0+1)}{2(n-p_0+1)})$ satisfying

$$p_1 - r_1 < \frac{nq_0}{n-2q_0}.$$

Noting that $\frac{n}{2}(\frac{1}{q_0} - \frac{1}{p_1-r_1}) < 1$, we deduce from Lemma 2.4 (i) and (3.11) that there exists a constant $c_1 > 0$ such that

$$\|v(\cdot,t)\|_{L^{p_1-r_1}(\Omega)} \le C\Big(1 + \sup_{s\in(0,\infty)} \|u(\cdot,s)\|_{L^{q_0}(\Omega)}\Big) \le c_1 \qquad \text{for all } t > 0$$

and Lemma 3.1 yields that there exists a constant $c_1' > 0$ fulfilling

$$\int_\Omega u^{p_1}v^{-r_1} \le c_1' \qquad \text{for all } t > 0.$$

Using a similar estimate as the first iteration, we have that for all $q_1 \in (1, \min\{p_1, \frac{n(p_1-r_1)}{n-2r_1}\})$ there exists a constant $c_1'' > 0$ such that

$$\int_\Omega u^{q_1} \le c_1'' \qquad \text{for all } t > 0.$$

If we can choose $p_1 > \frac{n}{2}$, then we can pick $q_1 > \frac{n}{2}$ and establish (3.7). Moreover if $p_1 \le \frac{n}{2}$, then we have that for all $q_1 \in (1, \frac{n(p_1+1)}{2(n-p_1+1)})$ there exists a constant $c_1'' > 0$ satisfying

$$\int_\Omega u^{q_1} \le c_1'' \qquad \text{for all } t > 0.$$

Consequently, we can define a pair (p_k, r_k) $(k \in \mathbb{N})$:

$$\text{(3.13)} \qquad \begin{cases} p_k \in \Big(p_{k-1}, \min\Big\{\dfrac{1}{{\chi_0}^2}, n+1, \dfrac{p_{k-1}(n+2)}{n-2p_{k-1}}\Big\}\Big), \\ r_k := \dfrac{p_k - 1}{2}, \end{cases}$$

and if $p_k \le \frac{n}{2}$, then we deduce that for all $q_k \in (1, \frac{n(p_k+1)}{2(n-p_k+1)})$

$$\int_\Omega u^{q_k} \le c_k'' \qquad \text{for all } t > 0$$

with constant $c_k'' > 0$. Because $\frac{2}{n} < \min\{\frac{1}{\chi^2}, n+1\}$ due to the condition $\chi_0 < \sqrt{\frac{2}{n}}$ and the increasing function $f(x) := \frac{x(n+2)}{n-2x}$ satisfies $f(x) > 1$ $(x > 1)$ and $f(x) \to \infty$ as $x \to \frac{n}{2}$, we can obtain some k_0 large enough such that $p_{k_0} > \frac{n}{2}$ and hence $q_{k_0} > \frac{n}{2}$. Therefore we prove (3.7).

(Step 2) In light of L^p-boundedness of solutions (Step 1), we show L^∞-boundedness in this step. Building on Lemma 2.4 (ii), we invoke the standard semigroup technique (e.g. [**21**, Lemma 3.4]) to imply that there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0.$$

Thus we can complete the proof. □

Remark 3.1. Our method in this section can be applied to the general case:

$$\begin{cases} u_t = \Delta u - \chi_0 \nabla \cdot (\frac{u}{v^k} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{3.14}$$

with $k > 1$. Indeed, instead of $h(p, r, \chi_0)$ in (3.4), set

$$\begin{aligned} h(p, r, \chi_0, v) :&= r(r+1) + pr\chi_0 \cdot \frac{1}{v^{k-1}} - \frac{\left[2pr + p(p-1)\chi_0 \cdot \frac{1}{v^{k-1}}\right]^2}{4p(p-1)} \\ &\geq r(r+1) + pr\chi_0 \cdot \frac{1}{\eta^{k-1}} - \frac{\left[2pr + p(p-1)\chi_0 \cdot \frac{1}{\eta^{k-1}}\right]^2}{4p(p-1)}. \end{aligned}$$

Replacing χ_0 with $\bar{\chi_0} := \frac{\chi_0}{\eta^{k-1}}$, we can argue similarly as our proofs. Hence, if

$$\chi_0 < \sqrt{\frac{2}{n}} \cdot \eta^{k-1}$$

we can establish boundedness of solutions to (3.14) with $k > 1$.

4. Proof of Theorem 1.2

In this section we focus on the strongly singular case $\chi(v) = \frac{\chi_0}{v^k}$ ($\chi_0 > 0,\ k > 1$). Firstly, we consider the following regularization of (1.1):

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon \chi_\varepsilon(v_\varepsilon) \nabla v_\varepsilon), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x, 0) = u_0(x), \quad v_\varepsilon(x, 0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where $\varepsilon \in (0, 1)$ and

$$\chi_\varepsilon(s) := \chi(s + \varepsilon), \quad s \geq 0.$$

Then χ_ε belongs to $C^{1+\delta}_{\mathrm{loc}}([0, \infty))$ for some $\delta > 0$ and

$$0 < \chi_\varepsilon(s) = \chi(s + \varepsilon) \leq \frac{\chi_0}{(s + \varepsilon)^k} = \frac{\varepsilon^{-k}\chi_0}{(1 + \frac{1}{\varepsilon}s)^k}.$$

Therefore we can invoke the method in [**19**] to obtain global classical solutions of (4.1). Moreover, we can easily find that u_ε fulfils the mass conservation property

$$\int_\Omega u_\varepsilon(x, t)\, dx \equiv \int_\Omega u_0.$$

In light of Lemma 2.2, we can find a positive constant $\eta > 0$ satisfying

$$\inf_{x \in \Omega} v_\varepsilon(x, t) \geq \eta > 0 \qquad \text{for all } t \geq 0,\ \varepsilon \in (0, 1),$$

where η does not depend on ε and t.

We are now in a position to prove Theorem 1.2. We will apply Winkler's method [**19**] to the approximate problem (4.1) and accomplish the passage to the limit of approximate solutions.

PROOF OF THEOREM 1.2 The proof is divided into three steps.

(Step 1) In this step we prove an independent-in-ε bound on the L^p norm for the approximate solutions u_ε. Using the same method as in [**19**, Lemma 3.1], we see that there exists a constant $C_1 > 0$ such that

$$\sup_{t>0} \|u_\varepsilon(t)\|_{L^p(\Omega)} \le C_1 \quad \text{for all } \varepsilon \in (0,1),\ p > 1.$$

Indeed, from Lemma 2.2 it suffices to make the following upper estimate for χ_ε on $[\eta, \infty)$:

$$\chi_\varepsilon(s) \le \frac{\chi_0}{(s+\varepsilon)^k} \le \frac{\chi_0}{s^k} = \frac{2^k\chi_0}{(s+s)^k} \le \frac{2^k\chi_0}{(s+\eta)^k} \quad \text{for all } s \ge \eta.$$

We remark that in the proof of [**19**, Lemma 3.1] the constant C_1 depends only on the dominating function $\frac{2^k\chi_0}{(s+\eta)^k}$, so that the constant C_1 is independent of ε.

(Step 2) Using Lemma 2.2, we can proceed as in the proof of [**19**, Theorem 3.2] to deduce an independent-in-ε bound on the L^∞ norm for u_ε: there exists a constant $C_2 > 0$ such that

$$\sup_{t>0} \|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{for all } \varepsilon \in (0,1).$$

(Step 3) Finally we construct a solution of (1.1) as the limit of a sequence of solutions to (4.1). This method is due to the proof of [**21**, Theorem 3.5]. For convenience we recall the proof. Since $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty(\bar\Omega \times [0,\infty))$, parabolic Schauder estimate ([**14**]) entails that both $(u_\varepsilon)_{\varepsilon\in(0,1)}$ and $(v_\varepsilon)_{\varepsilon\in(0,1)}$ are bounded in $C^{2+\theta,1+\frac{\theta}{2}}_{\mathrm{loc}}(\bar\Omega \times (0,\infty))$ for some $\theta > 0$. We apply the Arzelà-Ascoli theorem and then infer that there exist a suitable sequence of numbers $\varepsilon_k \searrow 0$ and a pair (u, v) such that $u_{\varepsilon_k} \to u$ and $v_{\varepsilon_k} \to v$ in $C^{2,1}_{\mathrm{loc}}(\bar\Omega \times (0,\infty))$. This pair (u, v) solves the PDEs and the Neumann conditions in (1.1). The initial condition is also checked by parabolic regularity theory and semigroup techniques. Consequently, we have a global classical solution (u, v) of (1.1) such that u belongs to $L^\infty(\bar\Omega \times [0,\infty))$ in light of boundedness of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^\infty(\bar\Omega \times [0,\infty))$; note that this boundedness property is uniform with respect to ε. □

Remark 4.1. By the *time-independent* pointwise lower bound for v (Lemma 2.2), global existence and boundedness are proved in some nonlinear diffusion and cross-diffusion case (F.-Nishiyama-Yokota [**3**]).

Remark 4.2. In [**4**] (joint work with Takasi Senba), global existence and boundedness in the parabolic-elliptic system are established for general sensitivity $\chi \in C^1((0,\infty))$ satisfying $\chi > 0$ and $\chi(s) \to 0$ as $s \to \infty$ in the two dimensional setting.

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