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QUASI-VARIATIONAL INEQUALITIES IN ECONOMIC GROWTH MODELS WITH TECHNOLOGICAL DEVELOPMENT

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1. Introduction

This is a report on the recent work [5] with A. Kadoya and M. Niezgodka. The main objective of this paper is to reconsider economic growth models (cf. [7,8]) in the macroeconomics from a viewpoint of the mathematical theory on quasi-variational inequalities. As many economists pointed out, the technological innovation brings various changes to production systems. Especially it enables to get big output by rather small labor force and this is a very important point for our aging society in the future.

In this paper we propose new economic growth models, taking account of technological development, in which we investigate its influence on the growth of economics. Moreover, we discuss it in a closed system between major economic elements which are capital, technological level, labor force and output; in the classical growth model due to R. M. Solow [8] the most important one was "capital" as well as its dynamics on condition that the evolution of technological level and labor force are prescribed independently each other and the output is prescribed by the other three elements. However, in a complex structure of our future society it is quite natural to suppose that these elements depend on each other and is expected that the production system is formulated as a closed loop between them. Along such a direction we shall propose a simplified model promoting the economic growth.

2. Formulation and theorems

We consider an economic model that includes the formation and dynamics of knowledge-technological ($K\&T$) region in the system:

$$w'(t)+bw(t)=\sigma P(\boldsymbol{L}\cdot\boldsymbol{A}(t))w(t)^{\alpha},\quad t>0, \tag{1}$$

$$r_k'(t)+\partial\psi_k(r_k(t))\ni f_k(w(t),r_k(t)),\quad t>-\tau_0,\ k=1,2,\cdots,N, \tag{2}$$

$$\begin{gathered}\boldsymbol{A}'(t)+\partial I_{K_0(\boldsymbol{r}(t))}(\boldsymbol{A}(t))\ni \boldsymbol{g}(\boldsymbol{r}(t);w(t),\boldsymbol{A}(t))\ \text{ in } \mathbf{R}^N,\quad t>0,\\ \text{with }\ \boldsymbol{r}(t):=(r_1(t),r_2(t),\cdots,r_N(t)),\end{gathered} \tag{3}$$

$$w(t)=w_0(t)\ \text{for}\ t\in[-\tau_0,0],\quad \boldsymbol{A}(0)=\boldsymbol{A}_0,\quad \boldsymbol{r}(-\tau_0)=0. \tag{4}$$

In the macro-economics, $w(t)$ is the capital, $K_0(\boldsymbol{r}(t))$ is the knowledge-technology ($K\&T$) region cultivated by making continuous investiment, and $\boldsymbol{A}(t)$ is the technological level.

The objective of this paper is to construct a global in time solution $\{w, \boldsymbol{r}, \boldsymbol{A}\}$ of (1)-(4) which posseses some properties from the economic point of view; for instance,

(e1) the capital $w(t)$ is non-decreasing in t and the effective labor $\boldsymbol{L} \cdot \boldsymbol{A}(t)$ is non-decreasing in t, too,

(e2) the $K\&T$ region $K_0(\boldsymbol{r}(t))$ is non-decreasing in t and $\boldsymbol{A}(t) \in K_0(\boldsymbol{r}(t))$ for all t,

(e3) $w(t), \boldsymbol{r}(t)$ and $\boldsymbol{L} \cdot \boldsymbol{A}(t)$ converges as $t \to \infty$ to $w_\infty, \boldsymbol{r}_\infty$ and $\boldsymbol{L} \cdot \boldsymbol{A}_\infty$ for any cluster point $\boldsymbol{A}_\infty$ of $\boldsymbol{A}(t)$.

In this work one of important questions is how to set up some specified classes of functions $f_k(w, r_k)$, $\boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A})$ and $K_0(\boldsymbol{r})$ so that the solution posseses the above mentioned properties.

Our problem is treated under the following assumptions:

(A1) $b > 0$, $0 < \sigma < 1$, $0 < \alpha < 1$, $r_0 > 0$ are constants, $\boldsymbol{L} := (L_1, L_2, \cdots, L_N)$ with $L_k > 0$ $(1 \le k \le N)$ and $P(r)$ is a smooth function on $\mathbf{R}_+$ such that

$$P(0) = 0, \quad P'(r) > 0, \ \forall r > 0, \quad \lim_{r \downarrow 0} P'(r) = \infty.$$

(A2) To each vector $\boldsymbol{r} := (r_1, r_2, \cdots, r_N) \in \mathbf{R}_+^N$ a compact and convex subset $K_0(\boldsymbol{r})$ of $\mathbf{R}_+^N$ is asigned so that

$$\text{Int}.K_0(\boldsymbol{r}) \neq \emptyset, \quad \forall \boldsymbol{r} \in \mathbf{R}_+^N \text{ with } r_k > 0, \ k = 1, 2, \cdots, N,$$

and the mapping $\boldsymbol{r} \to K_0(\boldsymbol{r})$ is Lipschitz continuous in the sense of Hausdorff distance in $\mathbf{R}^N$ and monotone increasing in the sense that $K_0(\boldsymbol{r}) \subset K_0(\boldsymbol{r}')$ if $r_k \le r'_k$ $(1 \le k \le N)$ for all $\boldsymbol{r} = (r_1, r_2, \cdots, r_N)$ and $\boldsymbol{r}' = (r'_1, r'_2, \cdots, r'_N) \in \mathbf{R}_+^N$.

(A3) $\psi_k(\cdot)$ is a proper l.s.c., non-negative convex function on $\mathbf{R}$ such that $\psi_k(r) = 0$ for all $r \le r_0$ with a fixed positive number r_0 and $D(\psi_k)$ is bounded from above, say $D(\psi_k) \subset (-\infty, \gamma_k]$ or $(-\infty, \gamma_k)$ for a positive finite number γ_k; hence $\partial\psi_k(r) = 0$ for $r < r_0$ and $R(\partial\psi_k) = \mathbf{R}_+$.

(A4) For each $k = 1, 2, \cdots, N$, $f_k(\cdot, \cdot)$ is a positive, non-decreasing (in each variable) and Lipschitz continuous function on $\mathbf{R}_+^2$. If $\boldsymbol{A} := (A_1, A_2, \cdots, A_N)$ with $0 < A_k < \gamma_k$ and $1 \le k \le N$ and if w is a positive number satisfying $bw = \sigma P(L \cdot \boldsymbol{A}) w^\alpha$, then

$$f_k(w, A_k) > \sup \partial\psi_k(A_k).$$

(A5) $\boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A}) := (g_1(\boldsymbol{r}; w, \boldsymbol{A}), g_2(\boldsymbol{r}; w, \boldsymbol{A}), \cdots, g_N(\boldsymbol{r}; w, \boldsymbol{A}))$ is a Lipschitz continuous function from $\mathbf{R}_+^N \times \mathbf{R}_+ \times \mathbf{R}_+^N$ into $\mathbf{R}_+^N$. If $\boldsymbol{r} \in \mathbf{R}_+^N$ and $\boldsymbol{A} = (A_1, A_2, \cdots, A_N) \in \partial K_0(\boldsymbol{r})$ with $A_k > 0$ for all $k = 1, 2, \cdots, N$, then

$$\boldsymbol{N} \cdot \boldsymbol{L} \ge 0, \quad \forall \boldsymbol{N} \in \mathrm{N}_c(\boldsymbol{A}),$$

and

$$\left\{ \boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A}) - \left(\max_{\boldsymbol{N} \in \mathrm{N}_c(\boldsymbol{A})} \boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A}) \cdot \boldsymbol{N} \right)^+ \boldsymbol{N} \right\} \cdot \boldsymbol{L} \geq 0, \quad \forall w \geq 0,$$

where $\mathrm{N}_c(\boldsymbol{A})$ is the unit normal cone of $K_0(\boldsymbol{r})$ at $\boldsymbol{A}$, namely

$$\mathrm{N}_c(\boldsymbol{A}) := \{\boldsymbol{N} \in \mathbf{R}^N \mid |\boldsymbol{N}| = 1, \ \boldsymbol{N} \cdot (\boldsymbol{x} - \boldsymbol{A}) \leq 0, \ \forall \boldsymbol{x} \in K_0(\boldsymbol{r})\}.$$

Then we have:

Theorem 1. *For the initial data $w_0 \in W_+^{1,2}(-\tau_0, 0)$ and $\boldsymbol{A}_0$ assume that w_0 is positive and non-decreasing on $[-\tau_0, 0]$ and*

$$\boldsymbol{A}_0 \in \mathit{Int}.K_0(\boldsymbol{r}(0), \ bw_0(0) < \sigma P(\boldsymbol{L} \cdot \boldsymbol{A}_0) w_0(0)^\alpha.$$

Furthermore suppose that

$$\begin{gathered} \boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A}) \cdot \boldsymbol{L} > 0, \ \ \forall \boldsymbol{r} := (r_1, r_2, \cdots, r_N) \in \mathbf{R}_+^N \ \text{with} \ r_k > 0, \ k = 1, 2, \cdots, N, \\ \forall w \geq 0, \ \forall \boldsymbol{A} = (A_1, A_2, \cdots, A_N) \in K_0(\boldsymbol{r}) \ \text{with} \ A_k > 0, \ k = 1, 2, \cdots, N. \end{gathered} \tag{5}$$

Then problem (1)-(4) admits at least one global in time solution $\{w, \boldsymbol{r}, \boldsymbol{A}\}$ such that

$$w, \ \boldsymbol{r} \ \text{and} \ \boldsymbol{L} \cdot \boldsymbol{A} \ \text{are non} - \text{decreasing on} \ [0, \infty).$$

Theorem 2. *Under the same assumptions as in Theorem 1, let $\{w, \boldsymbol{r}, \boldsymbol{A}\}$ be any global in time solution of (1)-(4). Then we have:*

(i) $w_\infty := \lim_{t \to \infty} w(t)$, $\boldsymbol{r}_\infty := \lim_{t \to \infty} \boldsymbol{r}(t)$ and $\ell_\infty := \lim_{t \to \infty} \boldsymbol{L} \cdot \boldsymbol{A}(t)$ exist. Moreover, $K_0(\boldsymbol{r}(t))$ converges to $K_0(\boldsymbol{r}_\infty)$ in the sense of Hausdorff distance as $t \to \infty$.

(ii) Let $\boldsymbol{A}_\infty$ be any cluster point of $\boldsymbol{A}(t)$ as $t \to \infty$. Then $\ell_\infty = \boldsymbol{L} \cdot \boldsymbol{A}_\infty$. Moreover, with $\boldsymbol{r}_\infty = (r_{1\infty}, r_{2\infty}, \cdots, r_{N\infty})$ we have:

$$bw_\infty = \sigma P(\boldsymbol{L} \cdot \boldsymbol{A}_\infty) w_\infty^\alpha,$$

$$f_k(w_\infty, r_{k\infty}) \in \partial \psi_k(r_{k\infty}), \ k = 1, 2, \cdots, N,$$
$$\boldsymbol{g}(\boldsymbol{r}_\infty; w_\infty, \boldsymbol{A}_\infty) \cdot \boldsymbol{L} \in \partial I_{K_0(\boldsymbol{r}_\infty)}(\boldsymbol{A}_\infty) \cdot \boldsymbol{L},$$

where $\partial I_{K_0(\boldsymbol{r}_\infty)}(\boldsymbol{A}_\infty) \cdot \boldsymbol{L} = \{\boldsymbol{r}_\infty^* \cdot \boldsymbol{L} \mid \boldsymbol{r}_\infty^* \in \partial I_{K_0(\boldsymbol{r}_\infty)}(\boldsymbol{A}_\infty)\}$.

Remark.1 In general, it is not guaranteed that $\boldsymbol{A}(t)$ converges in $\mathbf{R}^N$ as $t \to \infty$. The uniqueness question of solutions to problem (1)-(4) remains open. See [2,3,4] for related works.

Remark 2. (Quasivariational structure) Let $\{\boldsymbol{r}, w, \boldsymbol{A}\}$ be a solution of our problem (1)-(4). Now we denote by $w = \boldsymbol{Q}\boldsymbol{A}$ the solution of (1) uniquely determined by $\boldsymbol{A}$ and by

$\boldsymbol{r} = \boldsymbol{R}w$ the solution of $\boldsymbol{r}$ of (2) uniquely determined by w. With these operator $\boldsymbol{Q}$ and $\boldsymbol{R}$, system (1)-(3) can be written in one evolution inclusion

$$\boldsymbol{A}'(t) + \partial I_{K_0(\boldsymbol{RQA}(t))}(\boldsymbol{A}(t)) \ni \boldsymbol{g}(\boldsymbol{RQA}(t); \boldsymbol{QA}(t), \boldsymbol{A}(t)).$$

We should note that the convex constraint $K_0(\boldsymbol{RQA}(t))$ depends on the unknown $\boldsymbol{A}(t)$. In this sence, system (1)-(3) includes the quasivariational structure and it is called a quasivariationl problem. For the general theory on quasivariational evolution inclusions, see [6].

3. Examples

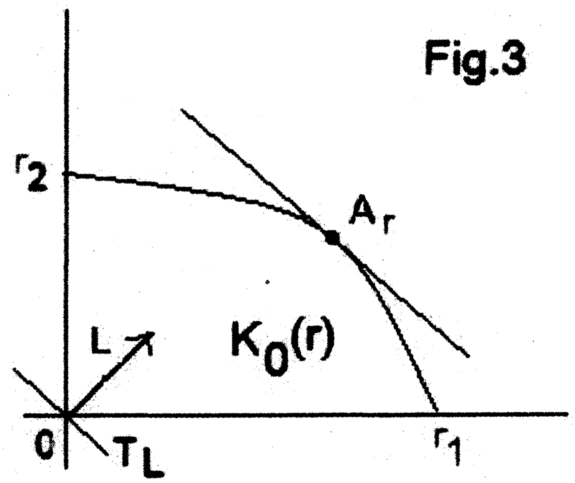
In this section we give some illustrative examples of data $K_0(\boldsymbol{r})$, $\psi_k(\cdot)$, $f_k(w, r_k)$ and $\boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A})$.

(Example of $K_0(\boldsymbol{r})$ and $\boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A})$)

For a finite number $p > 1$ and a large positive constant $M > \max\{\gamma_k \mid k = 1, 2, \cdots, N\}$, we put

$$K_0(\boldsymbol{r}) := \left\{ \boldsymbol{r}' = (r'_1, r'_2, \cdots, r'_N) \in \mathbf{R}^N_+ \;\middle|\; \sum_{i=1}^{N} \frac{{r'_k}^p}{\min\{(r_k)^p, M^p\}} \le 1 \right\}, \quad \forall \boldsymbol{r} \in \mathbf{R}^N_+. \qquad (6)$$

Then it is clear that $\cup_{\boldsymbol{r} \in \mathbf{R}^N_+} K_0(\boldsymbol{r})$ is bounded and the mapping $\boldsymbol{r} \to K_0(\boldsymbol{r})$ satisfies condition (A2). Next, let $\mathbf{T}_{\boldsymbol{L}}$ be the hyperplane cantaining 0, which is orthogonal to $\boldsymbol{L}$. See Fig. 3.



As is easily seen, for each vector $\boldsymbol{r} := (r_1, r_2, \cdots, r_N) \in \mathbf{R}^N_+$, there is one and only one point $\boldsymbol{A_r} = (A_{\boldsymbol{r}1}, A_{\boldsymbol{r}2}, \cdots, A_{\boldsymbol{r}N})$ on $\partial K_0(\boldsymbol{r})$ and a hyperplane parallel to $\mathbf{T}_{\boldsymbol{L}}$ meets with $\partial K_0(\boldsymbol{r})$ at $\boldsymbol{A_r}$. We now define

$$\boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A}) := c_1(w)\boldsymbol{L} + c_2(w)(\boldsymbol{A_r} - \boldsymbol{A}), \quad \forall \boldsymbol{r} \in \mathbf{R}^N_+, \ \forall w \ge 0, \ \forall \boldsymbol{A} \in \mathbf{R}^N_+, \qquad (7)$$

where $c_1(\cdot)$ and $c_2(\cdot)$ are positive, globally bounded and Lipschitz continuous functions on $\mathbf{R}_+$. The mapping $\boldsymbol{r} \to K_0(\boldsymbol{r})$ and the vector field $\boldsymbol{g}$ defined by (6) and (7) satisfy (A5). To check it we observe that

$$\boldsymbol{A_r} \cdot \boldsymbol{L} = \max_{\boldsymbol{A} \in \partial K_0(\boldsymbol{r})} \boldsymbol{A} \cdot \boldsymbol{L} = \max_{\boldsymbol{A} \in K_0(\boldsymbol{r})} \boldsymbol{A} \cdot \boldsymbol{L},$$

whence

$$\boldsymbol{g}(\boldsymbol{r}; w, \boldsymbol{A}) \cdot \boldsymbol{L} = c_1(w)|\boldsymbol{L}|^2 + c_2(w)(\boldsymbol{A_r} - \boldsymbol{A}) \cdot \boldsymbol{L} > 0, \quad \forall \boldsymbol{r} \in \mathbf{R}_+^N,\ \forall w \geq 0,\ \forall \boldsymbol{A} \in K_0(\boldsymbol{r}).$$

Also, for any point $\boldsymbol{A} = (A_1, A_2, \cdots, A_N)$ of $\partial K_0(\boldsymbol{r})$ with $A_k > 0,\ k = 1, 2, \cdots, N$, and any $\boldsymbol{N} \in \mathrm{N}_c(\boldsymbol{A})$ we see from (6) that $\boldsymbol{N} \cdot \boldsymbol{L} \geq 0$. Since $(\boldsymbol{A_r} - \boldsymbol{A}) \cdot \boldsymbol{N} \leq 0$, it turns out that

$$\begin{aligned} & (c_1(w)\boldsymbol{L} + c_2(w)(\boldsymbol{A_r} - \boldsymbol{A})) \cdot \boldsymbol{L} - \{(c_1(w)\boldsymbol{L} + c_2(w)(\boldsymbol{A_r} - \boldsymbol{A})) \cdot \boldsymbol{N}\} \boldsymbol{N} \cdot \boldsymbol{L} \\ = \ & c_1(w)(|\boldsymbol{L}|^2 - |\boldsymbol{L} \cdot \boldsymbol{N}|^2) + c_2(w)\left((\boldsymbol{A_r} - \boldsymbol{A}) \cdot \boldsymbol{L}) - (\boldsymbol{A_r} - \boldsymbol{A}) \cdot \boldsymbol{N}(\boldsymbol{N} \cdot \boldsymbol{L})\right) \\ \geq \ & 0. \end{aligned}$$

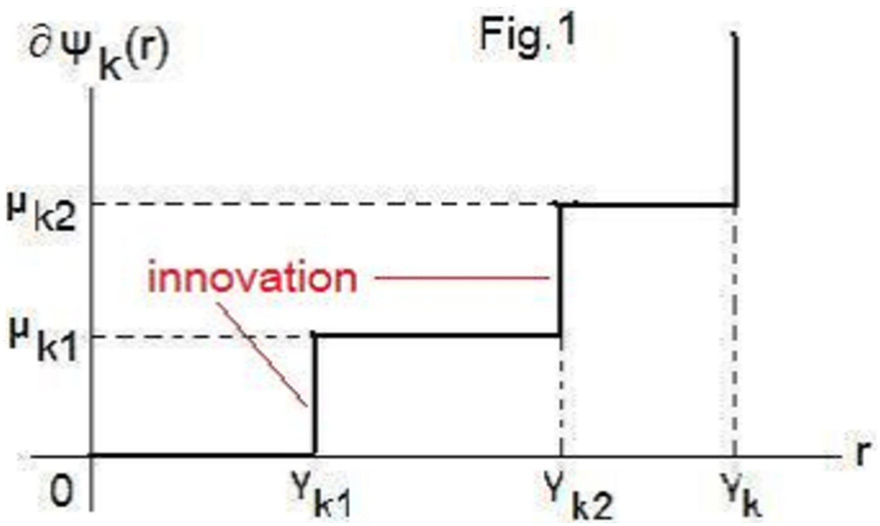
Thus (A5) was checked.

(Example of $\psi_k(\cdot)$ and $f_k(w, r_k)$)

For each $k = 1, 2, \cdots, N$, we give an example of $\psi_k(\cdot)$ appearing in (2) accompanied with scientific innovation. Consider the proper, l.s.c. convex function $\psi_k(\cdot)$ on $\mathbf{R}$ given by

$$\psi_k(r) := \begin{cases} 0, & \text{for } r \leq \gamma_{k1}, \\ \mu_{k1}(r - \gamma_{k1}), & \text{for } \gamma_{k1} < r \leq \gamma_{k2}, \\ \mu_{k2}(r - \gamma_{k2}) + \mu_{k1}(\gamma_{k2} - \gamma_{k1}), & \text{for } \gamma_{k2} < r \leq \gamma_k, \\ \infty, & \text{for } r > \gamma_k, \end{cases} \tag{8}$$

In this case the graph of the subdifferential ψ_k is given by Fig.1:



where $\gamma_{k1},\ \gamma_{k2},\ \mu_{k1}$ and μ_{k2} are constants such that

$$0 < \gamma_{k1} < \gamma_{k2} < \gamma_k < \infty, \quad 0 < \mu_{k1} < \mu_{k2} < \infty.$$

Moreover consider a positive, Lipschitz continuous and non-decreasing function $f_k(w,r)$ on $\mathbf{R}^2_+$ such that

$$f_k(w,r) \geq \varepsilon_0, \quad \forall w \geq 0,\ \forall r \in [0, \gamma_{k1}), \tag{9}$$

$$f_k(w,r) \geq \mu_{k1} + \varepsilon_0, \quad \forall w \geq 0,\ \forall r \in [\gamma_{k1}, \gamma_{k2}), \tag{10}$$

$$f_k(w,r) \geq \mu_{k2} + \varepsilon_0, \quad \forall w \geq 0,\ \forall r \in [\gamma_{k2}, \gamma_k], \tag{11}$$

$$f_k(\mu_{k1}, \gamma_{k1}) > \mu_{k1}, f_k(\mu_{k2}, \gamma_{k2}) > \mu_{k2}. \tag{12}$$

for a positive number ε_0. It is easy to see (A4) from (9)-(12). As a concrete example of $f_k(w,r)$ there is the follwoing function:

$$f_k(w,r) = \varepsilon_0 w + \mu_{k1} f_{k1}(w,r) + (\mu_{k2} - \mu_{k1}) f_{k2}(w,r),$$

where

$$f_{k1}(w,r) = \begin{cases} 0, & \text{for } r \leq r_{k1} - \varepsilon_1, \\ \frac{1}{\varepsilon_1}(r - r_{k1}) + 1, & \text{for } r_{k1} - \varepsilon_1 < r < r_{k1}, \\ 1, & \text{for } r \geq r_{k1}, \end{cases}$$

$$f_{k2}(w,r) = \begin{cases} 0, & \text{for } r \leq r_{k2} - \varepsilon_1, \\ \frac{1}{\varepsilon_1}(r - r_{k2}) + 1, & \text{for } r_{k2} - \varepsilon_1 < r < r_{k2}, \\ 1, & \text{for } r \geq r_{k2}. \end{cases}$$

See Fig.5 which shows the graph of $y = f_k(w,r)$ for each fixed $w > 0$.

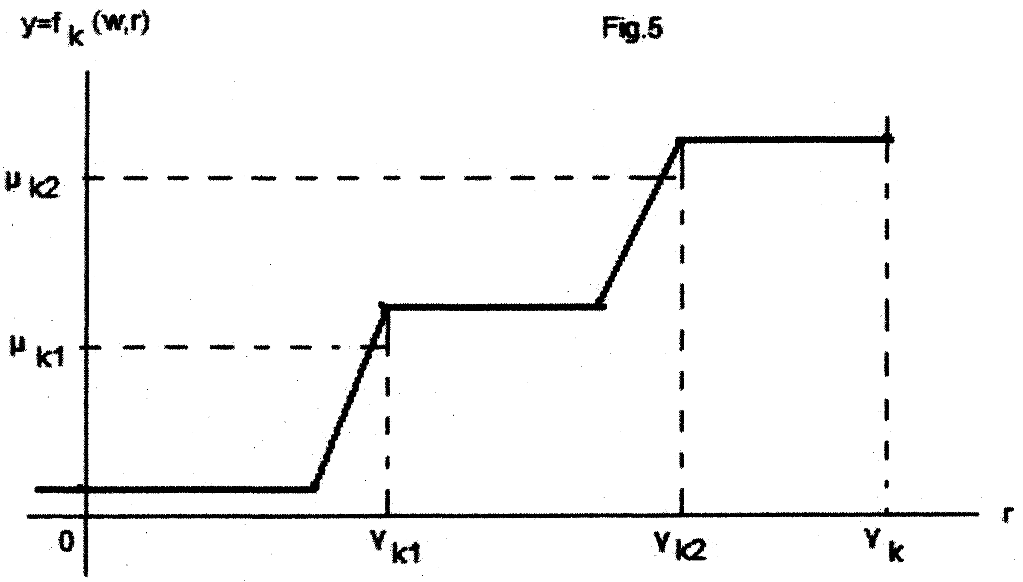


Fig.5

Remark 1. In the above Fig. 1 and 5, it is illustrated that there happen scientific innovations at $r = \gamma_{k1}$ and $r = \gamma_{k2}$ on the subject r_k and it costs a greate deal to recompose their knowledges obtained by innovations as industrial technologies. Also, there is a infinite scientific wall at $r = \gamma_k$, which is the limit of knowledge and one has no ideas how to get it over.

Remark 2. Even in the above example, the uniqueness question of solutions to problem (1)-(4) remains open.

4. Outline of the proof

In this section we mention the outline of the proofs of Theorems 1 and 2; for the complete proofs we refer to the paper [5].

(Existence proof)

We construct a local in time solution of (1)-(4) by the fixed point arguement. Let $\boldsymbol{L}, w_0$ and $\boldsymbol{A}_0$ be as in the statement of Theorem 1, and for a finite time $T > 0$ and a positive constant

$$C_1 > |w_0|_{W^{1,2}(-\tau_0,0)},$$

put

$$X_T(w_0, C_1) := \left\{ w \in W^{1,2}_+(-\tau_0, T) \mid w = w_0 \text{ on } [-\tau_0, 0], |w|_{W^{1,2}(-\tau_0,T)} \le C_1 \right\}. \tag{13}$$

Now, for each $w \in X_T(w_0, C_1)$, solve the problem

$$\begin{cases} r_k'(t) + \partial\psi_k(r_k(t)) \ni f_k(w(t), r_k(t)), & \text{for a.e. } t \in (-\tau_0, T), \; r_k(-\tau_0) = 0, \\ \boldsymbol{A}'(t) + \partial I_{K_0(\boldsymbol{r}(t))}(\boldsymbol{A}(t)) \ni \boldsymbol{g}(\boldsymbol{r}(t); w(t), \boldsymbol{A}(t)) & \text{for a.e. } t \in (0, T), \;\; \boldsymbol{A}(0) = \boldsymbol{A}_0, \end{cases} \tag{14}$$

where $\boldsymbol{r}(t) := (r_1(t), r_2(t), \cdots, r_N(t))$; in fact, problem (14) can be solved by the general theory on evolution equations generated by time-dependent subdifferentials, since $t \to K_0(\boldsymbol{r}(t))$ and $t \to \boldsymbol{g}(\boldsymbol{r}(t); w(t), \boldsymbol{A}(t))$ are regular enough in t. Moreover, we see that there is a positive constant C_2 depending only on C_1, T, $L_{\boldsymbol{g}}$ (Lipschitz constant of $\boldsymbol{g}$) and the initial data w_0, $\boldsymbol{A}_0$ such that the solutions $\boldsymbol{r}$ and $\boldsymbol{A}$ satisfy the following inequality:

$$|\boldsymbol{r}|_{W^{1,2}(0,T;\mathbf{R}^N)} + |\boldsymbol{A}|_{W^{1,2}(0,T;\mathbf{R}^N)} \le C_2, \tag{15}$$

as long as $w \in X_T(w_0, C_1)$. By the way, there is a positive constant C_3 depending only on C_2 and T such that

$$\sup_{0 \le t \le T} \boldsymbol{L} \cdot \boldsymbol{A}(t) \le C_3, \tag{16}$$

as long as $w \in X_T(w_0, C_1)$.

Now, consider the Cauchy problem associated with the solutions $\boldsymbol{r}$ and $\boldsymbol{A}$ of (14)

$$\tilde{w}'(t) + b\tilde{w}(t) = \sigma P(\boldsymbol{L} \cdot \boldsymbol{A}(t))\tilde{w}(t)^\alpha, \quad t \in [0, T], \quad \tilde{w} = w_0 \text{ on } [-\tau_0, 0].$$

Then, this problem has one and only one C^1–solution $\tilde{w}$, and

$$0 < \tilde{w}(t) \le \max\left\{ w_0(0), \left\{\frac{\sigma P(C_3)}{b}\right\}^{\frac{1}{1-\alpha}} \right\} =: C_4, \quad t \in [0,T]. \tag{17}$$

These inequalities are obtained as follows. Since $\sigma P(\boldsymbol{L}\cdot \boldsymbol{A}(t)) \le \sigma P(C_3)$ by (16), comparing $\tilde{w}$ with the solution $\hat{w}$ of

$$\hat{w}'(t) + b\hat{w}(t) = \sigma P(C_3)\hat{w}(t)^\alpha, \quad t \ge 0, \quad \tilde{w} = w_0 \text{ on } [-\tau_0, 0],$$

we get by virtue of the usual comparison results that

$$\tilde{w}(t) \le \hat{w}(t) \le \max\left\{ w_0(0), \left\{\frac{\sigma P(C_3)}{b}\right\}^{\frac{1}{1-\alpha}} \right\} =: C_5; \tag{18}$$

note that $\xi := \left\{\frac{\sigma P(C_3)}{b}\right\}^{\frac{1}{1-\alpha}}$ is the positive root of equation $\sigma P(C_3)X^\alpha - bX = 0$ of X. Thus (18) holds. We see immediately from (17) and (18) that

$$\tilde{w}'(t) \le \sigma P(C_3)C_5^\alpha =: C_6, \quad \forall t \in [0,T].$$

Accordingly, with a small time $T > 0$ satisfying $|w_0|_{W^{1,2}(-\tau_0,0)} + (C_5 + C_6)\sqrt{T} \le C_1$ we see that

$$\begin{aligned} |\tilde{w}|_{W^{1,2}(-\tau_0,T)} &= \left(|w_0|^2_{W^{1,2}(-\tau_0,0)} + \int_0^T (|\tilde{w}(t)|^2 + |\tilde{w}'(t)|^2)dt \right)^{\frac{1}{2}} \\ &\le |w_0|_{W^{1,2}(-\tau_0,0)} + \left(\int_0^T |\tilde{w}(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int_0^T |\tilde{w}'(t)|^2 dt\right)^{\frac{1}{2}} \\ &\le |w_0|_{W^{1,2}(-\tau_0,0)} + (C_5 + C_6)\sqrt{T} \le C_1. \end{aligned}$$

This shows that the mapping $\mathcal{S}$, which is defined by $\mathcal{S}(w) = \tilde{w}$ via the solution $\boldsymbol{A}$ of (3), maps $X_T(w_0, C_1)$ into itself. Moreover it is not difficult to derive that the set $X(w_0; C_1)$ is non-empty, compact and convex in $C([-\tau_0, T])$ and $\mathcal{S}$ is continuous in it with respect to the topology of $C([-\tau_0, T])$.

We now apply the Schauder's fixed point theorem to the mapping $\mathcal{S}$ in $X_T(w_0, C_1)$ to find at least one fixed point $w \in X_T(w_0, C_1)$ of $\mathcal{S}$, namely $w = \mathcal{S}(w)$. By the definition of $\mathcal{S}$ it is easy to see that the tripret $\{w, \boldsymbol{r}, \boldsymbol{A}\}$, with the solutions $\boldsymbol{r}$ and $\boldsymbol{A}$ of (14), gives solutions of (1)-(3) on $[-\tau_0, T]$ or $[0,T]$ with (4). Furthermore, it is a standard work to extend this local in time solution on the whole time interval $[-\tau_0, \infty)$ or $[0, \infty)$

(The monotonicity properties of w, $\boldsymbol{r}$ and $\boldsymbol{L}\cdot\boldsymbol{A}$ in time)

Let $\{w, \boldsymbol{r}, \boldsymbol{A}\}$ be a global in time solution of (1)-(4). In our proof the main point is to show that $\boldsymbol{L}\cdot\boldsymbol{A}(t)$ is non-decreasing in $t \ge 0$.

First of all, we prepare the statement:

Lemma 1. *If w is non-decreasing on an interval $[0, t']$, then the solution r_k of (2) is also non-decreasing on $[0, t']$.*

We shall use this below.

(Step 1) Now, we put

$$t_0 := \sup\{t \geq 0 \mid \boldsymbol{L} \cdot \boldsymbol{A} \text{ is non} - \text{decreasing on } [0, t]\}.$$

Since $\boldsymbol{A}_0$ is given in the interior of $K_0(\boldsymbol{r}(0))$ and $t \to K_0(\boldsymbol{r}(t))$ is continuous in the sense of Hausdorff distance (cf. (A2)), we see that $\boldsymbol{A}(t)$ is in the interior of $K_0(\boldsymbol{r}(t))$ for an small time interval $[0, t']$, $t' > 0$, so that $\partial I_{K_0(\boldsymbol{r}(t))}(\boldsymbol{A}(t)) = 0$ for all $t \in [0, t']$. This implies by (3) that $\boldsymbol{A}'(t) = \boldsymbol{g}(t) := \boldsymbol{g}(\boldsymbol{r}(t); w(t), \boldsymbol{A}(t))$ for a.e. $t \in [0, t']$. From our assumption we have that $\boldsymbol{g}(t) \cdot \boldsymbol{L} \geq 0$, whence $\boldsymbol{A}'(t) \cdot \boldsymbol{L} \geq 0$ for a.e. $t \in [0, t']$. This implies that $\boldsymbol{A}(t) \cdot \boldsymbol{L}$ is non-decreasing on $[0, t']$. Hence $t_0 > 0$. Our claim is to show that $t_0 = \infty$. For a contradiction, suppose that $t_0 < \infty$. In this case we have $\boldsymbol{A}(t_0) \in \partial K_0(\boldsymbol{r}(t_0))$. Otherwise, since $\boldsymbol{A}(t_0) \in$ Int.$K_0(\boldsymbol{r}(t_0))$, by repeating the same argument as above we deduce that $\boldsymbol{L} \cdot \boldsymbol{A}(t)$ is non-decreasing on an interval $[t_0, t_0']$, $t_0' > t_0$. This contradicts the definition of t_0.

Since $\sigma P(\boldsymbol{L} \cdot \boldsymbol{A}(t))$ is non-decreasing on $[0, t_0]$ by (A1), it follows from the usual comparison result that w is non-decreasing and of C^1 on $[0, t_0]$, namely $w' \geq 0$ on $[0, t_0]$. If $w'(t_0) > 0$, then w is non-decreasing $[-\tau_0, t_0 + \delta_0]$ for some positive number δ_0. Therefore it follows from Lemma 1 that the solutions $r_k(t)$, $k = 1, 2, \cdots, N$, of (2), namely $\boldsymbol{r}(t)$ is non-decreasing on $[-\tau_0, t_0 + \delta_0]$. This implies by (A2) that the mapping $t \to K_0(\boldsymbol{r}(t))$ is non-decreasing in $\mathbf{R}_+^N$ with respect to $t \in [-\tau_0, t_0 + \delta_0]$. In another case of $w'(t_0) = 0$, it holds that $\boldsymbol{A}(t_0) \in \partial K_0(\boldsymbol{r}(t_0))$ and $bw(t_0) = \sigma P(\boldsymbol{L} \cdot \boldsymbol{A}(t_0)) w(t_0)^\alpha$. Therefore, by (A4),

$$f_k(w(t_0), r_k(t_0)) > \sup \partial \psi_k(r_k(t_0)) \qquad (19).$$

Here, we apply Theorem 3.5 in [1] to see that the right-derivative $\frac{d^+}{dt} r_k(t)$ exists at every $t \geq 0$ and

$$\frac{d^+}{dt} r_k(t) = \inf_{\xi \in \partial \psi_k(r_k(t_0))} |f_k(w(t), r_k(t)) - \xi|.$$

In the present case, by (19) it turns out that

$$\frac{d^+}{dt} r_k(t_0) = \inf_{\xi \in \partial \psi_k(r_k(t_0))} (f_k(w(t_0), r_k(t_0)) - \xi) > 0,$$

which implies that $r_k(t)$ is increasing on an interval $[t_0, t_0 + \delta_0]$ for a small $\delta_0 > 0$, $k = 1, 2, \cdots, N$. As a consequence, we observe that $\boldsymbol{r}(t)$ is non-decreasing on $[0, t_0 + \delta_0]$, and so is $t \to K_0(\boldsymbol{r}(t))$ on $[0, t_0 + \delta_0]$.

As was seen above, in any case the mapping $t \to K_0(\boldsymbol{r}(t))$ is non-decreasing in $\mathbf{R}_+^N$ on $[0, t_0 + \delta_0]$ for a small positive number $\delta_0 > 0$.

(Step 2) Now, put

$$E = \{t \in [0, t_0 + \delta_0] \mid \boldsymbol{A}(t) \in \partial K_0(\boldsymbol{r}(t))\}.$$

We pay our attention to the equation of $\boldsymbol{A}$ which is written in the form:

$$\boldsymbol{A}^*(t) := \boldsymbol{g}(\boldsymbol{r}(t); w(t), \boldsymbol{A}(t)) - \boldsymbol{A}'(t) \in \partial I_{K_0(\boldsymbol{r}(t))}(\boldsymbol{A}(t)) \quad \text{for a.e. } t \in [0, t_0 + \delta_0],$$

subject to the initial condition $\boldsymbol{A}(0) = \boldsymbol{A}_0$. Here we note from the definition of subdifferentials of indicator functions that

$$\partial I_{K_0(\boldsymbol{r}(t))}(\boldsymbol{A}) = \begin{cases} 0, & \text{if } \boldsymbol{A} \in \text{Int.}K_0(\boldsymbol{r}(t)), \\ \{c\boldsymbol{N} \mid c \in \mathbf{R}_+,\ \boldsymbol{N} \in \mathrm{N}_c(\boldsymbol{A})\}, & \text{if } \boldsymbol{A} \in \partial K_0(\boldsymbol{r}(t)). \end{cases}$$

Therefore $\boldsymbol{A}^*(t)$ should be of the form:

$$\boldsymbol{A}^*(t) = \begin{cases} 0, & \text{if } t \neq E, \\ c(t)\boldsymbol{N}(t), & \text{if } t \in E, \end{cases} \tag{20}$$

where $c(t)$ is non-negative function of $t \in E$ and $\boldsymbol{N}(t)$ is an element in the normal cone $\mathrm{N}_c(\boldsymbol{A}(t))$ for $t \in E$; note that E is a closed in $\mathbf{R}_+$ and $c(\cdot)\boldsymbol{N}(\cdot) \in L^2(E; \mathbf{R}^N)$.

Next, at each point $\boldsymbol{A}(t) \in \partial K_0(\boldsymbol{r}(t))$ we decompose the forcing term $\boldsymbol{g}$ into the normal and tangetial components:

$$\boldsymbol{g}(t) := \boldsymbol{g}(\boldsymbol{r}(t); w(t), \boldsymbol{A}(t)) = g_N(t)\boldsymbol{N}(t) + \boldsymbol{g}_T(t), \quad \boldsymbol{g}_T(t) := \boldsymbol{g}(t) - g_N(t)\boldsymbol{N}(t),$$

where $\boldsymbol{N}(t)$ is the same normal vector as in (20) and $g_N(t) := \boldsymbol{g}(t) \cdot \boldsymbol{N}(t)$.

Lemma 2. *We have that* $0 \leq c(t) \leq g_N(t)$ *for a.e.* $t \in E$.

Proof. At each point $\boldsymbol{A}(t) \in \partial K_0(\boldsymbol{r}(t))$ we observe that $\boldsymbol{A}^*(t) \cdot \boldsymbol{A}'(t) \geq 0$. In fact, since $K_0(\boldsymbol{r}(t))$ is non-decreasing, it follows from the definition of subdifferential of $I_{K_0(\boldsymbol{r}(t))}$ that

$$\boldsymbol{A}^*(t) \cdot (\boldsymbol{A}(t) - \boldsymbol{A}(t - \delta)) \geq 0$$

for all $\delta > 0$. Hence, by deviding the both sides by δ and taking the limit as $\delta \downarrow 0$, we get $\boldsymbol{A}^*(t) \cdot \boldsymbol{A}'(t) \geq 0$.

Next, we multiply equation (3) by $\boldsymbol{A}^*(t)$ to obtain

$$|\boldsymbol{A}^*(t)|^2 \leq (g_N(t)N(t) + \boldsymbol{g}_T(t)) \cdot \boldsymbol{A}^*(t)$$

for any $t \in E$. Since $\boldsymbol{A}^*(t) = c(t)\boldsymbol{N}(t)$ and $\boldsymbol{g}(t) \cdot \boldsymbol{A}^*(t) = 0$, it follows from the above inequality that $c(t)^2 \leq g_N(t)c(t)$, hence $0 \leq c(t) \leq g_N(t)$. ◇

Here, taking the inner product between $\boldsymbol{L}$ and the both sides of

$$\boldsymbol{A}'(t) = \boldsymbol{g}_T(t) + (g_N(t) - c(t))\boldsymbol{N}(t)$$

at any point $\boldsymbol{A}(t)$ with $t \in E$, we derive from contition (A5) and Lemma 2 that

$$\boldsymbol{A}'(t) \cdot \boldsymbol{L} = \boldsymbol{g}_T(t) \cdot \boldsymbol{L} + (g_N(t) - c(t))\boldsymbol{N}(t) \cdot \boldsymbol{L} \geq 0.$$

Also, at any point $\boldsymbol{A}(t)$, $t \neq E$, namely $\boldsymbol{A}(t) \in \text{Int.}K_0(\boldsymbol{r}(t))$ we have by (5) that

$$\boldsymbol{A}'(t) \cdot \boldsymbol{L} = \boldsymbol{g}(t) \cdot \boldsymbol{L} \geq 0.$$

As a cosequence the inequality $\boldsymbol{A}'(t) \cdot \boldsymbol{L} \geq 0$ holds for a.e. $t \in [0, t_0 + \delta_0]$, and thus $\boldsymbol{A} \cdot \boldsymbol{L}$ is non-decreasing on $[0, t_0 + \delta_0]$. This contradicts the definition of t_0. Thus $t_0 = \infty$ holds.

(Step 3) Finally we show that w is non-decreasing on $[0, \infty)$. In fact, since $\boldsymbol{L} \cdot \boldsymbol{A}(t)$ is non-decreasing on $[0, \infty)$, the coefficient $\sigma P(\boldsymbol{L} \cdot \boldsymbol{A}(t))$ of the equation

$$w'(t) + bw(t) = \sigma P(\boldsymbol{L} \cdot \boldsymbol{A}(t))w(t)^{\alpha}$$

is non-decreasing on $[0, \infty)$, too. Hence, w is non-decreasing on $[0, \infty)$.

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