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# Time-periodic problem for the compressible Navier-Stokes-Korteweg system on $\mathbb{R}^3$

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## 1 Introduction

We consider time periodic problem for the following compressible Navier-Stokes-Korteweg system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t\rho+\operatorname{div}M=0, & (1.1)\\ \partial_tM+\operatorname{div}\left(\dfrac{M\otimes M}{\rho}\right)=\operatorname{div}\left(\mathcal{S}(\dfrac{M}{\rho})+\mathcal{K}(\rho)\right)+\rho g, & (1.2)\\ \partial_t(\rho E)+\operatorname{div}(ME)+\operatorname{div}\left(P(\rho,\theta)\dfrac{M}{\rho}\right) & \\ \qquad\qquad =\tilde{\alpha}\Delta\theta+\operatorname{div}\left(\left(\mathcal{S}(\dfrac{M}{\rho})+\mathcal{K}(\rho)\right)\dfrac{M}{\rho}\right)+Mg. & (1.3)\end{cases}$$

Here $\rho=\rho(x,t)$, $M=(M_1(x,t),M_2(x,t),M_3(x,t))$ and $E=E(x,t)>0$ denote the unknown density, momentum, and total energy respectively, at time $t\in\mathbb{R}$ and position $x\in\mathbb{R}^3$; $\theta$ denotes the absolute temperature of fluid satisfying

$$E=C_v\theta+\frac{1}{2}\frac{|M|^2}{\rho^2},$$

where $C_v$ denotes the heat capacity at the constant volume, that is assumed to be a positive constant; $\mathcal{S}$ and $\mathcal{K}$ denote the viscous stress tensor and the Korteweg stress tensor that are given by

$$\begin{cases} \mathcal{S}(\frac{M}{\rho})=\left(\mu'\operatorname{div}\frac{M}{\rho}\right)\delta_{i,j}+2\mu d_{ij}\left(\frac{M}{\rho}\right),\\ \mathcal{K}(\rho)=\frac{\kappa}{2}(\Delta\rho^2-|\nabla\rho|^2)\delta_{i,j}-\kappa\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j},\end{cases}\tag{1.4}$$

where $d_{ij}\left(\frac{M}{\rho}\right)=\frac{1}{2}\left(\frac{\partial}{\partial x_i}\left(\frac{M}{\rho}\right)_j+\frac{\partial}{\partial x_j}\left(\frac{M}{\rho}\right)_i\right)$; $\mu$ and $\mu'$ are the viscosity coefficients that are assumed to be constants satisfying

$$\mu>0,\quad \frac{2}{3}\mu+\mu'\geq 0;$$

$P = P(\rho, \theta)$ is the pressure that is assumed to be a smooth function of $\rho$ and $\theta$ satisfying

$$P_\rho(\rho_*, \theta_*) > 0, \quad P_\theta(\rho_*, \theta_*) > 0,$$

where $\rho_*$ and $\theta_*$ are given positive constants; $\kappa$ and $\tilde{\alpha}$ denote the capillary constant and the heat conductivity coefficient respectively, that are assumed to be positive constants; and $g = g(x,t)$ is a given external force periodic in $t$. We assume that $g = g(x,t)$ satisfies the condition

$$g(x, t+T) \quad = \quad g(x,t) \quad (x \in \mathbb{R}^3,\ t \in \mathbb{R}) \tag{1.5}$$

for some constant $T > 0$.

The system (1.1)-(1.3) is known to be a model system for two phase flow with phase transition between liquid and vapor in compressible fluid. In deriving (1.1)-(1.3), phase transition boundary is regarded as a diffuse interface. So (1.1)-(1.3) describes fluid state by the changes of the density. (Cf., [4, 6, 11] for the derivation of (1.1)-(1.3).)

As for the mathematical analysis for (1.1)-(1.3), most of literatures treated the system in terms of the density $\rho$, velocity $v = M/\rho$ and absolute temperature $\theta$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, & (1.6) \\ \rho(\partial_t v + (v \cdot \nabla) v) + \nabla P(\rho, \theta) = \mu \Delta v + (\mu + \mu') \nabla \operatorname{div} v + \kappa \rho \nabla \Delta \rho + \rho g, & (1.7) \\ \rho C_v (\theta_t + (v \cdot \nabla)\theta) + \theta P_\theta(\rho, \theta) \operatorname{div} v = \tilde{\alpha} \Delta \theta + \Psi(v) + \tilde{\Phi}(\rho, v), & (1.8) \end{cases}$$

where $\Psi(v)$ and $\tilde{\Phi}(\rho, v)$ are given by

$$\begin{cases} \Psi(v) = \mu' (\operatorname{div} v)^2 + 2\mu \mathbb{D}v : \mathbb{D}v, \quad \mathbb{D}v = (d_{ij}(v))_{i,j=1}^3, \\ \tilde{\Phi}(\rho, v) = \kappa \left( \frac{|\nabla \rho|^2}{2} + \rho \Delta \rho \right) \operatorname{div} v - \kappa (\nabla \rho \otimes \nabla \rho) : \nabla v. \end{cases}$$

Chen and Zhao ([3]) considered the stationary problem (1.6)-(1.8) for $g$ of the form $g(x) = \operatorname{div} g_1(x) + g_2(x)$ around $(\rho_*, 0, \theta_*)$. It was shown in [3] that if $g$ satisfies

$$\begin{aligned} &\sum_{k=1}^{3} \|(1+|x|)^{k+1} \nabla^k g\|_{L^2} + \sum_{k=0}^{1} \|(1+|x|)^{3+k} \nabla^k g\|_{L^\infty} \\ &\quad + \|(1+|x|)^2 g_1\|_{L^\infty} + \|(1+|x|)^{-1} g_2\|_{L^1} \ll 1, \end{aligned} \tag{1.9}$$

then there exists a stationary solution for problem (1.6)-(1.8) in the weighted $L^\infty \cap L^2$ space. The stability of the stationary solution was also considered in [3]. It was shown in [3] that if $g$ satisfies (1.9), then the stationary solution $(\rho^*, v^*, \theta^*)$ is asymptotically stable under sufficiently small initial perturbations, and the perturbation satisfies

$$\|(\rho(t), v(t), \theta(t)) - (\rho^*, v^*, \theta^*)\|_{L^\infty} \to 0$$

as $t \to \infty$. Chen, Xiao and Zhao ([2]) and Cai, Tan and Xu ([1]) then considered time periodic problem for the barotropic and non-barotropic system of (1.6)-(1.8), respectively, on $\mathbb{R}^n$ with $n \geq 5$. They proved that there exists a time periodic solution $(\rho_{per}, v_{per}, \theta_{per})$ around $(\rho_*, 0, \theta_*)$ for a sufficiently small $g \in C^0(\mathbb{R}; H^{N-1} \cap L^1)$ satisfying (1.5), where

$N \in \mathbb{Z}$ satisfying $N \geq n+2$. Furthermore, the time periodic solution is stable under sufficiently small perturbations and it holds that

$$\|(\rho(t)-\rho_{per}(t), v(t)-v_{per}(t), \theta(t)-\theta_{per}(t))\|_{L^\infty} \to 0 \quad (t \to \infty).$$

In this paper we consider time periodic problem for (1.1)-(1.3) instead of (1.6)-(1.8). We will show the existence of a time periodic solution for (1.1)-(1.3) around $(\rho_*, 0, E_*)$ on $\mathbb{R}^3$ with $E_* = C_v\theta_*$. It will be proved that if $g$ satisfies (1.5) and

$$\|g\|_{C([0,T];L^1)} + \|(1+|x|^3)g\|_{C([0,T];L^\infty)} + \|(1+|x|^2)g\|_{L^2(0,T;H^{s-1})} \ll 1$$

for an integer $s \geq 2$, then there exists a time periodic solution $(\rho_{per}-\rho_*, M_{per}, E_{per}-E_*) \in C([0,T];H^s)$ with period $T$ for (1.1)-(1.3), and $(\rho_{per}-\rho_*, M_{per}, E_{per}-E_*)$ satisfies the estimate

$$\begin{aligned}
&\sup_{t\in[0,T]}\Big\{\sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j(\rho_{per}-\rho_*)(t)\|_{L^\infty} + \sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j M_{per}(t)\|_{L^\infty} \\
&\quad + \sum_{j=0}^{1}\|(1+|x|^{1+j})\partial_x^j(E_{per}-E_*)(t)\|_{L^\infty}\Big\} \\
&\leq C(\|g\|_{C([0,T];L^1)} + \|(1+|x|^3)g\|_{C(0,T;L^\infty)} + \|(1+|x|^2)g\|_{L^2(0,T;H^{s-1})}). \qquad (1.10)
\end{aligned}$$

Furthermore, the time periodic solution $(\rho_{per}, M_{per}, E_{per})$ for (1.1)-(1.3) is asymptotically stable under sufficiently small initial perturbations and the perturbation satisfies

$$\|(\rho(t), M(t), E(t)) - (\rho_{per}(t), M_{per}(t), E_{per}(t))\|_{L^\infty} \to 0 \quad (t \to \infty).$$

The precise statements of our results are given in Theorem 2.1 and Theorem 2.2 below.

The existence of time periodic solution is proved by using the time-$T$-map for the linearized semigroup at $(\rho_*, 0, E_*)$. We will employ a function space of hybrid type which, roughly speaking, consists of functions whose low frequency parts belong to a weighted $L^\infty \cap L^2$ space and high frequency parts belong to a weighted $L^2$-Sobolev space. For the low frequency part we introduce a function space similar to that employed in the study of the stationary problem in [3], that is, a set of periodic functions with values in a weighted $L^\infty \cap L^2$ space similar to (1.9). We investigate the spatial decay properties of the integral kernel of the time-$T$-map, and establish the estimates for the low frequency part by a potential theoretic method. Due to the conservation form of momentum and total energy we can estimate the nonlinear terms for the low frequency part directly. As for the high frequency part, we employ the weighted energy method to obtain the a priori estimates. Note that by making use of the smoothing effect for $\rho$ due to the term $\kappa\nabla\Delta\rho$ arising in the Korteweg tensor, the derivative loss due to the term $v\cdot\nabla\rho$ does not occur for the high frequency part and we can directly treat (1.1)-(1.3).

The asymptotic stability of the time periodic solution $(\rho_{per}, M_{per}, E_{per})$ is proved by the energy method using the Hardy inequality as in [3, 7].

## 2 Main results

To state our results, we define function spaces with spatial weight.

For a nonnegative integer $\ell$ and $1 \leq p \leq \infty$, we denote by $L^p_\ell$ the weighted $L^p$ space defined by

$$L^p_\ell = \{u \in L^p; \|u\|_{L^p_\ell} := \|(1+|x|)^\ell u\|_{L^p} < \infty\}.$$

Let $k$ and $\ell$ be nonnegative integers. We define the weighted $L^2$-Sobolev space $H^k_\ell$ by

$$H^k_\ell = \{u \in H^k; \|u\|_{H^k_\ell} < +\infty\},$$

where

$$H^k_\ell = \left\{ u \in H^k; \|u\|_{H^k_\ell} := \left( \sum_{|\alpha| \leq k} \|\partial_x^\alpha u\|^2_{L^2_\ell} \right)^{\frac{1}{2}} < +\infty \right\}$$

We also introduce function spaces of $T$-periodic functions in $t$. We denote by $C_{per}(\mathbb{R}; X)$ the set of all $T$-periodic continuous functions with values in $X$ equipped with the norm $\|\cdot\|_{C([0,T];X)}$; and we denote by $L^2_{per}(\mathbb{R}; X)$ the set of all $T$-periodic locally square integrable functions with values in $X$ equipped with the norm $\|\cdot\|_{L^2(0,T;X)}$.

Our result on the existence of a time periodic solution is stated as follows.

**Theorem 2.1.** *Let $s$ be an integer satisfying $s \geq 2$. Assume that $g(x,t)$ satisfies* (1.5) *and $g \in C_{per}(\mathbb{R}; L^1 \cap L^\infty_3) \cap L^2_{per}(\mathbb{R}; H^{s-1}_2)$. Set*

$$[g]_s = \|g\|_{C([0,T];L^1 \cap L^\infty_3)} + \|g\|_{L^2(0,T;H^{s-1}_2)}.$$

*Then there exists a constant $\delta > 0$ such that if $[g]_s \leq \delta$, then the system* (1.1)-(1.3) *has a time periodic solution $u_{per} = {}^\top(\rho_{per} - \rho_*, M_{per}, E_{per} - E_*) \in C_{per}(\mathbb{R}; L^\infty)$ with $\nabla u_{per} \in C_{per}(\mathbb{R}; H^s \times H^{s-1})$ satisfying*

$$\sup_{t \in [0,T]} \left( \|(1+|x|) u_{per}(t)\|_{L^\infty} + \|(1+|x|)^2 \nabla u_{per}(t)\|_{L^\infty} \right) \leq C[g]_s.$$

Our next issue to study the stability of the time periodic solution obtained in Theorem 2.1. Let ${}^\top(\rho_{per}, M_{per}, E_{per})$ be the time-periodic solution obtained in Theorem 2.1, let the perturbation be denoted by $\tilde{u} = {}^\top(\tilde{\rho}, \tilde{M}, \tilde{E})$, where $\tilde{\rho} = \rho - \rho_{per}, \tilde{M} = M - M_{per}, \tilde{E} = E - E_{per}$ and let the initial perturbation be denoted by

$$\tilde{u}_0 = \tilde{u}|_{t=0} = {}^\top(\rho(0) - \rho_{per}(0), M(0) - M_{per}(0), E(0) - E_{per}(0)).$$

We have the following stability result of the time periodic solution.

**Theorem 2.2.** *Let $s$ be an integer satisfying $s \geq 2$. Assume that $g(x,t)$ satisfies* (1.5) *and $g \in C_{per}(\mathbb{R}; L^1 \cap L^\infty_3) \cap L^2_{per}(\mathbb{R}; H^s_2)$. Let ${}^\top(\rho_{per}, M_{per}, E_{per})$ be the time-periodic solution obtained in Theorem* 2.1 *and let $\tilde{u}_0 \in H^{s+1} \times H^s$. Then there exist constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that if*

$$[g]_{s+1} \leq \epsilon_1, \quad \|\tilde{u}_0\|_{H^{s+1}\times H^s} \leq \epsilon_2,$$

*then $\tilde{u}(t)$ exists globally in time and $\tilde{u}(t)$ satisfies*

$$\begin{aligned}
&\tilde{u} \in C([0,\infty); H^{s+1} \times H^s),\\
&\|\tilde{u}(t)\|^2_{H^{s+1}\times H^s} + \int_0^t \|\nabla \tilde{u}(\tau)\|^2_{H^{s+1}\times H^s} d\tau \leq C\|\tilde{u}_0\|^2_{H^{s+1}\times H^s} \quad (t \in [0,\infty)),\\
&\|\tilde{u}(t)\|_{L^\infty} \to 0 \quad (t \to \infty).
\end{aligned}$$

Theorem 2.2 is proved as follows. We write (1.1)-(1.3) into (1.6)-(1.8). Let ${}^\top(\rho_{per}, M_{per}, E_{per})$ be the periodic solution given in Theorem 2.1. We set $v_{per}$, $\theta_{per}$ and $U_{per}$ by

$$v_{per} = \frac{M_{per}}{\rho_{per}}, \ \theta_{per} = \frac{1}{C_v}\Big(E_{per} - \frac{|M_{per}|^2}{2\rho_{per}^2}\Big), \ U_{per} = {}^\top(\rho_{per}, v_{per}, \theta_{per}).$$

It follows from Theorem 3.1 that $U_{per}$ satisfies the estimate

$$\|{}^\top(v_{per}, \theta_{per} - \theta_*)\|_{C([0,T];L^\infty_1)} \leq C[g]_{s+1}, \tag{2.1}$$

$$\|\nabla\{{}^\top(v_{per}, \theta_{per} - \theta_*)\}\|_{C([0,T];L^\infty_2)} \leq C[g]_{s+1}. \tag{2.2}$$

Let the perturbation be denoted by $U = {}^\top(\phi, w, \vartheta)$, where $\phi = \rho - \rho_{per}$, $w = v - v_{per}$, $\vartheta = \theta - \theta_{per}$. Then the perturbation $U = {}^\top(\phi, w, \vartheta)$ is governed by

$$\left\{\begin{aligned}
&\partial_t\phi + v_{per}\cdot\nabla\phi + \phi\mathrm{div}v_{per} + \rho_{per}\mathrm{div}w + w\cdot\nabla\rho_{per} = f^1,\\
&\partial_t w - \tfrac{1}{\rho_{per}}\{\mu\Delta w + (\mu+\mu')\nabla\mathrm{div}\, w\}\\
&\qquad + B_1(U, U_{per})\nabla\phi - \kappa\nabla\Delta\phi + B_2(U, U_{per})\nabla\vartheta = f^2,\\
&\partial_t\vartheta - \tilde{\alpha}B_3(U_{per})\Delta\vartheta + B_4(U, U_{per})\mathrm{div}\, w = f^3,
\end{aligned}\right. \tag{2.3}$$

where

$$\begin{aligned}
f^1 &= -\mathrm{div}(\phi w),\\
f^2 &= -(v_{per}\cdot\nabla)w - (w\cdot\nabla)(v_{per}+w) - (B_1(U,U_{per}) - B_1(U_{per}))\nabla\rho_{per}\\
&\quad - (B_2(U,U_{per}) - B_2(U_{per}))\nabla\theta_{per}\\
&\quad - \frac{\phi}{\rho_{per}(\rho_{per}+\phi)}\Big\{\mu\Delta(v_{per}+w) + (\mu+\mu')\nabla\mathrm{div}\,(v_{per}+w)\Big\},
\end{aligned}$$

$$
\begin{aligned}
f^3 = &-(v_{per}\cdot\nabla)\vartheta-(w\cdot\nabla)(\theta_{per}+\vartheta)+\tilde{\alpha}(B_3(U,U_{per})-B_3(U_{per}))\Delta(\theta_{per}+\vartheta)\\
&+(B_3(U,U_{per})-B_3(U_{per}))(\Psi(v_{per})+\tilde{\Phi}(\rho_{per},v_{per}))\\
&+B_3(U,U_{per})\Big\{\Psi(v)-\Psi(v_{per})+\tilde{\Phi}(\rho,v)-\tilde{\Phi}(\rho_{per},v_{per})\Big\}\\
&-(B_4(U,U_{per})-B_4(U_{per}))\mathrm{div}\, v_{per},
\end{aligned}
$$

$$
\begin{aligned}
B_1(U,U_{per})&=\frac{P_\rho(\rho_{per}+\phi,\theta_{per}+\theta)}{\rho_{per}+\phi},\quad B_2(U,U_{per})=\frac{P_\theta(\rho_{per}+\phi,\theta_{per}+\theta)}{\rho_{per}+\phi},\\
B_3(U,U_{per})&=\frac{1}{C_v(\rho_{per}+\phi)},\quad B_4(U,U_{per})=\frac{(\theta_{per}+\theta)P_\theta(\rho_{per}+\phi,\theta_{per}+\theta)}{C_v(\rho_{per}+\phi)}
\end{aligned}
$$

with

$$
\begin{aligned}
B_1(U_{per})&=\frac{P_\rho(\rho_{per},\theta_{per})}{\rho_{per}},\quad B_2(U_{per})=\frac{P_\theta(\rho_{per},\theta_{per})}{\rho_{per}},\\
B_3(U_{per})&=\frac{1}{C_v\rho_{per}},\quad B_4(U_{per})=\frac{\theta_{per}P_\theta(\rho_{per},\theta_{per})}{C_v\rho_{per}}.
\end{aligned}
$$

We consider the initial value problem for (2.3) under the initial condition

$$
U|_{t=0}=U_0={}^\top(\phi_0,w_0,\vartheta_0).
$$

One can show that if $[g]_{s+1}$ and $\|U_0\|_{H^{s+1}\times H^s}$ are sufficiently small, then $U(t)$ exists globally in time and $U(t)$ satisfies

$$
\begin{aligned}
&U\in C([0,\infty);H^{s+1}\times H^s),\\
&\|U(t)\|^2_{H^{s+1}\times H^s}+\int_0^t\|\nabla U(\tau)\|^2_{H^{s+1}\times H^s}d\tau\le C\|U_0\|^2_{H^{s+1}\times H^s}\quad (t\in[0,\infty)),\\
&\|U(t)\|_{L^\infty}\to 0\quad (t\to\infty).
\end{aligned}
$$

These can be proved by similar methods as those in [3, 7], since the Hardy inequality works well to deal with the linear terms including ${}^\top(\rho_{per},v_{per},\theta_{per})$ due to the estimates for ${}^\top(\rho_{per},v_{per},\theta_{per})$ in Theorem 3.1 and (2.1)-(2.2). We thus omit the details.

# 3 Outline of the proof of the main result

## 3.1 Formulation

To prove Theorem 3.1, we rewrite (1.1)-(1.3) as follows. Let

$$
\gamma_1=\sqrt{P_\rho(\rho_*,\theta_*)},\quad \gamma_2=\gamma_1\sqrt{\frac{C_vP(\rho_*,\theta_*)}{P_\theta(\rho_*,\theta_*)}},\quad \gamma_3=\frac{P_\theta(\rho_*,\theta_*)\gamma_2}{\gamma_1C_v}.
$$

We define $\phi$, $m$ and $\varepsilon$ by $\phi = \rho - \rho_*$, $m = \frac{M}{\gamma_1}$ and $\varepsilon = (\rho_* + \phi)\frac{E-E_*}{\gamma_2}$, respectively. Then (1.1)-(1.3) is rewritten as

$$\partial_t u + Au = F(u,g), \tag{3.1}$$

where

$$u = {}^{\top}(\phi, m, \varepsilon), \quad A = \begin{pmatrix} 0 & \gamma_1 \mathrm{div} & 0 \\ \gamma_1\nabla - \kappa_0\nabla\Delta & -\nu\Delta - \tilde{\nu}\nabla\mathrm{div} & \zeta\nabla \\ 0 & \zeta\mathrm{div} & -\alpha_0\Delta \end{pmatrix}, \tag{3.2}$$

$$\nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu+\mu'}{\rho_*}, \quad \zeta = \frac{\gamma_1 P(\rho_*, \theta_*)}{\gamma_2\rho_*}, \quad \kappa_0 = \frac{\kappa\rho_*}{\gamma_1}, \quad \alpha_0 = \frac{\tilde{\alpha}}{C_v\rho_*}$$

and

$$\begin{aligned}
F(u,g) &= \begin{pmatrix} 0 \\ F^2(u,g) \\ F^3(u) \end{pmatrix}, &(3.3)\\
F^2(u,g) &= -\Big\{\frac{\rho_*}{\gamma_1}\mathrm{div}\,(m\otimes m) + \gamma_1\mathrm{div}\,(P^{(1)}(\phi)\phi m\otimes m) \\
&\quad + \rho_*\nu\Delta(P^{(1)}(\phi)\phi m) + \rho_*\tilde{\nu}\nabla\mathrm{div}\,(P^{(1)}(\phi)\phi m) + \gamma_3\nabla(P^{(1)}(\phi)\phi\varepsilon) \\
&\quad + \frac{1}{\gamma_1}\nabla(P^{(2)}(\phi)\phi^2) - \frac{1}{\gamma_1}\nabla\Big(P_\theta(\rho_*,\theta_*)\frac{\gamma_1^2|m|^2}{2C_v(\rho_*+\phi)^2}\Big) \\
&\quad + \frac{1}{C_v^2\gamma_1}\nabla\Big\{P^{(3)}(\theta)\Big(\Big(\frac{\gamma_1^2|m|^2}{2(\rho_*+\phi)^2}\Big)^2 - \frac{\gamma_1^2\gamma_2\varepsilon|m|^2}{(\rho_*+\phi)^3} + \frac{\gamma_2^2\varepsilon^2}{(\rho_*+\phi)^2}\Big)\Big\} \\
&\quad + \frac{1}{C_v\gamma_1}\nabla\Big\{P^{(4)}(\theta)\Big(\frac{\gamma_2\phi\varepsilon}{\rho_*+\phi} - \frac{\gamma_1^2|m|^2\phi}{2(\rho_*+\phi)^2}\Big)\Big\} \\
&\quad - \frac{1}{\gamma_1}\mathrm{div}\,\Phi(\phi) - \frac{1}{\gamma_1}(\rho_*+\phi)g\Big\}, &(3.4)\\
\theta &= \frac{1}{C_v}\Big(E_* + \frac{\gamma_2\varepsilon}{\rho_*+\phi} - \gamma_1^2\frac{|m|^2}{2(\rho_*+\phi)^2}\Big), \\
P^{(1)}(\phi) &= \int_0^1 f'(\rho_*+\tau\phi)d\tau, \quad f(\tau) = \frac{1}{\tau} \quad (\tau\in\mathbb{R}), \\
P^{(2)}(\phi,\theta) &= \int_0^1 (1-\tau)P_{\rho\rho}\Big(\rho_*+\tau\phi, \theta\Big)d\tau, \\
P^{(3)}(\theta) &= \int_0^1 (1-\tau)P_{\theta\theta}\Big(\rho_*, \theta_*+\tau(\theta-\theta_*)\Big)d\tau, \\
P^{(4)}(\theta) &= \int_0^1 P_{\rho\theta}\Big(\rho_*, \theta_*+\tau(\theta-\theta_*)\Big)d\tau, \\
\Phi(\phi) &= \kappa\Big\{\phi\Delta\phi I + (\nabla\phi)\cdot(\nabla\phi)I - \frac{|\nabla\phi|^2}{2}I - \nabla\phi\otimes\nabla\phi\Big\}, \\
F^3(u) &= -\Big\{\frac{\gamma_1}{\rho_*}\mathrm{div}\,(\varepsilon m) + \gamma_1\mathrm{div}\,(P^{(1)}(\phi)\phi\varepsilon) - \alpha_0\rho_*\Delta(P^{(1)}(\phi)\phi\varepsilon),
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha_0}{C_v\gamma_2}\Delta\Big(\frac{\gamma_1^2|m|^2}{2(\rho_*+\phi)^2}\Big)+\frac{\gamma_1}{\gamma_2}\mathrm{div}\,(P^{(1)}(\phi)\phi m P(\rho_*+\phi,\theta))\\
&+\frac{\gamma_1}{\rho_*\gamma_2}\mathrm{div}\,(mP^{(5)}(\phi,\theta)\phi)\\
&+\frac{\gamma_1}{C_v\rho_*\gamma_2}\mathrm{div}\,\Big(mP^{(6)}(\theta)\Big(\frac{\gamma_2\varepsilon}{(\rho_*+\phi)}-\frac{\gamma_1^2|m|^2}{2(\rho_*+\phi)^2}\Big)\Big)\\
&-\frac{\gamma_1}{\gamma_2}\mathrm{div}\,\Big(\Big(\mathcal{S}(\frac{\gamma_1 m}{\rho_*+\phi})+\mathcal{K}(\rho_*+\phi)\Big)\frac{m}{\rho_*+\phi}\Big)-\frac{1}{\gamma_2}mg\Big\}, \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
P^{(5)}(\phi,\theta) &= \int_0^1 P_\rho(\rho_*+\tau\phi,\theta)d\tau,\\
P^{(6)}(\theta) &= \int_0^1 P_\theta(\rho_*,\theta_*+\tau(\theta-\theta_*))d\tau.
\end{aligned}
$$

Let us introduce a semigroup $S(t)=e^{-tA}$ generated by $A$;

$$S(t)=e^{-tA}=\mathcal{F}^{-1}e^{-t\hat{A}_\xi}\mathcal{F},$$

where

$$\hat{A}_\xi=\begin{pmatrix} 0 & i\gamma_1{}^\top\xi & 0\\ i\gamma_1\xi+i\kappa_0|\xi|^2\xi & \nu|\xi|^2I_n+\tilde{\nu}\xi^\top\xi & i\zeta\xi\\ 0 & i\zeta^\top\xi & \alpha_0|\xi|^2\end{pmatrix}\quad(\xi\in\mathbb{R}^3).$$

Then $S(t)$ has the following properties.

**Proposition 3.1.** *Let $s$ be a nonnegative integer satisfying $s\geq 2$. Then $S(t)=e^{-tA}$ is a contraction semigroup on $H^s\times H^{s-1}\times H^{s-1}$. In addition, for each $u\in H^s\times H^{s-1}\times H^{s-1}$ and all $T'>0$, $S(t)$ satisfies*

$$S(\cdot)u\in C([0,T'];H^s\times H^{s-1}\times H^{s-1}),\ S(0)u=u$$

*and there hold the estimates*

$$\|S(t)u\|_{H^s\times H^{s-1}\times H^{s-1}}\leq\|u\|_{H^s\times H^{s-1}\times H^{s-1}} \qquad (3.6)$$

*for $u\in H^s\times H^{s-1}\times H^{s-1}$ and $t\geq 0$.*

We set an operator $\Gamma$ using the time-$T$-map by

$$\Gamma[F]=S(t)(I-S(T))^{-1}\mathscr{S}(T)F+\mathscr{S}(t)F\quad(t\in[0,T]), \qquad (3.7)$$

where

$$\mathscr{S}(t)F:=\int_0^t S(t-\tau)F(\tau)d\tau.$$

To solve the time periodic problem for (3.1), as in [9], we look for a fixed point $u$ of $\Gamma[F(u,g)]$, i.e.,

$$u = \Gamma[F(u,g)] \quad (t \in [0,T]), \tag{3.8}$$

where $u = {}^{\top}(\phi, m, \varepsilon)$ and $F(u,g)$ is given by (3.3)-(3.5). From (3.7) and (3.8), it holds that $u(T) = u(0)$. Therefore, we will investigate properties of the map $\Gamma$.

We next introduce function spaces. We define a space $\mathscr{X}$ by

$$\mathscr{X} = \{w \in L_1^\infty, \nabla w \in H^1; \|w\|_{\mathscr{X}} < +\infty\},$$

where

$$\|w\|_{\mathscr{X}} := \sum_{j=0}^{1} \|(1+|x|)^{1+j}\nabla^j w\|_{L^\infty} + \sum_{j=1}^{2} \|(1+|x|)^{j-1}\nabla^j w\|_{L^2}.$$

Note that $w$ decays in the same order as the fundamental solution of the Laplace equation. Accordingly, this space is a similar one to that introduced in the stationary problem [3].

Let $\ell$ be a nonnegative integer and let $s$ be a nonnegative integer satisfying $s \geq 2$. We define the weighted $L^2$-Sobolev space $\mathscr{Y}_\ell^s(a,b)$ by

$$\begin{aligned} \mathscr{Y}_\ell^s(a,b) = &\left[C([a,b]; H_\ell^{s+1}) \cap L^2(a,b;; H_\ell^{s+2})\right] \\ &\times \left[C([a,b]; H_\ell^s) \cap L^2(a,b; H_\ell^{s+1})\right]. \end{aligned}$$

Let us introduce operators which decompose a function into its low and high frequency parts. Operators $P_1$ and $P_\infty$ on $L^2$ are defined by

$$P_j f = \mathcal{F}^{-1}\hat{\chi}_j \mathcal{F}[f] \quad (f \in L^2, j = 1, \infty),$$

where

$$\begin{aligned} &\hat{\chi}_j(\xi) \in C^\infty(\mathbb{R}^n) \quad (j = 1,\infty), \quad 0 \leq \hat{\chi}_j \leq 1 \quad (j = 1, \infty), \\ &\hat{\chi}_1(\xi) = \begin{cases} 1 & (|\xi| \leq r_1), \\ 0 & (|\xi| \geq r_\infty), \end{cases} \\ &\hat{\chi}_\infty(\xi) = 1 - \hat{\chi}_1(\xi), \ 0 < r_1 < r_\infty. \end{aligned}$$

We fix $0 < r_1 < r_\infty < \frac{2\gamma}{\nu+\tilde{\nu}}$ in such a way that the estimate (3.10) in Lemma 3.7 below holds for $|\xi| \leq r_\infty$.

Let $s$ be a nonnegative integer satisfying $s \geq 2$. We define a solution space $\mathscr{Z}^s(a,b)$ by

$$\mathscr{Z}^s(a,b) = \left\{u;\ P_1 u \in C(a,b;\mathscr{X}),\ P_\infty u \in \mathscr{Y}_2^s(a,b)\right\},$$

and the norm is defined by

$$\|u\|_{\mathscr{Z}^s(a,b)} = \|P_1 u\|_{C(a,b;\mathscr{X})} + \|P_\infty u\|_{\mathscr{Y}_2^s(a,b)}.$$

Observe that

$$P_j \Gamma[F(u,g)] = \Gamma[P_j F(u,g)] \quad (j = 1, \infty)$$

and

$$\text{supp } \widehat{P_1 F}(u,g) \subset \{|\xi| \le r_\infty\},$$
$$\text{supp } \widehat{P_\infty F}(u,g) \subset \{|\xi| \ge r_1\}.$$

So we will investigate the restriction of $\Gamma$ to the space of functions whose Fourier transforms have support in $\{|\xi| \le r_\infty\}$ and will then establish estimates for $\Gamma P_1$ in subsection 3.2. Likewise, the restriction of $\Gamma$ to the high frequency part will be investigated to establish estimates for $\Gamma P_\infty$ in subsection 3.3.

In the remaining of this subsection we introduce some lemmas which will be used in the proof of Theorem 2.1.

We will use the following lemma for the estimates for the integral kernels which will appear in the analysis of the low frequency part.

**Lemma 3.2.** [16, Theorem 2.3] *Let $\ell$ be a nonnegative integer and let $E(x) = \mathscr{F}^{-1}\hat{\Phi}_\ell$ ($x \in \mathbb{R}^3$), where $\hat{\Phi}_\ell \in C^\infty(\mathbb{R}^3 - \{0\})$ is a function satisfying*

$$\partial_\xi^\alpha \hat{\Phi}_\ell \in L^1 \quad (|\alpha| \le \ell),$$
$$|\partial_\xi^\beta \hat{\Phi}_\ell| \le C|\xi|^{-2-|\beta|+\ell} \quad (\xi \ne 0,\ |\beta| \ge 0).$$

*Then $E(x)$ $(x \ne 0)$ satisfies the estimate*

$$|E(x)| \le C|x|^{-(1+\ell)}.$$

The following lemma is related to the estimates for the convolutions which appear in the analysis of the low frequency part.

**Lemma 3.3.** (i) [17, Lemma 2.5] *Let $E(x)$ $(x \in \mathbb{R}^3)$ be a function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C}{(1+|x|)^{|\alpha|+1}} \quad (|\alpha| = 0, 1, 2). \tag{3.9}$$

*Assume that $f$ is a function satisfying $\|f\|_{L_3^\infty \cap L^1} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.*

$$|[\partial_x^\alpha E * f](x)| \le \frac{C}{(1+|x|)^{|\alpha|+1}} \|f\|_{L_3^\infty \cap L^1}.$$

(ii) [17, Lemma 2.5] *Let $E(x)$ $(x \in \mathbb{R}^3)$ be a function satisfying* (3.9). *Assume that $f$ is a function of the form: $f = \partial_{x_j} f_1$ for some $1 \le j \le n$ satisfying $\|\partial_{x_j} f_1\|_{L_3^\infty} + \|f_1\|_{L_2^\infty} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.*

$$|[\partial_x^\alpha E * f](x)| \le \frac{C}{(1+|x|)^{|\alpha|+1}}(\|\partial_{x_j} f_1\|_{L_3^\infty} + \|f_1\|_{L_2^\infty}).$$

(iii) [14, Lemma 4.9] *Let $E(x)$ $(x \in \mathbb{R}^3)$ be a function satisfying*

$$|\partial_x^\alpha E(x)| \le \frac{C}{(1+|x|)^{|\alpha|+2}} \quad (|\alpha| = 0, 1).$$

*Assume that $f$ is a function satisfying $\|f\|_{L_3^\infty} < \infty$. Then there holds the following estimate for $|\alpha| = 0, 1$.*

$$|[\partial_x^\alpha E * f](x)| \le \frac{C \log|x|}{(1+|x|)^{|\alpha|+2}} \|f\|_{L_3^\infty}.$$

## 3.2 Estimate of $\Gamma$ for the low frequency part

In this subsection we estimate $\Gamma$ for the low frequency part. We introduce a $L^2$ space for the low frequency part. The symbol $L^2_{(1)}$ stands for the set of all $u \in L^2$ satisfying supp $\hat{f} \subset \{|\xi| \le r_\infty\}$. For any nonnegative integer $k$, we see that $H^k \cap L^2_{(1)} = L^2_{(1)}$. (Cf., Lemma 3.4 (ii) bellow.)

We next state some properties of $P_1$.

**Lemma 3.4.** ([9, Lemma 4.3]) (i) *Let $k$ be a nonnegative integer. Then $P_1$ is a bounded linear operator from $L^2$ to $H^k$ and $P_1$ satisfies the estimates*

$$\|\nabla^k P_1 f\|_{L^2} \le C\|f\|_{L^2} \qquad (f \in L^2).$$

*As a result, for any $2 \le p \le \infty$, $P_1$ is bounded from $L^2$ to $L^p$.*

(ii) *Let $k$ be a nonnegative integer. Then there hold the estimates*

$$\|\nabla^k f_1\|_{L^2} + \|f_1\|_{L^p} \le C\|f_1\|_{L^2} \quad (f \in L^2_{(1)}),$$

*where $2 \le p \le \infty$.*

We derive the following inequalities for the weighted $L^p$ norm of the low frequency part.

**Lemma 3.5.** *Let $k$ and $\ell$ be nonnegative integers and let $1 \le p \le \infty$. Then there holds the estimate*

$$\|\nabla^k f_1\|_{L^p_\ell} \le C\|f_1\|_{L^p_\ell} \quad (f \in L^2_{(1)} \cap L^p_\ell).$$

The proof of the estimate is given in [14, Lemma 4.3].

We define a space $\mathscr{X}_{(1)}$ by

$$\mathscr{X}_{(1)} = \{u \in \mathscr{X}; \mathrm{supp}\ \hat{u} \subset \{|\xi| \le r_\infty\}\}.$$

We set operators $S_1(t)$ and $\mathscr{S}_1(t)$ by

$$S_1(t) = S(t)|_{\mathscr{X}_{(1)}}, \quad \mathscr{S}_1(t)F_1 = \int_0^t S_1(t-\tau)F_1(\tau)\,d\tau.$$

Then we have the following

**Proposition 3.6.** (i) *$S_1(t)$ is a uniformly continuous semigroup on $\mathscr{X}_{(1)}$. In addition, for each $u_1 \in \mathscr{X}_{(1)}$ and all $T' > 0$, $S_1(t)$ satisfies*

$$S_1(t)u_1 \in C^1([0,T']; \mathscr{X}_{(1)}),$$

$$\partial_t S_1(t)u_1 = -A_1 S_1(t)u_1 \ (= -AS_1(t)u_1), \ S_1(0)u_1 = u_1,$$

*and there hold the estimates*

$$\|\partial_t^k S_1(\cdot)u_1\|_{C([0,T'];\mathscr{X}_{(1)})} \le C\|u_1\|_{\mathscr{X}_{(1)}}$$

*for $u_1 \in \mathscr{X}_{(1)}$, $k = 0, 1$, where $T' > 0$ is any given positive number and $C$ is a positive constant depending on $T'$.*

(ii)

$$\mathscr{S}_1(t) : L^2(0,T;\mathscr{X}_{(1)}) \to C([0,T];\mathscr{X}_{(1)}) \cap H^1(0,T;\mathscr{X}_{(1)})$$

*is a bounded linear operator for $t \in [0,T]$ satisfying*

$$\partial_t \mathscr{S}_1(t)F_1 + A_1\mathscr{S}_1(t)F_1 = F_1(t), \ \mathscr{S}_1(0)F_1 = 0,$$

$$\|\mathscr{S}_1(\cdot)F_1\|_{C([0,T];\mathscr{X}_{(1)})} \le C\|F_1\|_{L^2(0,T;\mathscr{X}_{(1)})},$$

$$\|\partial_t\mathscr{S}_1(\cdot)F_1\|_{L^2(0,T;\mathscr{X}_{(1)})} \le C\|F_1\|_{L^2(0,T;\mathscr{X}_{(1)})}.$$

*for $F_1 \in L^2(0,T;\mathscr{X}_{(1)})$, where $C$ is a positive constant depending on $T$.*

(iii) *It holds that*

$$S_1(t)\mathscr{S}_1(t')F_1 = \mathscr{S}_1(t')[S_1(t)F_1]$$

*for any $t \ge 0$, $t' \in [0,T]$ and $F_1 \in L^2(0,T;\mathscr{X}_{(1)})$.*

Proposition 3.6 can be proved in a similar manner to the proof of [14, Proposition 5.1]; and we omit the proof.

To estimate $\Gamma$, we prepare some lemmas. The following lemma plays an important role to investigate the spatial decay properties of the time-$T$-map.

**Lemma 3.7.** (i) *Let*

$$\hat{A}_\xi = \begin{pmatrix} 0 & i\gamma_1{}^{\mathsf{T}}\xi & 0 \\ i\gamma_1\xi + i\kappa_0|\xi|^2\xi & \nu|\xi|^2 I_3 + \tilde{\nu}\xi^{\mathsf{T}}\xi & i\zeta\xi \\ 0 & i\zeta^{\mathsf{T}}\xi & \alpha_0|\xi|^2 \end{pmatrix} \quad (\xi \in \mathbb{R}^3).$$

*Then there exists $\delta_0 > 0$ such that if $0 < r_\infty \le \delta_0$, the set of all eigenvalues of $-\hat{A}_\xi$ consists of $\lambda_j(\xi)\,(j = 1, \cdots 4)$, where*

$$\begin{cases} \lambda_1(\xi) = -\nu|\xi|^2 + O(|\xi|^3), \\ \lambda_2(\xi) = -\frac{\alpha_0\gamma_1^2}{\gamma_1^2+\zeta^2}|\xi|^2 + O(|\xi|^3), \\ \lambda_3(\xi) = i\sqrt{\gamma_1^2 + \zeta^2}|\xi| - \frac{\nu+\tilde{\nu}}{2}|\xi|^2 - \frac{\alpha_0\zeta^2}{2(\gamma_1^2+\zeta^2)}|\xi|^2 + O(|\xi|^3), \\ \lambda_4(\xi) = \bar{\lambda}_3 \text{ (complex conjugate)}. \end{cases}$$

(ii) *For $|\xi| \le \delta_0$, $e^{-t\hat{A}_\xi}$ has the spectral resolution*

$$e^{-t\hat{A}_\xi} = \sum_{j=1}^{4} e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

*where $\Pi_j(\xi)$ is eigenprojections for $\lambda_j(\xi)\,(j = 1, \cdots, 4)$, and $\Pi_j(\xi)\,(j = 1, \cdots, 4)$ satisfy*

$$\Pi_1(\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_3 - \frac{\xi^{\mathsf{T}}\xi}{|\xi|^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(|\xi|),$$

$$\Pi_2(\xi) = \begin{pmatrix} 1 - \frac{\gamma_1^2}{\gamma_1^2+\zeta^2} & 0 & -\frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} \\ 0 & 0 & 0 \\ -\frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} & 0 & 1 - \frac{\zeta^2}{\gamma_1^2+\zeta^2} \end{pmatrix} + O(|\xi|),$$

$$\Pi_3(\xi) = \frac{1}{2}\begin{pmatrix} \frac{\gamma_1^2}{\gamma_1^2+\zeta^2} & -\frac{i\gamma_1{}^{\mathsf{T}}\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} \\ -\frac{i\gamma_1\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\xi^{\mathsf{T}}\xi}{|\xi|^2} & -\frac{i\zeta\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} \\ \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} & -\frac{i\zeta^{\mathsf{T}}\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\zeta^2}{\gamma_1^2+\zeta^2} \end{pmatrix} + O(|\xi|),$$

$$\Pi_4(\xi) = \frac{1}{2}\begin{pmatrix} \frac{\gamma_1^2}{\gamma_1^2+\zeta^2} & \frac{i\gamma_1{}^{\mathsf{T}}\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} \\ \frac{i\gamma_1\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\xi^{\mathsf{T}}\xi}{|\xi|^2} & \frac{i\zeta\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} \\ \frac{\gamma_1\zeta}{\gamma_1^2+\zeta^2} & \frac{i\zeta^{\mathsf{T}}\xi}{i\sqrt{\gamma_1^2+\zeta^2}|\xi|} & \frac{\zeta^2}{\gamma_1^2+\zeta^2} \end{pmatrix} + O(|\xi|).$$

*Furthermore, there exist a constant $C > 0$ such that the estimates*

$$\|\Pi_j(\xi)\| \le C\,(j = 1, \cdots, 4) \tag{3.10}$$

*hold for $|\xi| \le r_\infty$.*

Lemma 3.7 is proved by the analytic perturbation theory ([10]). We set

$$\xi = |\xi|\omega, \quad \omega = \frac{\xi}{|\xi|}, \quad -\hat{A}_\xi = r\tilde{A}_\xi, \quad \tilde{A}_\xi = L_1 + rL_2 + r^2 L_3,$$

where $r = |\xi|$,

$$L_1 \quad = \quad -i\begin{pmatrix} 0 & {\gamma_1}^\top\omega & 0 \\ \gamma_1\omega & 0 & \zeta\omega \\ 0 & \zeta^\top\omega & 0 \end{pmatrix}, \quad L_2 = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu I_3 + \omega^\top\omega & 0 \\ 0 & 0 & \alpha_0 \end{pmatrix}$$

and

$$L_3 = -\begin{pmatrix} 0 & 0 & 0 \\ i\kappa_0\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying the reduction process ([10, Section II-2-3]), we can prove Lemma 3.7. See also [12, Lemma 3.1].

Hereafter we fix $0 < r_1 < r_\infty \le \delta_0$ so that (3.10) in Lemma 3.7 holds for $|\xi| \le r_\infty$.

**Lemma 3.8.** *Let $\alpha$ be a multi-index. Then the following estimates hold true uniformly for $\xi$ with $|\xi| \le r_\infty$ and $t \in [0, T]$.*

(i) $|\partial_\xi^\alpha \lambda_j| \le C|\xi|^{2-|\alpha|} \quad ((|\alpha| \ge 0,\ j = 1, 2),\ |\partial_\xi^\alpha \lambda_j| \le C|\xi|^{1-|\alpha|} \quad ((|\alpha| \ge 0,\ j = 3, 4)$.

(ii) $|(\partial_\xi^\alpha \Pi_j)\hat{F}| \le C|\xi|^{-|\alpha|}|\hat{F}|\ (|\alpha| \ge 0)$.

(iii) $|\partial_\xi^\alpha(e^{\lambda_j t})| \le C|\xi|^{2-|\alpha|}\ (|\alpha| \ge 1,\ j = 1, 2)$.

(iv) $|\partial_\xi^\alpha(e^{\lambda_j t})| \le C|\xi|^{1-|\alpha|}\ (|\alpha| \ge 1,\ j = 3, 4)$.

(v) $|(\partial_\xi^\alpha e^{-t\hat{A}_\xi})\hat{F}| \le C|\xi|^{-|\alpha|}|\hat{F}|\ (|\alpha| \ge 1)$.

(vi) $|\partial_\xi^\alpha(I - e^{\lambda_j t})^{-1}| \le C|\xi|^{-2-|\alpha|}\ (|\alpha| \ge 0,\ j = 1, 2)$.

(vii) $|\partial_\xi^\alpha(I - e^{\lambda_j t})^{-1}| \le C|\xi|^{-1-|\alpha|}\ (|\alpha| \ge 0,\ j = 3, 4)$.

Lemma 3.8 can be verified by direct computations based on Lemma 3.7.

We are now in a position to give an estimate of $\Gamma$ for the low frequency part.

**Proposition 3.9.** *Let $s$ be a nonnegative integer satisfying $s \ge 2$. Assume that $u = {}^\top(\phi, m, \varepsilon)$ satisfies*

$$\|u\|_{\mathscr{Z}^s(0,T)} << 1.$$

*Then it holds that*

$$\|\Gamma[P_1 F(u, g)]\|_{C([0,T];\mathscr{X})} \le C\|u\|^2_{\mathscr{Z}^s(0,T)} + C(1 + \|u\|_{\mathscr{Z}^s(0,T)})[g]_s$$

*uniformly for $u$.*

**Proof.** We set

$$\begin{aligned}\Gamma_1[P_1F(u,g)] &:= S(t)(I-S(T))^{-1}\mathscr{S}(T)[P_1F(u,g)],\\ \Gamma_2[P_1F(u,g)] &:= \mathscr{S}(t)[P_1F(u,g)].\end{aligned}$$

By using Lemma 3.8, one can easily obtain the required estimates for $\|\nabla^k\Gamma_j[P_1F(u,g)]\|_{L^2_{k-1}}$ $(j,k=1,2)$.

We estimate $\Gamma_j$ in the weighted $L^\infty$ space. As for the term $\Gamma_1[P_1F(u,g)]$, by Proposition 3.6 we have

$$\begin{aligned}\Gamma_1[P_1F(u,g)] &= S_1(t)(I-S_1(T))^{-1}\mathscr{S}_1(T)[P_1F(u,g)]\\ &= \mathscr{F}^{-1}\Big\{e^{-t\hat{A}_\xi}(I-e^{-T\hat{A}_\xi})^{-1}\int_0^T e^{-(T-\tau)\hat{A}_\xi}\hat{\chi}_1\hat{F}(\tau,u,g)d\tau\Big\}\\ &=: \int_0^T E_1(t,\tau)*P_1F(\tau,u,g)d\tau, \end{aligned}\tag{3.11}$$

where

$$E_1(t,\tau)=\mathcal{F}^{-1}\{\hat{\chi}_0e^{-t\hat{A}_\xi}(I-e^{-T\hat{A}_\xi})^{-1}e^{-(T-\tau)\hat{A}_\xi}\},$$

$\chi_0$ is a cut-off function defined by $\chi_0=\mathcal{F}^{-1}\hat{\chi}_0$ with $\hat{\chi}_0$ satisfying

$$\hat{\chi}_0\in C^\infty(\mathbb{R}^n),\ \ 0\le\hat{\chi}_0\le1,\ \ \hat{\chi}_0=1\ \text{ on }\ \{|\xi|\le r_\infty\}\ \text{ and }\ \operatorname{supp}\hat{\chi}_0\subset\{|\xi|\le 2r_\infty\}.$$

By Lemma 3.7, $e^{-t\hat{A}_\xi}$ has the spectral resolution

$$e^{-t\hat{A}_\xi}=\sum_{j=1}^4 e^{t\lambda_j(\xi)}\Pi_j(\xi),$$

where $\lambda_j$ and $\Pi_j$ $(j=1,\cdots,4)$ are the same ones in Lemma 3.7. Therefore, we see that

$$(I-e^{-T\hat{A}_\xi})^{-1}=\sum_{j=1}^4(I-e^{T\lambda_j})^{-1}\Pi_j. \tag{3.12}$$

Let $\alpha$ be a multi-index satisfying $|\alpha|\ge0$. It follows from Lemma 3.8 that

$$\sum_j|\partial_x^\alpha E_1(x)|\le C\int_{|\xi|\le2r_\infty}|\xi|^{-2}d\xi\ \ (x\in\mathbb{R}^3).$$

Since $\int_{|\xi|\le2r_\infty}|\xi|^{-2}d\xi<\infty$, we see that

$$\sum_j|\partial_x^\alpha E_1(x)|\le C\ \ (x\in\mathbb{R}^3), \tag{3.13}$$

where $C>0$ is a constant depending on $\alpha$, $T$. By Lemma 3.8, we have

$$|\partial_\xi^\beta((i\xi)^\alpha\hat{\chi}_0(I-e^{\lambda_jT})^{-1}\Pi_j)|\ \le\ C|\xi|^{-2+|\alpha|-|\beta|}\ \text{ for } j=1,2,\ |\beta|\ge0,$$

$$|\partial_\xi^\beta((i\xi)^\alpha\hat{\chi}_0(I-e^{\lambda_j T})^{-1}\Pi_j)| \;\le\; C|\xi|^{-1+|\alpha|-|\beta|} \text{ for } j=3,4,\ |\beta|\ge 0.$$

It then follows from Lemma 3.2 and (3.12) that

$$|\partial_x^\alpha E_1(x)| \le C|x|^{-(1+|\alpha|)}. \tag{3.14}$$

From (3.13) and (3.14), we obtain that

$$|\partial_x^\alpha E_1(x)| \le C(1+|x|)^{-(1+|\alpha|)} \tag{3.15}$$

uniformly for $x \in \mathbb{R}^3$.

Concerning the estimate for the nonlinear term $P_1\mathrm{div}\,(m\otimes m)$ in the estimate of $\Gamma_1[P_1F(u,g)]$, due to the conservation form, applying Lemma 3.3, Lemma 3.8, (3.11) and (3.15) with $|\alpha|\ge 1$, we see that

$$\|\Gamma_1[F_1(u)]\|_{\mathscr{X}_{(1)}(0,T)} \le C\|u\|^2_{\mathscr{X}^s(0,T)}, \tag{3.16}$$

where $F_1(u) = {}^\top(0, P_1\mathrm{div}\,(m\otimes m), 0)$. Similarly to (3.16), the remaining terms can be estimated. Hence, we obtain the desired estimate for $\Gamma_1$. The estimate for $\Gamma_2$ can be proved in a similar manner to the proof of the estimate for $\Gamma_1$. This completes the proof. $\square$

### 3.3 Estimate of $\Gamma$ for the high frequency part

In this subsection we establish an estimate $\Gamma$ for the high frequency part. The following function spaces are introduced for the high frequency part. Let $k$ and $\ell$ be nonnegative integers. The symbol $H^k_{(\infty)}$ stands for the set of all $u \in H^k$ satisfying supp $\hat{u} \subset \{|\xi| \ge r_1\}$ and the space $H^k_{(\infty),\ell}$ is defined by

$$H^k_{(\infty),\ell} = \{u \in H^k_{(\infty)};\, \|u\|_{H^k_\ell} < +\infty\}.$$

We prepare some lemmas for the high frequency part.

**Lemma 3.10.** [9, Lemma 4.4] (i) *Let $k$ be a nonnegative integer. Then $P_\infty$ is a bounded linear operator on $H^k$.*

(ii) *There hold the inequalities*

$$\begin{aligned} \|P_\infty f\|_{L^2} &\le C\|\nabla f\|_{L^2} \quad (f \in H^1), \\ \|f_\infty\|_{L^2} &\le C\|\nabla f_\infty\|_{L^2} \quad (f_\infty \in H^1_{(\infty)}). \end{aligned}$$

**Lemma 3.11.** [14, Lemma 4.13] *Let $\ell \in \mathbb{N}$. Then there exists a positive constant $C$ depending only on $\ell$ such that*

$$\|P_\infty f\|_{L^2_\ell} \le C\|\nabla f\|_{L^2_\ell}.$$

Let $s$ be a nonnegative integer satisfying $s \geq 2$. By Proposition 3.1, we define an operator

$$S_\infty(t) : H^{s+1}_{(\infty)} \times H^s_{(\infty)} \longrightarrow H^{s+1}_{(\infty)} \times H^s_{(\infty)} \quad (t \geq 0)$$

by $S_\infty(t)u_\infty = S(t)u_\infty$ for $u_\infty \in H^{s+1}_{(\infty)} \times H^s_{(\infty)}$. We also define

$$\mathscr{S}_\infty(t) : L^2(0,T; H^s_{(\infty)} \times H^{s-1}_{(\infty)}) \longrightarrow H^{s+1}_{(\infty)} \times H^s_{(\infty)} \quad (t \in [0,T])$$

by

$$\mathscr{S}_\infty(t)F_\infty = \int_0^t S_\infty(t-\tau)F_\infty(\tau)\,d\tau.$$

for $F_\infty \in L^2(0,T; H^s_{(\infty)} \times H^{s-1}_{(\infty)})$.

We have the following properties for $S_\infty$ and $\mathscr{S}_\infty$.

**Proposition 3.12.** (i) *It holds that $S_\infty(\cdot)u_{0\infty} \in C([0,\infty); H^{s+1}_{(\infty),2} \times H^s_{(\infty),2})$ for each $u_{0\infty} = {}^\top(\phi_{0\infty}, m_{0\infty}, \varepsilon_{0\infty}) \in H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}$ and there exist constants $a > 0$ and $C > 0$ such that $S_\infty(t)$ satisfies the estimate*

$$\|S_\infty(t)u_{0\infty}\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \leq Ce^{-at}\|u_{0\infty}\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}}$$

*for all $t \geq 0$ and $u_{0\infty} \in H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}$. Furthermore, $r_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}}(S_\infty(T)) < 1$, where $r_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}}(S_\infty(T))$ denotes the spectral radius of $S_\infty(T)$ on $H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}$; and $I - S_\infty(T)$ has a bounded inverse $(I - S_\infty(T))^{-1}$ on $H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}$ and $(I - S_\infty(T))^{-1}$ satisfies*

$$\|(I - S_\infty(T))^{-1}u\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \leq C\|u\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \quad \text{for} \quad u \in H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}.$$

(ii) *It holds that $\mathscr{S}_\infty(\cdot)F_\infty \in C([0,T]; H^{s+1}_{(\infty),2} \times H^s_{(\infty),2})$ for each $F_\infty = {}^\top(F^1_\infty, F^2_\infty, F^3_\infty) \in L^2(0,T; H^s_{(\infty),2} \times H^{s-1}_{(\infty),2})$ and $\mathscr{S}_\infty(t)$ satisfies the estimate*

$$\|\mathscr{S}_\infty(t)[F_\infty]\|_{H^{s+1}_{(\infty),2} \times H^s_{(\infty),2}} \leq C\left\{\int_0^t e^{-a(t-\tau)}\|F_\infty\|^2_{H^s_{(\infty),2} \times H^{s-1}_{(\infty),2}}\,d\tau\right\}^{\frac{1}{2}}$$

*for $t \in [0,T]$ and $F_\infty \in L^2(0,T; H^s_{(\infty),2} \times H^{s-1}_{(\infty),2})$ with a positive constant $C$ depending on $T$.*

Proposition 3.12 can be proved by the weighted $L^2$-energy method. In fact, Proposition 3.12 is an immediate consequence of the following proposition.

**Proposition 3.13.** *Let $s$ be a nonnegative integer satisfying $s \geq 2$. Assume that*

$$u_{0\infty} = {}^\top(\phi_{0\infty}, m_{0\infty}, \varepsilon_{0\infty}) \in H^{s+1}_{(\infty),2} \times H^s_{(\infty),2},$$

$$F_\infty = {}^\top(F^1_\infty, F^2_\infty, F^3_\infty) \in L^2(0,T'; H^s_{(\infty),2} \times H^{s-1}_{(\infty),2})$$

*for all $T' > 0$. Assume also that $u_\infty = {}^\top(\phi_\infty, m_\infty, \varepsilon_\infty)$ satisfies*

$$\begin{cases} \partial_t u_\infty + A u_\infty = F_\infty, \\ u_\infty|_{t=0} = u_{0\infty}. \end{cases} \tag{3.17}$$

*and*

$$\phi_\infty \in C([0,T']; H^{s+1}_{(\infty)}) \cap L^2(0,T'; H^{s+2}_{(\infty)}),\ {}^\top(m_\infty, \varepsilon_\infty) \in C([0,T']; H^s_{(\infty)}) \cap L^2(0,T'; H^{s+1}_{(\infty)})$$

*Then $u_\infty$ satisfies*

$$\phi_\infty \in C([0,T']; H^{s+1}_{(\infty),2}) \cap L^2(0,T'; H^{s+2}_{(\infty),2}),\ {}^\top(m_\infty, \varepsilon_\infty) \in C([0,T']; H^s_{(\infty),2}) \cap L^2(0,T'; H^{s+1}_{(\infty),2})$$

*for all $T' > 0$ and there exists an energy functional $\mathcal{E}^s[u_\infty]$ such that there holds the estimate*

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}^s[u_\infty](t) + d(\|\phi_\infty(t)\|^2_{H^{s+2}_2} + \|m_\infty(t)\|^2_{H^{s+1}_2} + \|\varepsilon_\infty(t)\|^2_{H^{s+1}_2}) \\ &\qquad \le C\|F_\infty(t)\|^2_{H^s_2 \times H^{s-1}_2} \end{aligned} \tag{3.18}$$

*on $(0,T')$ for all $T' > 0$. Here $d$ is a positive constant; $C$ is a positive constant depending on $T$ but not on $T'$; $\mathcal{E}^s[u_\infty]$ is equivalent to $\|u_\infty\|^2_{H^{s+1}_2 \times H^s_2}$, i.e,*

$$C^{-1}\|u_\infty\|^2_{H^{s+1}_2 \times H^s_2} \le \mathcal{E}^s[u_\infty] \le C\|u_\infty\|^2_{H^{s+1}_2 \times H^s_2};$$

*and $\mathcal{E}^s[u_\infty](t)$ is absolutely continuous in $t \in [0,T']$ for all $T' > 0$.*

Making use of the smoothing effect of $\rho$ arising in the Korteweg stress tensor, we can prove Proposition 3.13 in a similar manner to the $L^2$-energy method as in [1, 5], and we omit the details here.

It follows from Proposition 3.12 that we obtain the following estimate of $\Gamma$ for the high frequency part.

**Proposition 3.14.** *Let $s$ be a nonnegative integer satisfying $s \ge 2$. Assume that $u = {}^\top(\phi, m, \varepsilon)$ satisfies*

$$\|u\|_{\mathscr{X}^s(0,T)} << 1.$$

*Then it holds that*

$$\|\Gamma[P_\infty F(u,g)]\|_{\mathscr{Y}^s_2(0,T)} \le C\|u\|^2_{\mathscr{X}^s(0,T)} + C(1 + \|u\|_{\mathscr{X}^s(0,T)})[g]_s$$

*uniformly for $u$.*

By Proposition 3.9 and Proposition 3.14, as in [9], we obtain Theorem 3.1.

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