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WELL-POSEDNESS OF THE COMPRESSIBLE NAVIER-STOKES-POISSON SYSTEM IN BESOV SPACES

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1. Introduction

This note is a summary of well-posedness results in [4] concerning the Cauchy problem of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^n$.

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) & \\ \qquad = \operatorname{div}(2\mu(\rho)\mathcal{D}(u)) + \nabla(\lambda(\rho)\operatorname{div} u) + \gamma\rho\nabla\psi, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ -\Delta\psi = \rho - \bar{\rho}, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\rho = \rho(t,x)$, $u = u(t,x)$ and $\psi = \psi(t,x)$ are the unknown functions, representing the fluid density, the velocity vector and the potential force, respectively, $P = P(\rho)$ denotes the pressure depending on only ρ, and $D(u)$ is the strain tensor. We denotes the tensor product of velocity vector u and u by $u \otimes u$. The Lamé constants μ, λ depend smoothly on ρ and satisfy $\mu > 0$ and $\lambda + \mu > 0$, which ensures that the operator $\operatorname{div}(2\mu(\rho)\mathcal{D}\cdot) + \nabla(\lambda(\rho)\operatorname{div}\cdot)$ is an operator of the elliptic type. The constant $\bar{\rho}$ is positive and describe the background density. The first equation represents the mass conservation law, the second one represents the equilibrium of momentum, and the third equation is a Helmholtz type elliptic equation that determines the potential force exerted by the electric field or the gravitational field.

The system (1.1) is the compressible Navier-Stokes-Poisson equation with a Coulomb potential, which describes various physical models. If $\gamma < 0$, (1.1) describes the transport of charged particles under the electric field of electrostatic potential force (cf. Markowich-Ringhofer-Schmeiser [14]). When $\gamma > 0$, (1.1) describes the dynamics of self-gravitating gaseous star (cf. Chandrasekhar [2]).

1.1. **Scale-critical functional framework.** The main purpose of this paper is to see the advantage of using the Lagrangian coordinate (or the method of characteristic) applied to the system (1.1) in in the *critical* or *near-critical* regularity framework. It is a well-known fact that if we ignore the pressure and the potential term, the system (1.1) is left invariant under the transformation $(\rho, u) \to (\rho_\ell, u_\ell)$ with

$$\rho_\ell(t,x) = \rho(\ell^2 t, \ell x) \quad \text{and} \quad u_\ell(t,x) = \nu u(\ell^2 t, \ell x). \tag{1.2}$$

The idea that the spaces that are norm-invariant under the above transformation should give a candidate for the largest possible space to find a unique solution has been noted by [10] for the incompressible Navier-Stokes system (with a constant density). This idea was then extended to the barotropic compressible viscous flow in [6].

Inspired by the recent papers [9], [8] on the compressible barotropic and the incompressible inhomogeneous fluids, we consider the solvability of the system (1.1) in the low-regularity function spaces using the Lagrangian coordinates. The principal merit in using the Lagrangian coordinates stems from the fact that it can be viewed locally-in-time as a parabolic system with a lower-order term (this lower-order term corresponds to the pressure), which has been noted by many authors. Effectively eliminating the pressure by the Lagrangian transformation, we may treat the system as a simple heat equation with variable coefficients, which enables us to use the contraction argument. Recently, the *flow estimates* in the Sobolev-subcritical Besov spaces are clarified so as to treat the *scale-critical* solvability (see [8,9] and the preliminaries below). The novelty of the two papers [8,9] is that the characteric is defined by a velocity vector only in the critical Besov space.

Hereafter, we denote L^p $(1 \le p \le \infty)$ as the Lebesgue space of p-th ordered integrable functions. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the homogeneous Littlewood-Paley dyadic decomposition of an unity. Namely, let $\widehat{\phi} \in \mathcal{S}$ is a non-negative radially symmetric function that satisfies

$$\operatorname{supp}\widehat{\phi} \subset \{\xi \in \mathbb{R}^n; 2^{-1} < |\xi| < 2\}, \quad \widehat{\phi}_j := \widehat{\phi}(2^{-j}\xi)\ (j \in \mathbb{Z}) \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi) = 1\ (\xi \neq 0).$$

We set $\widehat{\Phi}(\xi) := 1 - \sum_{j\ge 1} \widehat{\phi}_j(\xi)$ and $\widehat{\Phi}_j := \widehat{\Phi}(2^{-j}\xi)$.

Definition(the Besov spaces) Let $\mathcal{S}'$ be the space of all tempered distributions. For $s \in \mathbb{R}$ and $1 \le p \le \infty$ we define the homogeneous Besov space $\dot{B}^s_{p,1}(\mathbb{R}^n)$ to be:

$$\dot{B}^s_{p,1}(\mathbb{R}^n) := \{u \in \mathcal{S}'\ ;\ \sum_{j\in\mathbb{Z}} \phi_j * u = u \text{ in } \mathcal{S}', \quad \|u\|_{\dot{B}^s_{p,1}} < \infty\},$$

$$\text{with } \|u\|_{\dot{B}^s_{p,1}} := \sum_{j\in\mathbb{Z}} 2^{js} \|\phi_j * u\|_{L^p}$$

We define the hybrid Besov spaces $\widetilde{B}^{s,\sigma}_{p,1}$ for $s, s' \in \mathbb{R}$ and $1 \le p \le \infty$ by

$$\|u\|_{\widetilde{B}^{s,s'}_{p,1}} := \sum_{j<0} 2^{js} \|\phi_j * u\|_{L^p} + \sum_{j\ge 0} 2^{js'} \|\phi_j * u\|_{L^p}$$

We denote the low frequency of u by $u_L := \dot{S}_m u = \Phi_m * u$ for some fixed m and the high frequency of u by u_H. Then we may also express $\widetilde{B}^{s,s'}_{p,1}$ as the space in which u_L belongs to $\dot{B}^s_{p,1}$ and u_H belongs to $\dot{B}^{s'}_{p,1}$. The following relations hold:

$$\widetilde{B}^{s,s'}_{p,1} = \dot{B}^s_{p,1} \cap \dot{B}^{s'}_{p,1} \quad \text{if } s < s' \quad \text{and} \quad \widetilde{B}^{s,s'}_{p,1} = \dot{B}^s_{p,1} + \dot{B}^{s'}_{p,1} \quad \text{if } s > s'.$$

In the low-regularity Besov framework, Hao-Li [11] gave the unique global existence of the solution for (1.1) in the L^2-based Besov spaces, using the method of [6] in dimensions $n \ge 3$. Zheng [17] proved a global result, based on the work of [5], with a larger class of inital data with Besov regularity. In both [11] and [17], two-dimension is excluded. The main purpose of this paper is to prove the local solvability in the two and higher dimensions. Moreover, our result does not depend on the choice of $\gamma \in \mathbb{R}$; in other words, our main theorem also states a new local existence and uniqueness result for the barotropic compressible Navier-Stokes system.

1.2. **The Lagrangian coordinates.** For $n \times n$ matrices $A = (A_{ij})_{1\leq i,j\leq n}$ and $B = (B_{ij})_{1\leq i,j\leq n}$, we define the trace product $A : B$ by $A : B = \mathrm{tr}AB = \sum_{ij} A_{ij}B_{ji}$. By $\mathrm{adj}(A)$, we denote the adjugate matrix of A, i.e. the transpose of the cofactor matrix of A. If A is invertible then $\mathrm{adj}(A) = (\det A)A^{-1}$. Given some matrix A, we define the transformed deformation tensor and divergence operator by

$$D_A(u) := \frac{1}{2}(DuA + {}^tA\nabla u) \quad \text{and} \quad \mathrm{div}_A u := {}^tA : \nabla u = Du : A.$$

The flow $X = X_u$ of u is defined by

$$X_u(t,y) = y + \int_0^t u(\tau, X_u(\tau,y))d\tau. \tag{1.3}$$

We denote $\bar{\rho}(t,y) := \rho(t, X_u(t,y))$ and $\bar{u}(t,y) := u(t, X_u(t,y))$. With the notation $J = J_u := \det(DX_u)$ and $A = A_u := (D_yX_u)^{-1}$, the system (1.1) in Lagrangean coordinate writes as follows

$$\begin{cases} \partial_t(J\bar{\rho}) = 0, \\ \rho_0\partial_t\bar{u} - \mathrm{div}\,(\mathrm{adj}(DX)(2\mu D_A\bar{u} + \lambda \mathrm{div}_A\bar{u} - P(\bar{\rho})) +^t \mathrm{adj}(DX)\nabla\overline{\psi} = 0, \\ -\,\mathrm{div}\,(\mathrm{adj}(DX)A^t\nabla\overline{\psi}) = \rho_0 - J, \\ (\bar{\rho},\bar{u})|_{t=0} = (\rho_0, u_0). \end{cases} \tag{1.4}$$

From hereon, we may forget any reference to the initial Eulerian vector-field u in the equations and redefine the *flow* of $\overline{u}$ as

$$X_{\overline{u}}(t,y) = y + \int_0^t \overline{u}(\tau,y)d\tau. \tag{1.5}$$

We are going to solve the above system in homogeneous Besov spaces that are similar to the critical space for the barotropic model.

1.3. **Main result.** In the following, we occasionally denote by I the time interval $[0,T]$. We define $E_p(T)$ as the space in which the tempered distribution $v \in \widetilde{B}^{s,\frac{n}{p}-1}_{p,1}$ satisfies

$$\begin{aligned} &v \in C(I; \widetilde{B}^{s,\frac{n}{p}-1}_{p,1}) \cap L^2(I; \widetilde{B}^{s+1,\frac{n}{p}}_{p,1}) \\ &\text{and} \quad \partial_t v_H, \nabla^2 v_H \in L^1(I; \dot{B}^{\frac{n}{p}-1}_{p,1}). \end{aligned} \tag{1.6}$$

The norm of $E_p(T)$ is defined by

$$\|v\|_{E_p(T)} := \|v\|_{L^\infty(I;\widetilde{B}^{s,\frac{n}{p}-1}_{p,1})} + \|Dv\|_{L^2(I;\widetilde{B}^{s,\frac{n}{p}-1}_{p,1})} + \|\partial_t v_H, \nabla^2 v_H\|_{L^1(I;\dot{B}^{\frac{n}{p}-1}_{p,1})}.$$

The first result concerns the existence and uniquenesss of the local-in-time solution $(\overline{\rho}, \overline{u}, \overline{\psi})$ to the system (1.4):

Theorem 1.1 ([4]). *Let* $1 < p < \dfrac{2n}{1+\frac{n}{p}-s}$,

$$\frac{n}{p}-1 \le s \le \frac{n}{p} \quad \text{if} \quad n \ge 3 \quad \text{and} \quad \frac{n}{p}-1 \lneq s \le \frac{n}{p} \quad \text{if} \quad n=2. \tag{1.7}$$

Let u_0 *be a vector field in* $\widetilde{B}_{p,1}^{s,\frac{n}{p}-1}$. *Assume that the initial density* ρ_0 *satisfies* $a_0 := (\rho_0 - 1) \in \widetilde{B}_{p,1}^{s-1,\frac{n}{p}}$ *and*

$$\inf_x \rho_0(x) > 0. \tag{1.8}$$

Then the system (1.4) *admits a unique local solution* $(\overline{\rho}, \overline{u}, \overline{\psi})$ *with* $\overline{a} := \overline{\rho} - 1$ *in* $C(I; \widetilde{B}_{p,1}^{s-1,\frac{n}{p}})$, $\overline{u}$ *in* $E_p(T)$ *and* $\nabla^2\overline{\psi}$ *in* $C(I; \dot{B}_{p,1}^{s-1})$. *Moreover, the flow map* $(a_0, u_0) \mapsto (\overline{a}, \overline{u})$ *is Lipschitz continuous from* $\widetilde{B}_{p,1}^{s-1,\frac{n}{p}} \times \widetilde{B}_{p,1}^{s,\frac{n}{p}-1}$ *to* $C(I; \widetilde{B}_{p,1}^{s-1,\frac{n}{p}}) \times E_p(T)$.

As one can see easily from the above, in terms of the admissibility of the exponent p, taking $s = \frac{n}{p}$ gives the best result. Now, Theorem 1.1 can be written as follows in the Euclidian coordinate:

Theorem 1.2 ([4]). *Under the same assumptions as in Theorem 1.1, the system* (1.1) *has a unique local solution* (ρ, u, ψ) *with* $u \in E_p(T)$, ρ *bounded away from 0 and* $\rho - 1 \in C(I; \widetilde{B}_{p,1}^{s-1,\frac{n}{p}})$, *and* $\nabla^2\psi \in C(I; \widetilde{B}_{p,1}^{s-1,\frac{n}{p}})$.

Remark 1.3. One would expect $\overline{\psi}$ to have the natural regularity $C(I; \widetilde{B}_{p,1}^{s-1,\frac{n}{p}+2})$ since $\overline{\psi}$ is a solution to the second order elliptic equation with the outer force $\overline{a} \in C(I; \widetilde{B}_{p,1}^{s-1,\frac{n}{p}})$. This is not attainable due to the failure of elliptic estimate (see Proposition 3.1) with the high regularity. However, when reverting back to Eulerian coordinate, one may prove by the lifting property of $(-\Delta)^{-1}$ that $\nabla^2\psi$ (in Eulerian coordinate) does belong to $C(I; \widetilde{B}_{p,1}^{s-1,\frac{n}{p}})$.

1.4. **Banach's fixed point argument.** In the rest of this section, we drop the bars on the functions in the Lagrangian coordinate. We assume $a_0 = (\rho_0 - 1) \in \widetilde{B}_{p,1}^{s-1,\frac{n}{p}}$ and $u_0 \in \widetilde{B}_{p,1}^{s,\frac{n}{p}-1}$ and solve the system (1.4) in the function space $E_p(T)$. Let us first linearize the system (1.4) into a quasi-linear parabolic system with variable coefficients. We denote $L_{\rho_0}u := \partial_t u - \rho_0^{-1}\mathrm{div}\,(2\mu D(u) + \lambda \mathrm{div}\, u\, \mathrm{Id})$ and write

$$\begin{cases} L_{\rho_0}u = \rho_0^{-1}\Big(\mathrm{div}\,(I_1(u,u) + I_2(u,u) + I_2(u,u) + I_3(u)) + I_4(u,\psi)\Big), \\ -\,\mathrm{div}\,(\mathrm{adj}(DX)^t A \nabla\psi) = \rho_0 - J, \end{cases}$$

where

$$\begin{aligned}
&I_1(v,w) := (\mathrm{adj}(DX_v) - \mathrm{Id})\left(\mu(DwA_v +^t A_v\nabla w) + \lambda({}^tA_v : \nabla w)\mathrm{Id}\right),\\
&I_2(v,w) := \mu(Dw(A_v - \mathrm{Id}) +^t (A_v - \mathrm{Id})\nabla w) + \lambda({}^t(A_v - \mathrm{Id}) : \nabla w)\mathrm{Id},\\
(1.9)\qquad &I_3(v) := -\mathrm{adj}(DX_v)P(J_v^{-1}\rho_0),\\
&I_4(v,\psi) := {}^t\mathrm{adj}(DX_v)\nabla\psi\\
&\text{with } \psi \text{ determined by } -\mathrm{div}\,(\mathrm{adj}(DX_v)^tA_v\nabla\psi) = \rho_0 - J_v.
\end{aligned}$$

As we will prove later, the Poisson equation can be solved independently; for a given $v \in E_p(T)$, the solution ψ to the elliptic equation is uniquely determined. Hence, in order to solve (1.4), it suffices to show that the map

$$\Phi : v \mapsto u \tag{1.10}$$

with u the solution to the following linear system

$$\begin{cases} L_{\rho_0}u = \rho_0^{-1}\Big(\mathrm{div}\,(I_1(v,v) + I_2(v,v) + I_2(v,v) + I_3(v)) + I_4(v,\psi)\Big),\\ -\mathrm{div}\,(\mathrm{adj}(DX_v)^tA_v\nabla\psi) = \rho_0 - J_v \end{cases}$$

has a fixed point in $E_p(T)$ for small enough T.

2. Preliminaries

2.1. **Estimate for product, composition and commutator.**

For the proofs of the following propositions, see [1], [8] and [9].

Lemma 2.1. *Let $\nu \geq 0$ and $-\min(\frac{n}{p}, \frac{n}{p'}) < s \leq \frac{n}{p} - \nu$. The following product law holds:*

$$\|uv\|_{\dot B^s_{p,1}} \leq C\|u\|_{\dot B^{\frac{n}{p}-\nu}_{p,1}}\|v\|_{\dot B^{s+\nu}_{p,1}}.$$

Lemma 2.2. *Let I an open interval of $\mathbb{R}$ containing 0 and let $F : I \to \mathbb{R}$ be a smooth function vanishing at 0. Then for any $s > 0$, $1 \leq p \leq \infty$ and interval J compactly supported in I there exists a constant C such that*

$$\|F(a)\|_{\dot B^s_{p,1}} \leq C\|a\|_{\dot B^s_{p,1}}$$

for any $a \in \dot B^s_{p,1}$ with values in J.

2.2. **Lagrangean coordinates and estimates of flow.**

Proposition 2.3 ([8],[9]). *Let X be a globally bi-Lipschitz diffeomorphism of $\mathbb{R}^n$ and (s,p,q) with $1 \leq p < \infty$ and $-\frac{n}{p'} < s < \frac{n}{p}$ (or just $-\frac{n}{p'} < s \leq \frac{n}{p}$ if $q = 1$ and $-\frac{n}{p'} \leq s < \frac{n}{p}$ if $q = \infty$). Then $a \mapsto a \circ X$ is a self-map over $\dot B^s_{p,q}$ in the following cases:*

1. *$s \in (0,1)$,*
2. *$s \in (-1,0]$ and $J_{X^{-1}}$ is in the multiplier space $\mathcal{M}(\dot B^s_{p,q})$,*
3. *$s \geq 1$ and $(DX - \mathrm{Id}) \in \dot B^s_{p,q}$.*

Lemma 2.4 ([4]). *Let $1 \le p < \infty$, $-\min(\frac{n}{p}, \frac{n}{p'}) < s \le \frac{n}{p}$ and $v \in E_p(T)$. Assume that*

$$\int_0^T \|Dv\|_{\dot{B}^{\frac{n}{p}}_{p,1}} dt \le \widetilde{c}$$

holds for a small enough constant $\widetilde{c}$. Then for all $t \in [0,T]$, we have

$$\|\mathrm{Id} - \mathrm{adj}(DX_v(t))\|_{\dot{B}^s_{p,1}} \le C\|Dv\|_{L^1(I;\dot{B}^s_{p,1})}, \tag{2.1}$$

$$\|\mathrm{Id} - A_v(t)\|_{\dot{B}^s_{p,1}} \le C\|Dv\|_{L^1(I;\dot{B}^s_{p,1})}, \tag{2.2}$$

$$\|1 - J_v^{\pm 1}(t)\|_{\dot{B}^s_{p,1}} \le C\|Dv\|_{L^1(I;\dot{B}^s_{p,1})}. \tag{2.3}$$

Proof. The proofs are exactly the same as those in [8] and [9]. □

We also have the following difference estimate.

Lemma 2.5 ([4]). *Let $1 \le p < \infty$, $-\min\left(\frac{n}{p}, \frac{n}{p'}\right) < s \le \frac{n}{p}$. Assume that $\overline{v}_1$ and $\overline{v}_2 \in E_p(T)$ satisfy condition (3.2) and denote $\delta v := \overline{v}_2 - \overline{v}_1$. Then for all $t \in [0,T]$, we have*

$$\|A_2(t) - A_1(t)\|_{\dot{B}^s_{p,1}} \le C\|D\delta v\|_{L^1(I;\dot{B}^s_{p,1})}, \tag{2.4}$$

$$\|\mathrm{adj}(DX_2(t)) - \mathrm{adj}(DX_1(t))\|_{\dot{B}^s_{p,1}} \le C\|D\delta v\|_{L^1(I;\dot{B}^s_{p,1})}, \tag{2.5}$$

$$\|J_2^{\pm 1}(t) - J_1^{\pm 1}(t)\|_{\dot{B}^s_{p,1}} \le C\|D\delta v\|_{L^1(I;\dot{B}^s_{p,1})}. \tag{2.6}$$

3. A priori estimates for linearized systems

3.1. **A priori estimate for Poisson equation.** We first derive the a priori estimate for the potential term. Let ψ be a solution for

$$-\mathrm{div}\,(\mathrm{adj}(DX)^t A \nabla\psi) = \rho_0 - J. \tag{3.1}$$

Proposition 3.1 ([4]). *Let $a_0 \in \dot{B}^s_{p,1}$ and $v \in E_p(T)$. Assume*

$$\int_0^T \|Dv\|_{\dot{B}^{\frac{n}{p}}_{p,1}} dt \le \widetilde{c} \tag{3.2}$$

for a small enough $\widetilde{c}$. Then (3.1) admits a unique solution ψ that satisfies the estimate

$$\|\nabla^2\psi\|_{L^\infty(I;\dot{B}^s_{p,1})} \le C\left(\|a_0\|_{\dot{B}^s_{p,1}} + \|Dv\|_{L^1(I;\dot{B}^s_{p,1})}\right) \tag{3.3}$$

where s satisfies the condition

$$-n\min\left(\frac{1}{p}, \frac{1}{p'}\right) < s \le \frac{n}{p} - 1. \tag{3.4}$$

Proof. The existence of the solution ψ to (3.1) can be assured by fixed point argument under the assumptions above. To prove the estimate (3.3), note the equivalent expression

$$-\Delta\psi = \rho_0 - J_v - \operatorname{div}\Big((\operatorname{adj}(DX_v) - \mathrm{Id})({}^tA_v - \mathrm{Id})\nabla\psi \\ +(\operatorname{adj}(DX_v) - \mathrm{Id})\nabla\psi\mathrm{Id}) + ({}^tA_v - \mathrm{Id})\nabla\psi\Big).$$

We write $-\Delta\psi = a_0 + 1 - J_v + \operatorname{div} I_5(v,\psi)$ with

$$I_p(v,\psi) := (\operatorname{adj}(DX_v) - \mathrm{Id})({}^tA_v - \mathrm{Id})\nabla\psi + (\operatorname{adj}(DX_v) - \mathrm{Id})\nabla\psi + ({}^tA_v - \mathrm{Id})\nabla\psi.$$

Thus,

$$\|\nabla^2\psi\|_{L^\infty(I;\dot B^s_{p,1})} \le \|a_0\|_{L^\infty(I;\dot B^s_{p,1})} + \|1 - J_v\|_{L^\infty(I;\dot B^s_{p,1})} + \|I_5(v,\psi)\|_{L^\infty(I;\dot B^{s+1}_{p,1})}.$$

For $1 - J_v$, we have by Lemma 2.4,

$$\|1 - J_v\|_{L^\infty(I;\dot B^s_{p,1})} \le \|Dv\|_{L^1(I;\dot B^s_{p,1})},$$

where we need

$$-n\min(\frac{1}{p},\frac{1}{p'}) < s \le \frac{n}{p}. \tag{3.5}$$

By Lemma 2.1

$$\begin{aligned}
&\|I_p(u,\psi)\|_{L^\infty(I;\dot B^{s+1}_{p,1})} \\
&\quad\le C\|(\operatorname{adj}(DX_v) - \mathrm{Id})({}^tA_v - \mathrm{Id})\nabla\psi\|_{L^\infty(I;\dot B^{s+1}_{p,1})} \\
&\qquad + \|(\operatorname{adj}(DX_v) - \mathrm{Id})\nabla\psi)\|_{L^\infty(I;\dot B^{s+1}_{p,1})} + \|({}^tA_v - \mathrm{Id})\nabla\psi\|_{L^\infty(I;\dot B^{s+1}_{p,1})} \\
&\quad\le C\|\operatorname{adj}(DX_v) - \mathrm{Id}\|_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1})}\|{}^tA_v - \mathrm{Id}\|_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1})}\|\nabla\psi\|_{L^\infty(I;\dot B^{s+1}_{p,1})} \\
&\qquad + \|\operatorname{adj}(DX_v) - \mathrm{Id}\|_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1})}\|\nabla\psi\|_{L^\infty(I;\dot B^{s+1}_{p,1})} + \|{}^tA_v - \mathrm{Id}\|_{L^\infty(I;\dot B^{\frac{n}{p}}_{p,1})}\|\nabla\psi\|_{L^\infty(I;\dot B^{s+1}_{p,1})} \\
&\quad\le C(\|Dv\|^2_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})} + \|Dv\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})})\|\psi\|_{L^\infty(I;\dot B^{s+2}_{p,1})},
\end{aligned}$$

where we need

$$-n\min(\frac{1}{p},\frac{1}{p'}) - 1 < s \le \frac{n}{p} - 1. \tag{3.6}$$

By (3.5) and (3.6) we have the restriction (3.4). Hence, if $\widetilde{c}$ is taken suitably small, then we have

$$\|\psi\|_{L^\infty(I;\dot B^{s+2}_{p,1})} \le C\|a_0\|_{\dot B^s_{p,1}} + \|Dv\|_{L^1(I;\dot B^s_{p,1})}.$$

□

3.2. **The a priori estimate for the Lamé system.** We first look at the following Lamé system with nonconstant coefficients:

$$\partial_t u - 2a\mathrm{div}\,(\mu D(u)) - b\nabla(\lambda \mathrm{div}\, u) = f. \tag{3.7}$$

Both u and f are valued in $\mathbb{R}^n$. We assume throughout that the following uniform ellipticity condition is satisfied:

$$\alpha := \min\left(\inf_{(t,x)\in[0,T]\times\mathbb{R}^n}(a\mu)(t,x),\ \inf_{(t,x)\in[0,T]\times\mathbb{R}^n}(2a\mu+b\lambda)(t,x)\right) > 0 \tag{3.8}$$

For (3.7) with rough coefficients that are only in $L^\infty(I;\dot{B}^{\frac{n}{p}-1}_{p,1})$, we have the following proposition due to Danchin [9].

Proposition 3.2 ([9]). *Let a, b, λ and μ be bounded and uniformly continuous functions satisfying* (3.8). *Assume that $a\nabla\mu$, $b\nabla\lambda$, $\mu\nabla a$ and $\lambda\nabla b$ are in $L^\infty(I;\dot{B}^{\frac{n}{p}-1}_{p,1})$ for some $1<p<\infty$. There exist two constants η and κ such that if for some $m\in\mathbb{Z}$ we have*

$$\min\left(\inf_{(t,x)\in[0,T]\times\mathbb{R}^n} S_m(a\mu)(t,x),\ \inf_{(t,x)\in[0,T]\times\mathbb{R}^n} S_m(2a\mu+b\lambda)(t,x)\right) \geq \frac{\alpha}{2}, \tag{3.9}$$

$$\|(\mathrm{Id}-S_m)(a\nabla\mu, b\nabla\lambda, \mu\nabla a, \lambda\nabla b)\|_{L^\infty(I;\dot{B}^{\frac{n}{p}-1}_{p,1})} \leq \eta\alpha, \tag{3.10}$$

then the solutions to (3.7) *satisfy for all $t\in[0,T]$,*

$$\|u\|_{L^\infty(0,t;\dot{B}^s_{p,1})} + \alpha\|u\|_{L^1(0,t;\dot{B}^{s+2}_{p,1})}$$
$$\leq C(\|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^1(0,t;\dot{B}^s_{p,1})})\exp\left(\frac{C}{\alpha}\int_0^t (\|S_m(a\nabla\mu, b\nabla\lambda, \mu\nabla a, \lambda\nabla b)\|^2_{\dot{B}^{\frac{n}{p}}_{p,1}}\,d\tau\right)$$

whenever

$$-\min\left(\frac{n}{p},\frac{n}{p'}\right) < s \leq \frac{n}{p}-1. \tag{3.11}$$

The range of s in (3.11) of Proposition 3.2 does not include the case $\frac{n}{p}-1<s\leq\frac{n}{p}$. However, to close the estimate on the potential term, we are required to bound the velocity field u in $L^\infty(I;\widetilde{B}^{s,\frac{n}{p}-1}_{p,1})\cap L^2(I;\widetilde{B}^{s+1,\frac{n}{p}}_{p,1})$. To this end, we shall need the following estimate, the idea of which is to give up the full parabolic regularity so that the range of the regularity s may be taken higher.

For a starter, we shall look at the following heat equation with nonconstant coefficients:

$$\partial_t u - a\mathrm{div}\,(b\nabla u) = f. \tag{3.12}$$

Proposition 3.3 ([4]). *Let a and b be bounded functions satisfying $ab\geq\alpha>0$. Assume that $a\nabla b$ and $b\nabla a$ are in $L^\infty(I;\dot{B}^{\frac{n}{p}-1}_{p,1})$ and f in $L^1(I;\dot{B}^s_{p,1})$ for some $1<p<\infty$. There exist two constants η and κ such that if for some $m\in\mathbb{Z}$ we have*

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^n} S_m(ab)(t,x) \geq \frac{\beta}{2},$$

$$\|(\mathrm{Id}-S_m)(a\nabla b, b\nabla a)\|_{L^\infty(I;\dot B^{\frac{n}{p}-1}_{p,1})} \le \eta\beta,$$

then the solutions to (3.12) *satisfy for all* $t \in [0,T_1]$ ($T_1 \le T$),

$$\|u\|_{L^\infty(0,t;\dot B^s_{p,1})} + \beta\|u\|_{L^2(0,t;\dot B^{s+1}_{p,1})} \le C(p,a,b,m,T_1)(\|u_0\|_{\dot B^s_{p,1}} + \|f\|_{L^1(0,t;\dot B^s_{p,1})})$$

whenever

$$-\min(\frac{n}{p},\frac{n}{p'})+1 < s \le \frac{n}{p}. \tag{3.13}$$

For the proof of the above, we refer to [4]. The natural extension of this to the Lamé system is given by the following. To prove it, one must simply decompose the Lamé system into two heat equations in the manner of Proposition 3.2 in [9], and apply Proposition 3.3. We omit the proof of Proposition 3.4.

Proposition 3.4 ([4]). *Let* a*,* b*,* λ *and* μ *satisfy the same hypothesis as Proposition 3.2. Then the solutions to* (3.7) *satisfy for all* $t \in [0,T_1]$ ($T_1 \le T$)*, Then the solutions to* (3.7) *satisfy for all* $t \in [0,T]$,

$$\|u\|_{L^\infty(0,t;\dot B^s_{p,1})} + \beta\|u\|_{L^2(0,t;\dot B^{s+1}_{p,1})} \le C(p,a,b,\mu,\lambda,m,T_1)(\|u_0\|_{\dot B^s_{p,1}} + \|f\|_{L^1(0,t;\dot B^s_{p,1})}),$$

whenever s *satisfies* (3.13).

In practice, we will use the following proposition which bounds the low and high frequencies of the velocity field u with different regularity indices, in which there is a margin of higher admissibility of s for the high frequency.

Proposition 3.5 ([4]). *Let* a*,* b*,* λ *and* μ *satisfy the same hypothesis as Proposition 3.2. Let* u_0 *belongs to* $\widetilde{B}^{s_1,s_2}_{p,1}$*. Then the solutions to* (3.7) *satisfy for all* $t \in [0,T_1]$ ($T_1 \le T$),

$$\begin{aligned}\|u\|_{L^\infty(0,t;\widetilde{B}^{s_1,s_2}_{p,1})} &+ \beta\|u\|_{L^2(0,t;\widetilde{B}^{s_1+1,s_2+1}_{p,1})} + \alpha\|u_H\|_{L^1(0,t;\dot B^{s_2+2}_{p,1})}\\ &\le C(p,a,b,\mu,\lambda,m,T_1)(\|u_0\|_{\widetilde{B}^{s_1,s_2}_{p,1}} + \|f\|_{L^1(0,t;\widetilde{B}^{s_1,s_2}_{p,1})})\end{aligned}$$

whenever s_1 *satisfies* (3.13) *and* s_2 *satisfies* (3.11).

Proof. When $s_1 \le s_2$, it is obvious. When $s_1 > s_2$, We decompose u and f into $f = f_1 + f_2$ and $u = u_1 + u_2$ with $u_1, f_1 \in \dot B^{s_1}_{p,1}$ and $u_2, f_2 \in \dot B^{s_2}_{p,1}$. Then it is just a matter of applying Proposition 3.2 and Proposition 3.4 to each linear equation for u_1 and u_2, and adding the resulting inequalities. □

4. Outline of the proof of Theorem 1.1

We only give here the outline of the proof. For the details, see [4]. Let I denote the time interval $[0,T]$ as before. Let us note that for $v \in E_p(T)$, we have

$$\begin{aligned}\|Dv\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})} &\le \|Dv_L\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})} + \|Dv_H\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}\\ &\le T^{\frac12}\|Dv_L\|_{L^2(I;\dot B^s_{p,1})} + \|Dv_H\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})} \le C(T)\|v\|_{E_p(T)} < \infty,\end{aligned}$$

and

$$\|v\|_{L^1(I;\dot B^s_{p,1})} \le \|v_L\|_{L^1(I;\dot B^s_{p,1})} + \|v_H\|_{L^1(I;\dot B^s_{p,1})}$$
$$\le T\|v_L\|_{L^\infty(I;\dot B^s_{p,1})} + T^{\frac12}\|v_H\|_{L^2(I;\dot B^{\frac np}_{p,1})} \le C(T)\|v\|_{E_p(T)} < \infty,$$

with some $C(T)$ depending on T. These enable us to use the flow estimates (Lemma 2.4 and 2.5) in the same manner as [9]. We assume from now on that

$$\|Dv\|_{L^1(I;\dot B^{\frac np}_{p,1})} \le \widetilde{c}$$

is satisfied for a small enough constant $\widetilde{c}$.

We denote the linear part of the solution u by U, i.e.,

$$L_1 U = 0,\ U|_{t=0} = u_0.$$

Recall that L_1 is given by $L_{\rho_0}u := \partial_t u - \rho_0^{-1}\mathrm{div}\,(2\mu(\rho_0)D(u) + \lambda(\rho_0)\mathrm{div}\,u\mathrm{Id})$ with $\rho_0 \equiv 1$. Let $\widetilde{u} := u - U$ then $(\widetilde{u}, \psi)$ has to satisfy

$$(4.1)\qquad \begin{cases} L_{\rho_0}\widetilde{u} = \rho_0^{-1}\Big(\mathrm{div}\,(I_1(v,v) + I_2(v,v) + I_3(v)) + I_4(v,\psi)\Big) + (L_1 - L_{\rho_0})U, \\ -\,\mathrm{div}\,(\mathrm{adj}(DX_v)^t A_v \nabla\psi) = \rho_0 - J_v, \end{cases}$$

with $v \in E_p(T)$. We claim that the Banach fixed point theorem applies to the map Φ defined in (1.10) in some closed ball $\bar B_{E_p(T)}(U,R)$ with suitably small T and R.

If the right-hand side of the first equation is in $L^1(I;\dot B^{\frac np -1}_{p,1})$ and if there exists some $m \in \mathbb{Z}$ so that the conditions of Proposition 3.2 are satisfied then $\bar u \in E_p(T)$. Let α be defined by $\alpha := \inf_{x\in\mathbb{R}^n} \dfrac{1}{\rho_0(x)}$. Now, the existence of m so that

$$\inf_{x\in\mathbb{R}^n} \dot S_m\Big(\frac{1}{\rho_0}\Big) \ge \frac{\alpha}{2} \quad\text{and}\quad \Big\|(\mathrm{Id} - \dot S_m)\Big(\frac{\nabla\rho_0}{\rho_0^2}\Big)\Big\|_{L^\infty(I;\dot B^{\frac np -1}_{p,1})} \le \eta\alpha$$

is ensured by the fact that all the coefficients minus some constant belong to the space $\dot B^{\frac np}_{p,1}$ which is defined in terms of a convergent series and embeds continuously in the set of bounded continuous functions (that tend to 0 at infinity).

First step: Stability of the ball $\bar B_{E_p(T)}(U,R)$ for suitably small T and R . Applying Proposition 3.5 with $s_1 = s$ and $s_2 = \dfrac np - 1$ gives us

$$\|\widetilde{u}\|_{E_p(T)} \le Ce^{C_{\rho_0,m}T}\Big(\|(L_1 - L_{\rho_0})U\|_{L^1(I;\widetilde B^{s,\frac np -1}_{p,1})}$$
$$+ \|\rho_0^{-1}\|_{\mathcal{M}(\widetilde B^{s,\frac np -1}_{p,1})}\|\mathrm{div}\,(I_1(v,v) + I_2(v,v) + I_3(v)) + I_4(v,\theta)\|_{L^1(I;\widetilde B^{s,\frac np -1}_{p,1})}\Big)$$
$$(4.2)\qquad \le Ce^{C_{\rho_0,m}T}\Big(\|(L_1 - L_{\rho_0})U\|_{L^1(I;\dot B^{\frac np -1}_{p,1})}$$
$$+ \|\rho_0^{-1}\|_{\mathcal{M}(\widetilde B^{s,\frac np -1}_{p,1})}\Big(\|\mathrm{div}\,(I_1(v,v) + I_2(v,v) + I_3(v))\|_{L^1(I;\dot B^{\frac np -1}_{p,1})}$$
$$+ \|I_4(v,\theta)\|_{L^1(I;\dot B^s_{p,1})}\Big)\Big),$$

where we used the convenient property that $L^1(I;\widetilde{B}^{s,\frac{n}{p}-1}_{p,1}) = L^1(I;\dot{B}^s_{p,1}) + L^1(I;\dot{B}^{\frac{n}{p}-1}_{p,1})$. Here, the space $\mathcal{M}(\dot{B}^s_{p,1})$ is the multiplier space defined as the space of all tempered distributions such that $\|f\|_{\mathcal{M}(\dot{B}^s_{p,1})} := \sup_{\|h\|_{\dot{B}^s_{p,1}}=1} \|hf\|_{\dot{B}^s_{p,1}}$. With our assumption on p, we may confirm that ρ_0^{-1} belongs to $\mathcal{M}(\widetilde{B}^{s,\frac{n}{p}-1}_{p,1})$ when $\rho_0 - 1 \in \widetilde{B}^{s-1,\frac{n}{p}}_{p,1}$ by the product and composition estimates:

$$\|\rho_0^{-1}h\|_{\widetilde{B}^{s,\frac{n}{p}-1}_{p,1}} \le \|(\frac{a_0}{1+a_0} - 1)h\|_{\widetilde{B}^{s,\frac{n}{p}-1}_{p,1}} \le (\|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}} + 1)\|h\|_{\widetilde{B}^{s,\frac{n}{p}-1}_{p,1}}$$

if $1 \le p < \frac{2n}{1+\frac{n}{p}-s}$ and s as in (1.7).

<u>Estimate of $L_1 - L_{\rho_0}$ and I_j (j=1, 2, 3.)</u> By [9], we know that

$$\|(L_1 - L_{\rho_0})U\|_{L^1(I;\dot{B}^{\frac{n}{p}-1}_{p,1})} \le C\|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|DU\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})},$$

$$\|I_j(v,w)\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} \le C\|Dv\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})}\|Dw\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})},$$

for $j = 1, 2$ and

$$\|I_3(v)\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} \le CT(\|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}} + 1)(\|Dv\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} + 1).$$

<u>Estimate of I_4</u> For $I_4(v,\psi) = {}^tA_v : \nabla\psi$, we have

$$\|I_4(v,\psi)\|_{L^1(I;\dot{B}^s_{p,1})} \le CT(\|Dv\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} + 1)\|\nabla\psi\|_{L^\infty(I;\dot{B}^s_{p,1})}.$$

Now we use Proposition 3.1 with $s = s-1$ to bound $\nabla\psi$ in $L^\infty(I;\dot{B}^s_{p,1})$. We have

$$\|\nabla\psi\|_{L^\infty(I;\dot{B}^s_{p,1})} \le C\|a_0\|_{\dot{B}^{s-1}_{p,1}} + \|v\|_{L^1(I;\dot{B}^s_{p,1})}.$$

Therefore,

$$\|I_4(v,\psi)\|_{L^1(I;\dot{B}^{s+1}_{p,1})} \le CT(\|Dv\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} + 1)\big(\|a_0\|_{\dot{B}^{s-1}_{p,1}} + \|v\|_{L^1(I;\dot{B}^s_{p,1})}\big).$$

Plugging the above estimates in (4.2), we obtain

$$\|\widetilde{u}\|_{E_p(T)} \le Ce^{C_{\rho_0,m}T}(\|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}} + 1)^2\Big(T(1 + \|a_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}) + \|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|DU\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} + \|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|Dv\|^2_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} + T\|v\|_{L^1(I;\dot{B}^s_{p,1})}\Big).$$

Since $v \in B_{E_p(T)}(U,R)$, decomposing v into $\widetilde{v} + U$ gives us

$$\|\widetilde{u}\|_{E_p(T)} \le Ce^{C_{\rho_0,m}T}(\|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}} + 1)^2\Big(T(1 + \|a_0\|_{\dot{B}^{\frac{n}{p}-1}_{p,1}}) + \|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\|DU\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} + \|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}}(\|DU\|_{L^1(I;\dot{B}^{\frac{n}{p}}_{p,1})} + R)^2 + T(\|U\|_{L^1(I;\dot{B}^s_{p,1})} + R)\Big).$$

We first choose R so that for a small enough constant η,

$$C(\|a_0\|_{\dot{B}^{\frac{n}{p}}_{p,1}} + 1)^2 R \le \eta \tag{4.3}$$

and take T so that

$$\begin{aligned}&C_{\rho_0,m}T\le \log 2,\quad T(1+\|a_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}})\le R^2,\\ &T(\|U\|_{L^1(I;\dot B^s_{p,1})}+R)\le R,\quad \|a_0\|_{\dot B^{\frac{n}{p}}_{p,1}}\|DU\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}\le R^2,\end{aligned}\tag{4.4}$$

then we conclude that Φ is a self-map on $v\in B_{E_p(T)}(U,R)$.

Second step: contraction estimates. We next establish that, with suitably small R and T, Φ is contractive. We consider two vector-fields $v_1, v_2 \in \bar B_{E_p(T)}(u_L, R_1)$ and set $\delta v = v_2 - v_1$. We also set ψ_j to be a solution corresponding to the Poisson equation with given data v_j

$$-\mathrm{div}\,(\mathrm{adj}(DX_{v_j})^t A_{v_j}\nabla\psi_j)=\rho_0-J_{v_j}\tag{4.5}$$

for $j = 1,2$, and denote $\delta\psi := \psi_2 - \psi_1$. In order to prove that Φ is contractive, it is a matter of applying Proposition 3.5 and the potential estimate in the spirit of Proposition 3.1 to

$$\begin{cases} L_{\rho_0}(\Phi(v_2)-\Phi(v_1))\\ \qquad =\rho_0^{-1}\Big(\mathrm{div}\,\big((I_1(v_2,v_2)-I_1(v_1,v_1))+(I_2(v_2,v_2)-I_2(v_1,v_1))\\ \qquad\qquad +(I_3(v_2)-I_3(v_1))\big)+(I_4(v_2,\psi_2)-I_4(v_1,\psi_1))\Big),\\ -\mathrm{div}\,(\mathrm{adj}(DX_{v_j})^t A_{v_j}\nabla\psi_j)=\rho_0-J_{v_j}\ (j=1,2),\end{cases}\tag{4.6}$$

where I_j's are defined in (1.9).

Propositon 3.2 and the definition of the multiplier space $\mathcal{M}(\widetilde{B}^{s,\frac{n}{p}-1}_{p,1})$ give that

$$\begin{aligned}&\|\Phi(v_2)-\Phi(v_1)\|_{E_p(T)}\\ &\le Ce^{C_{\rho_0,m}T}\Big(\|\rho_0^{-1}\mathrm{div}\,\{(I_1(v_2,v_2)-I_1(v_1,v_1))+(I_2(v_2,v_2)-I_2(v_1,v_1))\\ &+(I_3(v_2)-I_3(v_1))+(I_4(v_2,\psi_2)-I_4(v_1,\psi_1))\}\|_{L^1(I;\dot B^{s,\frac{n}{p}-1}_{p,1})}\Big)\\ &\le Ce^{C_{\rho_0,m}T}\|\rho_0^{-1}\|_{\mathcal{M}(\widetilde{B}^{s,\frac{n}{p}-1}_{p,1})}\Big(\|I_1(v_2,v_2)-I_1(v_1,v_1)\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}\\ &\qquad+\|I_2(v_2,v_2)-I_2(v_1,v_1)\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}\\ &\qquad+\|I_3(v_2)-I_3(v_1)\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}+\|I_4(v_2,\theta_2)-I_4(v_1,\theta_1)\|_{L^1(I;\dot B^s_{p,1})}\Big).\end{aligned}$$

Thanks to Lemma 2.1 and 2.5, we may estimate all the terms appearing on the right-hand side to obtain

$$\begin{aligned}&\|\Phi(v_2)-\Phi(v_1)\|_{E_p(T)}\\ &\le Ce^{C_{\rho_0,m}T}(1+\|a_0\|_{\dot B^{\frac{n}{p}}_{p,1}})\Big(C_{\rho_0}\|(Dv_1,Dv_2)\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}\|D\delta v\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}\\ &\qquad+T(1+\|a_0\|_{\dot B^{\frac{n}{p}}_{p,1}})\|D\delta v\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}\\ &\qquad+T\big((\|a_0\|_{\dot B^{\frac{n}{p}-1}_{p,1}}+T\|v_2\|_{L^\infty(I;\dot B^s_{p,1})})\|D\delta v\|_{L^1(I;\dot B^{\frac{n}{p}}_{p,1})}+\|\delta v\|_{L^1(I;\dot B^s_{p,1})}\big)\Big).\end{aligned}$$

Given that $v_1, v_2 \in \bar{B}_{E_p(T)}(U,R)$ and our hypothesis over T and R (with smaller η in (4.3) if needed) thus ensure that,

$$\|\Phi(v_2)-\Phi(v_1)\|_{E_p(T)} \leq \frac{1}{2}\|\delta v\|_{E_p(T)}.$$

One can thus conclude that Φ admits a unique fixed point in $\bar{B}_{E_p(T)}(U,R)$.

Third step: Regularity of the density and the potential. Granted with the above velocity field u in $E_p(T)$, we set $\rho := J_u^{-1}\rho_0$, then proving that $a := \rho - 1$ is in $C(I;\dot{B}^{\frac{n}{p}}_{p,1})$ is easy by construction, thanks to product estimate. Moreover, Because $\dot{B}^{\frac{n}{p}}_{p,1}$ is continuously embedded in L^∞, condition $\inf_x \rho_0 > 0$ is fulfilled on $[0,T]$ (taking smaller T if needed).

To prove the regularity of ψ, it suffices to recal that $a_0 \in B^{\frac{n}{p}-1}_{p,1}$. Then by simply applying Proposition 3.1, with v replaced by u, we have that $\nabla\psi$ belongs to $C(I;\dot{B}^{\frac{n}{p}}_{p,1})$.

Last step: Uniqueness and continuity of the flow map. In order to prove the continuity of the flow map, we consider two couples (ρ_{01}, u_{01}) and (ρ_{02}, u_{02}) of data fulfilling the assumptions of Theorem 1.1 and we denote by (ρ_1, u_1) and (ρ_2, u_2) two solutions in $E_p(T)$ corresponding to those data. Making difference of the two equations corresponding to (ρ_1, u_1) and (ρ_2, u_2), it suffices to perform almost identical calculation to the second step.

4.1. **Proof of Theorem 1.2.** To prove Theorem 1.2, it suffices to use the following proposition.

Proposition 4.1 ([4]). *Assume that the triplet* (ρ,u,ψ) *with* $\rho-1 \in C(I;\widetilde{B}^{s-1,\frac{n}{p}}_{p,1})$, $u \in E_p(T)$ *and* $\nabla^2\psi \in C(I;\widetilde{B}^{s-1,\frac{n}{p}}_{p,1})$ *(with* $1<p<2n$*) is a solution for* (1.1) *such that*

$$\int_0^T \|\nabla u\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq \widetilde{c} \tag{4.7}$$

for a small enough constant $\widetilde{c}$. *Let* X *be the flow of* u *defined in* (1.3). *Then the triplet* $(\overline{\rho},\overline{u},\overline{\psi}) := (\rho\circ X, u\circ X, \psi\circ X)$ *belongs to the same functional space as* (ρ,u,ψ), *and satisfies* (1.4).

Conversely, if $\overline{\rho}-1 \in C(I;\widetilde{B}^{s-1,\frac{n}{p}}_{p,1})$, $\overline{u} \in E_p(T)$ *and* $\nabla^2\overline{\psi} \in C(I;\dot{B}^{s-1}_{p,1})$ $(\overline{\rho},\overline{u},\overline{\psi})$ *satisfies* (1.4) *and, for a small enough constant* $\widetilde{c}$,

$$\int_0^T \|\nabla \overline{u}\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \leq \widetilde{c} \tag{4.8}$$

then the map X *defined in* (1.5) *is a* C^1 *diffeomorphism over* $\mathbb{R}^n$ *and the triplet* $(\rho,u,\psi) := (\overline{\rho}\circ X^{-1}, \overline{u}\circ X^{-1}, \overline{\psi}\circ X^{-1})$ *satisfies* (1.1) *and has the same regularity as* $(\overline{\rho},\overline{u},\overline{\psi})$. *Moreover, one can prove by the potential estimate that* $\nabla^2\psi$ *actually belongs to* $C(I;\widetilde{B}^{s-1,\frac{n}{p}}_{p,1})$.

We consider data (ρ_0,u_0) with ρ_0 bounded away from 0, $(\rho_0-1) \in \widetilde{B}^{s-1,\frac{n}{p}}_{p,1}$ and $u_0 \in \widetilde{B}^{s,\frac{n}{p}-1}_{p,1}$. Then Theorem 1.1 provides a local solution $\overline{\rho},\overline{u},\overline{\psi}$ to system (1.4) with

$\overline{\rho} - 1 \in C(I; \widetilde{B}^{s-1,\frac{n}{p}}_{p,1})$, $\overline{u} \in E_p(T)$ and $\nabla^2\overline{\psi} \in C(I; \dot{B}^{s-1}_{p,1})$. If T is small enough then (4.7) is satisfied so Proposition 4.1 ensures that $(\rho, u, \psi) := (\overline{\rho} \circ X^{-1}, \overline{u} \circ X^{-1}, \overline{\psi} \circ X^{-1})$ is a solution of (1.1) in the desired functional space. In order to prove uniqueness, we consider two solutions (ρ_1, u_1, ψ_1) and (ρ_2, u_2, ψ_2) corresponding to the same data (ρ_0, u_0), and perform the Lagrangian change of variable, pertaining to the flow of u_1 and u_2 respectively. The obtained vector-fields $\overline{u}_1$ and $\overline{u}_2$ are in $E_p(T)$ and both satisfy (1.1) with the same ρ_0 and u_0. Hence they coincide, as a consequence of the uniqueness part of Theorem 1.1.

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