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Hereditary properties and obstructions of simplicial complexes

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Abstract

In this paper, we discuss the relation between shellability, sequentially Cohen-Macaulayness, and partitionability. Especially, our main concern is to see the difference of these properties when we require heredity. For a property $\mathcal{P}$, we say a simplicial complex satisfies hereditary-$\mathcal{P}$ if the simplicial complex itself and all the restrictions to subsets of its vertex set satisfy the property $\mathcal{P}$, and we want to see the difference between hereditary-shellability, hereditary-sequential Cohen-Macaulayness, and hereditary-partitionability. In this paper we briefly review on the relations and gaps between shellability, sequential Cohen-Macaulayness and partitionability, as well as their hereditary versions. Additionally, we provide a discussion on relations between these properties and h-triangles, especially, relations between partitionability and nonnegativity of h-triangles.

1 Introduction

Shellability is one of fundamental properties of simplicial complexes, studied in the area of combinatorics related to commutative algebra and topology. During the previous century, shellability and related properties are studied for pure simplicial complexes (i.e., simplicial complexes all of whose facets have the same dimension), in relation to polytopes or triangulations of manifolds (eg. [10, 14, 18]). But in recent days, it is generalized to nonpure simplicial complexes by Björner and Wachs [2, 3], and analogous theories are developed [2, 3, 4, 5]. In the study of shellability, its relation to many other properties such as Cohen-Macaulayness, partitionability, and so forth, is important. These related properties are also generalized to nonpure cases, and their relations and gaps are also well studied.

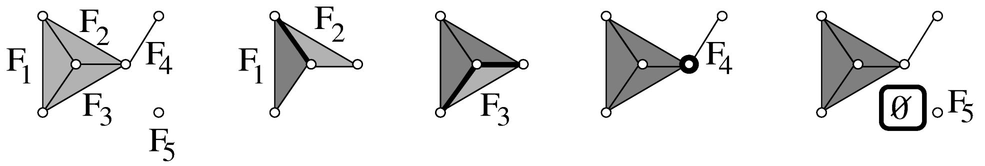
This paper treats hereditary versions of these properties. For a property $\mathcal{P}$ of simplicial complexes, we say a simplicial complex is hereditary-$\mathcal{P}$ if the simplicial complex as well as all the restrictions to subsets of its vertex set satisfy the property $\mathcal{P}$ [8]. Our main interest in this paper is the relation between hereditary-shellability, hereditary-sequential Cohen-Macaulayness, and hereditary-partitionability. It should be noted that matroid complexes are hereditary-pure simplicial complexes [1], which are also hereditary-shellable, hereditary-sequential Cohen-Macaulayness, and hereditary-partitionable. The view point to require the same property to all the restrictions started from the work of Wachs on obstructions to shellability [15]. An obstruction to a property $\mathcal{P}$ is a simplicial complex all of whose restrictions to its vertex set satisfies $\mathcal{P}$ except that the simplicial complex itself does not satisfy $\mathcal{P}$. There is a natural and obvious relation between obstructions and hereditary properties: obstructions to $\mathcal{P}$

are forbidden complexes of hereditary-$\mathcal{P}$ simplicial complexes with respect to restrictions, that is, a simplicial complex is hereditary-$\mathcal{P}$ if and only if no obstructions to $\mathcal{P}$ exist as its restrictions to subsets of the vertex set. The study of obstructions and hereditary properties are found in [8, 9, 15, 16, 17], however not so many is known currently. In Sections 2 and 3, we review what are known for the relations and gaps between shellability, sequentially Cohen-Macaulayness and partitionability, and their hereditary versions, mainly from the results of [8]. After that, in Section 4, we discuss nonnegativity of h-triangles and its use for showing the relations between these properties.

2 Shellability and related properties

In this paper, we assume the simplicial complexes are finite. Finite simplicial complexes can be treated as a set family on a finite set. Let V be a finite set. A *simplicial complex* Γ is a subset of 2^V satisfying that $F \subseteq F' \in \Gamma$ implies $F \in \Gamma$. The underlying set V is the vertex set of Γ. The elements of Γ are called *faces* and the maximal faces with respect to inclusion are *facets*. The *dimension* of a face F is $\dim F = |F| - 1$. A simplicial complex is *pure* if all the facets have the same dimension. Remark that we always assume $\emptyset \in \Gamma$ and $\dim \emptyset = -1$.

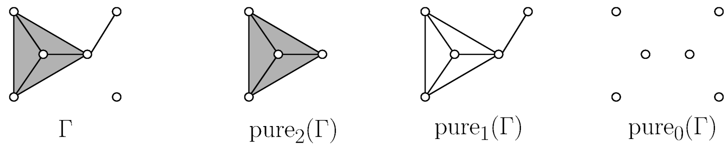
Shellability of simplicial complexes is a well-studied property, see [18, Lect. 8] for example. In this paper we adopt the definition for nonpure simplicial complexes, see [2, 3]. Here, a simplicial complex is *shellable* if its facets can be arranged into a sequence $F_1, F_2, \ldots, F_t$ satisfying the following condition: $(\overline{F_1} \cup \overline{F_2} \cup \cdots \cup \overline{F_{k-1}}) \cap \overline{F_k}$ is a pure simplicial complex of dimension $\dim F_k - 1$ for each $2 \leq k \leq t$, where $\overline{F}$ is a simplicial complex consisting of F and all its faces. This sequence of facets is called a *shelling*. The following figure shows an example of a shellable simplicial complex and one of its shelling. In the figure,the intersections $(\overline{F_1} \cup \overline{F_2} \cup \cdots \cup \overline{F_{k-1}}) \cap \overline{F_k}$ for $2 \leq k \leq 5$ are indicated.



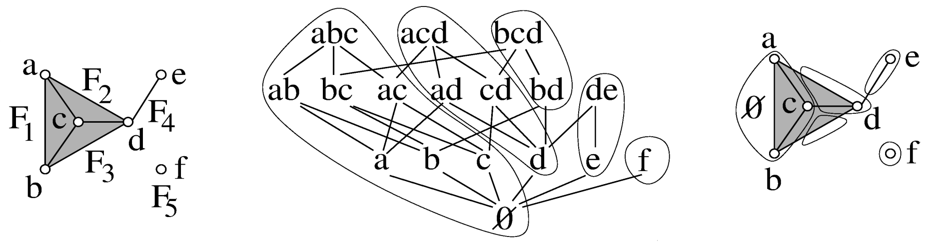
Related to shellability, the following implications are well-known.

vertex decomposable $\Rightarrow$ shellable $\nearrow$ sequentially Cohen-Macaulay
$\searrow$ partitionable

Sequential Cohen-Macaulayness is a property defined in terms of commutative algebra or topology, see [5, 14] for example. We here use the definition in terms of homology ([5]) as follows. A simplicial complex Γ is *Cohen-Macaulay* if $\widetilde{H}_i(\mathrm{link}_\Gamma(F)) = 0$ unless $i = \dim \mathrm{link}_\Gamma(F)$, where $\widetilde{H}_i$ denotes the i-dimensional reduced homology group, and $\mathrm{link}_\Gamma(F) = \{H \in \Gamma : H \cap F = \emptyset, H \cup F \in \Gamma\}$ is the link of F in Γ. (Note that $\mathrm{link}_\Gamma(F) = \Gamma$ if $F = \emptyset \in \Gamma$.) A simplicial complex Γ is *sequentially Cohen-Macaulay* if $\mathrm{pure}_i(\Gamma)$ is Cohen-Macaulay for all i, where $\mathrm{pure}_i(\Gamma)$, the i-dimensional *pure-skeleton* of Γ, is the subcomplex consisting of all the i-dimensional faces and its subfaces of Γ (following figure). By this definition, the implication "shellable $\Rightarrow$ sequentially Cohen-Macaulay" is easy to observe from the facts that pure shellable simplicial complexes are Cohen-Macaulay and that every $\mathrm{pure}_i(\Gamma)$ is shellable if Γ is shellable ([2]).



Partitionability is another well-known property implied by shellability. A simplicial complex Γ is *partitionable* if Γ is partitioned into the form $\Gamma = \dot{\bigcup}[\psi(F), F]$, where $[H, F] = \{A \in \Gamma : H \subseteq A \subseteq F\}$ is the interval from H to F, and the union is taken over all *facets* F of Γ with $\psi(F)$ some face of F. The implication "shellable $\Rightarrow$ partitionable is well-known for pure simplicial complexes, see [18, Lect. 8] for comprehensible explanation. The reasoning for nonpure simplicial complexes is completely the same: if Γ is shellable with a shelling $F_1, F_2, \ldots, F_t$, then a partition is given by setting $\psi(F_k)$ to be the minimal face of F_k that is not contained in $\overline{F_1} \cup \overline{F_2} \cup \cdots \cup \overline{F_{k-1}}$. In other words, if we consider an incremental construction of Γ by adding facets one by one according the shelling, then $[\psi(F_k), F_k]$ equals to the set of newly added faces when k-th facet F_k is added. In the following figure, for the shellable simplicial complex of the example before, the partition constructed from the shelling is shown both on the face poset (the partially ordered set over Γ ordered by inclusion relation) and on the geometric figure. (In the geometric figure, the empty set is explicitly indicated in order to prevent confusion. We use this geometric expression to show partitions for examples in Section 4.) However, it should be noted that there is a big difference between pure cases and nonpure cases. This will be explained in Section 4.



A simplicial complex is *vertex decomposable* if either of the following holds.

(a) Γ has only one facet.

(b) There exists $x \in V$ such that $\mathrm{link}_\Gamma(\{x\})$ and $\Gamma[V \backslash \{x\}]$ are both vertex decomposable, and no facet of $\mathrm{link}_\Gamma(\{x\})$ is a facet of $\Gamma[V \backslash \{x\}]$. Here, $\Gamma[V \backslash \{x\}]$ is the simplicial complex consisting of the faces contained in $V \backslash \{x\}$. ($\Gamma[V \backslash \{x\}]$ is the restriction of Γ to $V \backslash \{x\}$, used later in Section 3.) This definition of vertex decomposability is a version for nonpure simplicial complexes, see [3]. The fact that vertex decomposable simplicial complexes are shellable can be shown by induction ([3], as same as the classical pure cases [13].)

For the difference between the properties in the hierarchy of implications above, the following are known.

⟨⟨0-dimensional case⟩⟩
It is trivial that all 0-dimensional simplicial complexes are vertex decomposable, shellable, sequentially Cohen-Macaulay, and partitionable.

⟨⟨1-dimensional case⟩⟩
It is easy to observe that a 1-dimensional simplicial complex is partitionable if and only if at most one 1-dimensional connected component is a tree. On the other hand, a 1-dimensional simplicial complex is sequentially Cohen-Macaulay if and only if it has only one 1-dimensional connected component, and such a simplicial complex is shellable and vertex decomposable. Hence, vertex decomposability, shellability, and sequential Cohen-Macaulayness coincide in dimension 1, while partitionability and these three properties differ from dimension 1.

⟨⟨dimensions ≥ 2⟩⟩
2-dimensional shellable simplicial complexes that are not vertex decomposable can be found in [11]. 2-dimensional simplicial complexes that are sequentially Cohen-Macaulay but not shellable are provided in [14, Sec. III.2, p.84]: triangulations of the dunce hat are such examples. (See [7] for more detailed examples of dimension 2.) Such examples for dimensions ≥ 3 can be constructed by taking cones over 2-dimensional examples.

The difference between sequential Cohen-Macaulayness and partitionability has been a difficult problem. That the partitionability does not imply sequential Cohen-Macaulayness is easily observed from the 1-dimensional case, but whether sequential Cohen-Macaulayness implies partitionability or not has been an open problem. From 1980's, it was conjectured that Cohen-Macaulay (= pure and sequentially Cohen-Macaulay) simplicial complexes are partitionable, proposed by Stanley and Garsia independently. But recently, Duval, Goeckner, Klivans, and Martin [6] disproved this conjecture. They provided counterexamples of dimension 3, and they can be lifted to higher dimensions by taking cones. By their result, now it is known that there is no implication relation between (sequential) Cohen-Macaulayness and partitionability in dimensions ≥ 3. Whether there exist such counterexamples in dimension 2 or not is left open.

3 Hereditary properties and obstructions

For a simplicial complex Γ over a vertex set $V(\Gamma)$, the restriction of Γ to a subset $W \subseteq V(\Gamma)$, denoted by $\Gamma[W]$, is the subcomplex consisting of all the faces of Γ contained in W. We say a simplicial complex Γ satisfies *hereditary-$\mathcal{P}$* if $\Gamma[W]$ satisfies the property $\mathcal{P}$ for every $W \subseteq V(\Gamma)$. From the hierarchy given in the previous section, we naturally have the following implications:

$$\begin{array}{c} \text{hereditary-} \\ \text{vertex decomposable} \end{array} \Rightarrow \text{hereditary-shellable} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{l} \text{hereditary-sequentially} \\ \qquad\qquad \text{Cohen-Macaulay} \\ \text{hereditary-partitionable} \end{array}$$

Important example is a matroid complex. A *matroid complex* is a simplicial complex whose faces form the independent sets of a matroid. It is also known that a matroid complex is a simplicial complex every restriction is pure ([1]). That is, a matroid complex is a hereditary-pure simplicial complex. Also, it is well-known that matroid complexes are vertex decomposable ([1]). Since every restriction of a matroid complex is again a matroid complex, one can observe that a matroid complex is hereditary-vertex decomposable, hereditary-shellable, hereditary-sequentially Cohen-Macaulay, and also hereditary-partitionable. One more example is an independence complex of a chordal graph. Given a graph, an *independence complex* (or a *flag complex*) of the graph is a simplicial complex whose faces are independent sets of the graph. Independence complexes of chordal graphs are hereditary-vertex decomposable, see [16]. Hence they are also hereditary-shellable, hereditary-sequentially Cohen-Macaulay, and hereditary-partitionable.

For this hereditary version of the hierarchy, the situation of the gaps between properties are different. In [9], hereditary-vertex decomposability is shown to be different from hereditary-shellability in dimensions ≥ 2, as same as the normal case. But, in [8] it is shown that hereditary-shellability, hereditary-sequential Cohen-Macaulayness and hereditary-partitionability coincide in dimension 2. Currently, whether these three properties differ or not in dimensions ≥ 3 is open. (In [8], it is shown that these four hereditary properties are equivalent for the class of flag complexes.)

In the proof of showing the equivalence of the three properties, hereditary-shellability, hereditary-sequential Cohen-Macaulayness and hereditary-partitionability for dimensions ≤ 2 in [8], the key concepts are obstructions that are forbidden structures of hereditary properties. For a property $\mathcal{P}$ of simplicial complexes, a simplicial complex Γ is an obstruction to $\mathcal{P}$ if Γ does not satisfy $\mathcal{P}$ but $\Gamma[W]$ satisfies $\mathcal{P}$ for all $W \subsetneq V(\Gamma)$. This concept of obstructions is introduced by Wachs [15] in which obstructions to shellability is investigated intensively. (Obstructions to purity is also studied.)

The following is obvious but important.

Proposition 3.1. ([8]) A simplicial complex Γ is hereditary-$\mathcal{P}$ if and only if there exists no $W \subseteq V(\Gamma)$ such that $\Gamma[W]$ is an obstruction to $\mathcal{P}$.

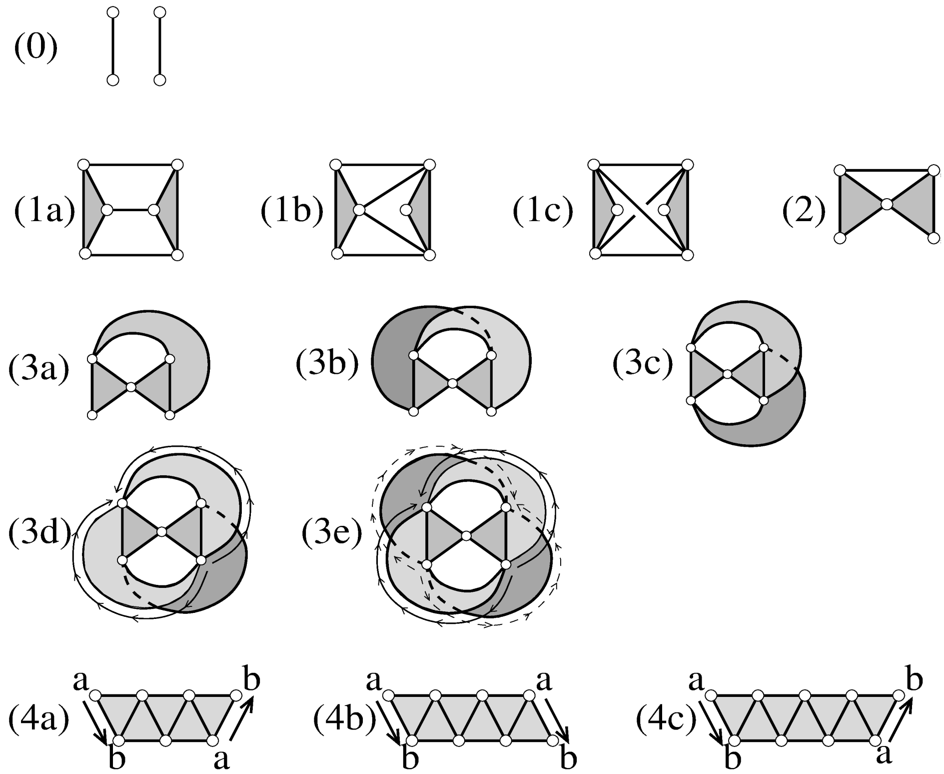
This implies that two properties $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if and only if the set of obstructions to $\mathcal{P}$ and that to $\mathcal{Q}$ coincide. More generally, if $\mathcal{X}$ is a class of simplicial complexes closed under restrictions, we can see that two properties $\mathcal{P}$ and $\mathcal{Q}$ are equivalent in the class $\mathcal{X}$ if and only if the set of obstructions to $\mathcal{P}$ and that to $\mathcal{Q}$ coincide in $\mathcal{X}$. (Typically, in the following, we set $\mathcal{X}$ to be the class of simplicial complexes of dimensions at most d.)

If there exists an implication relation between the two properties, we have the following useful proposition.

Proposition 3.2. ([8]) Let $\mathcal{X}$ be a class of simplicial complexes closed under restrictions. If $\mathcal{P} \Rightarrow \mathcal{Q}$, then the set of obstructions to $\mathcal{P}$ and that to $\mathcal{Q}$ coincide in $\mathcal{X}$ if and only if all the obstructions to $\mathcal{P}$ in $\mathcal{X}$ do not satisfy $\mathcal{Q}$.

By this proposition, to show the equivalence of the three properties, hereditary-shellability, hereditary-sequential Cohen-Macaulayness and hereditary-partitionability, the key is to list (or characterize) all the obstructions to shellability (in the class $\mathcal{X}$). The main result of [8] is to determine the list of all the obstructions to shellability of dimension at most 2.

Theorem 3.3. ([8]) The obstructions to shellability of dimension ≤ 2 are the 1-dimensional obstruction (0) of the following figure, and the simplicial complexes obtained by adding zero or more edges to one of the complexes (1a)–(1c), (2), (3a)–(3e), (4a)–(4c) of the figure. (In the figure, the pairs of edges indicated by the arrows are identified.)



The 1-dimensional obstruction to shellability was already identified in Wachs [15]. In Wachs [15], it is shown that only finitely many 2-dimensional obstructions to shellability exist. The problem to determine all the obstructions to shellability of dimension 2 was initially proposed by her.

For 3-dimensional and higher case, the list of obstructions to shellability or the characterization is not known. Wachs [15] asks whether the number of obstructions to shellability of dimension d is finite for each d ($d \geq 3$), and this is still left open.

Since all the obstructions to shellability determined in Theorem 3.3 are not sequentially Cohen-Macaulay nor partitionable, we have the following corollary.

Corollary 3.4. ([8]) The set of obstructions to shellability, that to sequential Cohen-Macaulayness, and that to partitionability coincide in dimensions ≤ 2. Hence the three properties, hereditary-shellability, hereditary-sequential Cohen-Macaulayness, and hereditary-partitionability are all equivalent in dimensions ≤ 2.

The following question is asked in [8] and it is still open.

Question. Do the set of obstructions to shellability, that to sequential Cohen-Macaulayness, and that to partitionability coincide for all dimensions? Equivalently, do the three properties hereditary-shellability, hereditary-sequential Cohen-Macaulay, and hereditary-partitionability are equivalent for all dimensions?

For Corollary 3.4, it is easy to see that the obstructions to shellability of dimensions ≤ 2 of Theorem 3.3 are not sequentially Cohen-Macaulay. On the other hand, the proof for showing they are not partitionable given in [8, Prop. 3.8] is rather incomprehensive.

4 Nonnegativity of h-triangles

Proposition 3.2 further implies the following trivially.

Corollary 4.1. ([8, p. 1620]) Let $\mathcal{X}$ be a class of simplicial complexes closed under restriction, and the properties $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ satisfies $\mathcal{P} \Rightarrow \mathcal{Q} \Rightarrow \mathcal{R}$. If hereditary-$\mathcal{P}$ and hereditary-$\mathcal{R}$ are equivalent in $\mathcal{X}$, then hereditary-$\mathcal{Q}$ is also equivalent to them in $\mathcal{X}$.

When trying to show that all the three properties, hereditary-shellability, hereditary-sequential Cohen-Macaulayness and hereditary-partitionability coincide, since there is no implication relation between sequential Cohen-Macaulayness and partitionability as is mentioned in Section 2, a possible strategy is to look for a property $\mathcal{Q}$ satisfying

$$\begin{array}{ccccc} & \Nearrow & \text{sequentially Cohen-Macaulay} & \Searrow & \\ \text{shellable} & & & & \mathcal{Q}, \\ & \Searrow & \text{partitionable} & \Nearrow & \end{array}$$

$$\begin{array}{ccccc} & \Nearrow & \text{hereditary-sequentially} & & \\ \text{hereditary-shellable} & & \text{Cohen-Macaulay} & \begin{array}{c}\Searrow\\ \Nearrow\end{array} & \text{hereditary-}\mathcal{Q}, \\ & \Searrow & \text{hereditary-partitionable} & & \end{array}$$

such that hereditary-shellability and hereditary-$\mathcal{Q}$ coincide.

A possible candidate for such a property $\mathcal{Q}$ will be nonnegativity of h-vectors, but we need a suitable modification. After reviewing some basic facts on h-vectors, we give a candidate of shch $\mathcal{Q}$ in the following.

For a simplicial complex Γ of dimension d, the *f-vector* is a sequence $(f_{-1}, f_0, f_1, \ldots, f_d)$ of integers, where f_i is the number of i-dimensional faces of Γ. The *h-vector* of Γ is a sequence $(h_0, h_1, \ldots, h_{d+1})$ of integers defined by

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d+1-j}{d+1-i} f_{j-1} \,. \tag{1}$$

In other word, if we define a polynomial $f(x) = f_{-1}x^{d+1} + f_0 x^d + \cdots + f_d x^0$, then $h(x) = h_0 x^{d+1} + h_1 x^d + \cdots + h_{d+1} x^0$ satisfies $h(x) = f(x-1)$. (For a convenient calculation, we can use the "Stanley's trick," see [18, Lect. 8] for example.) The h-vectors are commonly used in the study of *pure* simplicial complexes. Note that $(h_0, h_1, \ldots, h_{d+1})$

is given from $(f_{-1}, f_0, f_1, \ldots, f_d)$ by a non-singular linear transform. Especially, there is a one-to-one correspondence between $(h_0, h_1, \ldots, h_{d+1})$ and $(f_{-1}, f_0, f_1, \ldots, f_d)$. The inverse transform is given by

$$f_{i-1} = \sum_{j=0}^{i} \binom{d+1-j}{d+1-i} h_j \,. \tag{2}$$

Among the properties of h-vectors of pure simplicial complexes, the following two properties are well-known.

Proposition 4.2. If a *pure* simplicial complex is (sequentially) Cohen-Macaulay, then its h-vector is nonnegative.

Proposition 4.3. If a *pure* simplicial complex is partitionable, then its h-vector is nonnegative.

For the explanation of Proposition 4.2, we refer [10], [14]. Roughly, h_i are nonnegative because they coincide with the dimensions of some algebra.

For Proposition 4.3, there is a comprehensive explanation in [18, Lect. 8], but we also briefly explain here. Let Γ be a d-dimensional pure partitionable simplicial complex with a partition $\dot{\bigcup}[\psi(F), F]$, where F runs all the facets of Γ. Here, in one interval $[\psi(F), F]$, the number of $(i-1)$-dimensional faces (= subsets of F of size $|\psi(F)|$) equals to

$$\binom{|F| - |\psi(F)|}{i - |\psi(F)|} = \binom{d+1-|\psi(F)|}{i-|\psi(F)|} = \binom{d+1-|\psi(F)|}{d+1-i}$$

when $i \geq |\psi(F)|$, and equals to 0 if $i < |\psi(F)|$. Hence,

$$f_{i-1} = \sum_{F} \binom{d+1-|\psi(F)|}{d+1-i} = \sum_{j=0}^{d} \binom{d+1-j}{d+1-i} p_j,$$

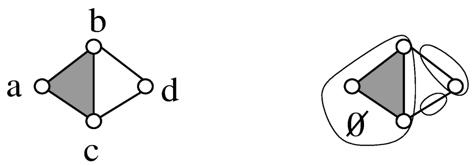
where $p_j = |\{\psi(F) : \dim \psi(F) = j-1\}|$. Here, by comparing with the formula (2), we have $h_j = p_j$, which implies that $h_j \geq 0$ for each j.

Important remark is that this explanation for nonnegativity relies on the fact that all the facets F of Γ satisfy $|F| = d+1$. In fact, it is not difficult to find counterexamples for nonpure simplicial complexes.

Example 4.4. In the following figure, the simplicial complex of 4 vertices is partitionable: it has a partition

$$[\emptyset, abc] \dot{\cup} [bd, d] \dot{\cup} [cd, cd]\,,$$

as indicated in the right of the figure.



For this simplicial complex, the f-vector is $(1, 4, 5, 1)$, and the h-vector is $(1, 1, 0, -1)$.

As this example shows, partitionable complexes can have negative elements in its h-vector if it is nonpure. This example is sequentially Cohen-Macaulay, so we can also see that sequentially Cohen-Macaulay complexes can have negative elements in its h-vector if it is nonpure.

For nonpure simplicial complexes, we can use h-triangles introduced in [2] instead of h-vectors. For a simplicial complex Γ, the *f-triangle* of Γ is defined as

$$f_{k,i} = \text{the number of faces of } \Gamma \text{ of degree } k \text{ and dimension } i-1$$

for $0 \le i \le k \le d+1$, where the *degree* of a face H is the maximum dimension of a face of Γ that contains H. The *h-triangle* of Γ is defined by

$$h_{k,i} = \sum_{j=0}^{i} (-1)^{i-j} \binom{k-j}{i-j} f_{k,j}\,.$$

(We refer this as the *h-transform* of f. For a convenient calculation, we can use "Stanley's trick" for each row.) For example, the f-triangle and the h-triangle of the simplicial complex in Example 4.4 are as follows:

$$f = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 1 & 2 & \\ 1 & 3 & 3 & 1 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 1 & 1 & \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The nonnegativity of h-vectors of pure Cohen-Macaulay simplicial complex (Proposition 4.2) is generalized for the nonpure case as follows.

Proposition 4.5. ([5]) If a simplicial complex is sequentially Cohen-Macaulay, then its h-triangle is nonnegative.

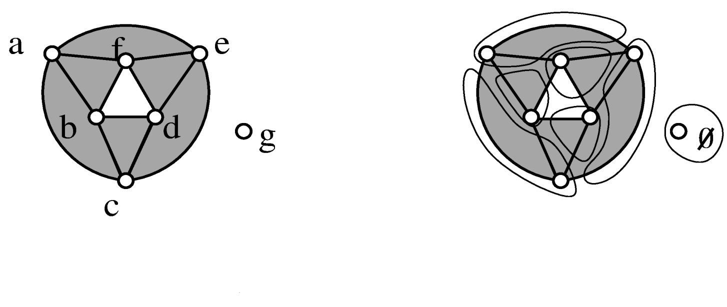
Proof. In [5] it is shown that $(h_{k,i})_{0\le i\le k\le d+1}$ is the h-triangle of a sequentially Cohen-Macaulay simplicial complex if and only if it is the h-triangle of a shellable simplicial complex, while the h-triangles of shellable simplicial complexes are nonnegative as shown in [2]. $\square$

On the other hand, unfortunately, the case of partitionability (Proposition 4.3) is not generalized to nonpure case.

Example 4.6. In the following figure, the simplicial complex of 7 vertices is partitionable: it has a partition

$$[c, abc]\dot\cup[d, bcd]\dot\cup[e, cde]\dot\cup[f, def]\dot\cup[a, efa]\dot\cup[b, fab]\dot\cup[\emptyset, g]\,,$$

as indicated in the right of the figure.



For this simplicial complex, the f-triangle and and the h-triangle are as follows:

$$f = \begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & 0 & 0 & \\ 1 & 6 & 12 & 6 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & 0 & 0 & \\ 1 & 3 & 3 & -1 \end{pmatrix}.$$

As this example shows, for nonpure case, partitionability does not imply nonnegativity of h-triangles.

Since partitionability does not imply nonnegativity of h-triangle, "to have nonnegative h-triangle" cannot be considered as a candidate of the property $\mathcal{Q}$ we are looking for. However, we have the following weaker nonnegativity claim of h-triangles.

Definition 4.7. For two nonnegative triangular arrays $f = (f_{k,i})_{0\le i\le k\le d+1}$ and $f' = (f'_{k,i})_{0\le i\le k\le d+1}$, we define $f \succeq f'$ if the following hold.

(i) $f_{ii} = f'_{ii}$ for each $0 \le i \le d+1$,

(ii) for each $0 \le i \le d+1$,
$\sum_{j=k}^{d+1} f_{j,i} \ge \sum_{j=k}^{d+1} f'_{j,i}$ for each k with $i+1 \le k \le d+1$, and
$\sum_{j=i}^{d+1} f_{j,i} = \sum_{j=i}^{d+1} f'_{j,i}$.

Proposition 4.8. If a d-dimensional simplicial complex Γ is partitionable, then there is a nonnegative triangular array $f' = (f'_{k,i})_{0\le i\le k\le d+1}$ with $f(\Gamma) \succeq f'$ such that $h_{f'}$ is nonnegative, where $f(\Gamma)$ is the f-triangle of Γ and $h_{f'}$ is the h-transform of f'.

For the simplicial complex in Example 4.6, we have

$$f = \begin{pmatrix} 0 & & & \\ 0 & 1 & & \\ 0 & 0 & 0 & \\ 1 & 6 & 12 & 6 \end{pmatrix} \succeq \begin{pmatrix} 0 & & & \\ 1 & 1 & & \\ 0 & 0 & 0 & \\ 0 & 6 & 12 & 6 \end{pmatrix} = f', \quad h_{f'} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 6 & 0 & 0 \end{pmatrix} \ge 0.$$

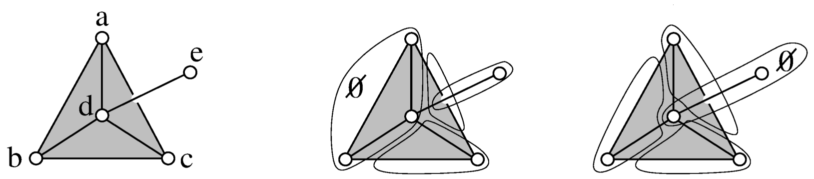
Instead of giving a formal proof for Proposition 4.8, we explain the meaning of "$\succeq$" and the reason why the statement holds. (After that, the proof will be obvious.)

First, let us see what the definition of the relation "$\succeq$" means. For a set of sequences of nonnegative integers with a constant total sum, consider an ordering defined by $(a_0, a_1, \dots, a_n) \succeq' (b_0, b_1, \dots, b_n)$ if and only if $\sum_{j=i}^{n} a_j \ge \sum_{j=i}^{n} b_j$ for each $1 \le i \le n$. Under this ordering, a sequence is larger if elements of larger indices have more weight. For example, we have $(a_1, \dots, a_i, \dots, a_j, \dots, a_n) \succeq' (a_1, \dots, a_i + t, \dots, a_j - t, \dots, a_n)$ if $t > 0$. Our ordering $\succeq$ is a variation of this, where the condition (ii) of Definition 4.7 requires that each column of the triangular array is required to be ordered by this ordering simultaneously. So, we have $f \succeq f'$ if f' is derived from f by moving some positive value from lower element to upper element in the same column. For example, in the example above, we have $f \succeq f'$ since "1" is moved from $f_{3,0}$ to $f'_{1,0}$.

The definition of f-triangles and h-triangles in relation to partitions is well-understood by considering the shellable case. If a d-dimensional simplicial complex Γ is shellable, by the Rearrangement Lemma ([2, Th. 2.6]), there is a shelling such that larger dimensional facets come first. According to this rearranged shelling, a partition is constructed as follows. First, $\psi(F)$ is assigned to each d-dimensional facets F such that $\mathrm{pure}_d(\Gamma)$ is partitioned. Next, $\psi(F)$ is assigned to each $(d-1)$-dimensional facets such that the partition is extended to $\mathrm{pure}_d(\Gamma) \cup \mathrm{pure}_{d-1}(\Gamma)$. Then proceed to $(d-2)$-dimensional facets, $(d-3)$-dimensional facets, and so forth. In such a partition, the h-triangle is given so that $h_{k,i}$ equals to the number of k-dimensional facets such that $\psi(F)$ is i-dimensional. (This follows from the fact that $f_{k,i}$ equals to the number of i-dimensional faces of $\mathrm{pure}_k(\Gamma) \setminus (\bigcup_{l=k+1}^{d} \mathrm{pure}_l(\Gamma))$, and the same calculation as the pure case.)

However, to have such a partition is of course very special. For example, in the partition of Example 4.6, the empty set is belonging to an interval of 0-dimensional facet. (If the partition is constructed according the rearranged shelling, the empty set should belong to an interval of d-dimensional facet.) Further, there even exist examples of partitionable simplicial complexes such that the number of k-dimensional facets such that $\psi(F)$ is i-dimensional is *not determined.*

Example 4.9. The simplicial complex of the following figure has two different partitions as shown.



First partition is

$$[\emptyset, abd] \dot\cup [c, bcd] \dot\cup [ac, acd] \dot\cup [e, de],$$

and the second partition is

$$[b, abd] \dot\cup [c, bcd] \dot\cup [a, acd] \dot\cup [\emptyset, de].$$

If we put the number of k-dimensional facets such that $\psi(F)$ is i-dimensional in the (k,i)-element of triangular arrays, these two partitions corresponds to the following two different arrays.

$$\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 1 & 0 & \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 3 & 0 & 0 \end{pmatrix}.$$

In the second partition of Example 4.9, the intervals of 2-dimensional facets do not cover $\mathrm{pure}_2(\Gamma)$ and left the vertex d, which is covered by an interval of 1-dimensional facet.

As these examples indicate, in general, partitions can be constructed as follows. First, $\psi(F)$ is assigned to each d-dimensional facets F so that $\mathrm{pure}_d(\Gamma)$ will be covered by the intervals $[\psi(F), F]$, but some faces of $\mathrm{pure}_d(\Gamma)$ may be left uncovered. Next,

$\psi(F)$ is assigned to each $(d-1)$-dimensional facets F so that $\text{pure}_{d-1}(\Gamma)$ together with the faces left uncovered before will be covered, but some faces may be left uncovered again. Then proceed to $(d-2)$-dimensional facets, then $(d-3)$-dimensional facets, and so forth. Here, in the f-triangle, if an i-dimensional face in $\text{pure}_k(\Gamma)\setminus(\bigcup_{l=k+1}^{d}\text{pure}_l(\Gamma))$ is covered by a j-dimensional facet with $j < k$, this corresponds moving a value 1 from $f_{k,i}$ to $f'_{j,i}$, which makes $f \succeq f'$. In the end of this process, if we apply h-transform to the final f', we get a triangular array showing the number of k-dimensional facets such that $\psi(F)$ is i-dimensional in the (k,i)-element, which assures the transformed triangular array is nonnegative.

Here, during the process above we always have $i < j$, since the i-dimensional face is not a facet and it is contained in an interval whose top is a j-dimensional facet. This implies that the diagonal elements $f_{i,i}$ do not change during the process. This is the reason the condition (i) in the Definition 4.7 is added.

Let us say that a simplicial complex Γ has a *dominated h-triangle* h' if h' is an h-transform of a triangular array f' such that $f' \preceq f(\Gamma)$. By Propositions 4.5 and 4.8, "to have a nonnegative dominated h-triangle" is a property commonly implied by sequential Cohen-Macaulayness and by partitionability. But, unfortunately, this property is too weak for the property $\mathcal{Q}$ we are looking for. The simplicial complex (3e) of Theorem 3.3 has the following f-triangle and h-triangle:

$$f = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 1 & 5 & 10 & 6 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 1 & 2 & 3 & 0 \end{pmatrix}.$$

The h-triangle itself is nonnegative, hence trivially has nonnegative dominated h-triangle. This means that hereditary-shellability and hereditarily "to have nonnegative dominated h-triangle" do not coincide.

This (3e) of Theorem 3.3 has a vertex (the vertex in the center of the figure in the theorem) such that its neighbor (= link of the vertex) has the structure of the obstruction (0) of Theorem 3.3 (the 1-dimensional obstruction to shellability with 4 vertices and 2 disjoint edges), and this immediately implies that this (3e) is not sequentially Cohen-Macaulay nor partitionable.

In general, we say a property $\mathcal{P}$ is *link-preserving* if Γ satisfies $\mathcal{P}$ implies that $\text{link}_\Gamma(F)$ satisfies $\mathcal{P}$ for all $F \in \Gamma$. (Recall that $\text{link}_\Gamma(F) = \{H \in \Gamma : H \cap F = \emptyset,\ H \cup F \in \Gamma\}$ is the link of the face F in Γ.) It is known that vertex decomposability, shellability, sequential Cohen-Macaulayness and partitionability are all link-preserving ([3, 5, 8, 12]). Since the obstruction (3e) has a vertex whose link is not sequentially Cohen-Macaulay nor partitionable, we can immediately find it is not sequentially Cohen-Macaulay nor partitionable. (The obstruction (0) is obviously not sequentially Cohen-Macaulay nor partitionable.)

On the other hand, the property "to have nonnegative dominated h-triangle" is not link-preserving, as the obstruction (3e) shows. So, for our purpose, we strengthen the property as follows.

property SNNH : every link has nonnegative h-triangle,

property SNNDH : every link has nonnegative dominated h-triangle.

("S" stands for "strongly", which means "to require for all links" as sams as in the remark after Theorem 4.10.) We have the following implications by Propositions 4.5 and 4.8 together with the fact that the properties are link-preserving as remarked above.

$$\text{shellable} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \text{sequentially Cohen-Macaulay} \Rightarrow \text{SNNH} \\ \\ \text{partitionable} \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \text{SNNDH},$$

$$\begin{array}{r} \text{hereditary-} \\ \text{shellable} \end{array} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \text{hereditary-sequentially} \\ \text{Cohen-Macaulay} \Rightarrow \begin{array}{r} \text{hereditary-} \\ \text{SNNH} \end{array} \\ \text{hereditary-partitionable} \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \begin{array}{r} \text{hereditary-} \\ \text{SNNDH}. \end{array}$$

That is, SNNDH is a candidate property for the Q we are looking for. If we can show hereditary-shellability and hereditary-SNNDH coincide, all the properties between them also coincide. In fact, this is the case for dimensions ≤ 2.

Theorem 4.10. For dimensions ≤ 2, hereditary-shellability and hereditary-SNNDH coincide.

Proof. We check each 2-dimensional simplicial complex of Theorem 3.3 does not satisfy the property SNNDH. Then the statement follows by Proposition 3.2.

(0): The f-triangle is $\begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 4 & 2 \end{pmatrix}$. Since the diagonal elements are fixed, there are only two choices for f' with $f \succeq f'$: the third row of f' is $(1,4,2)$ or $(0,4,2)$. The h-transform of these are $(1,2,-1)$ and $(0,3,-1)$. Hence, for neither choice the h-transform is not nonnegative.

(1a)-(1c): The f-triangle is $\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 3+a & \\ 1 & 6 & 6 & 2 \end{pmatrix}$, where $0 \leq a \leq 6$ for (1a) and $1 \leq a \leq 6$ for (1b) and (1c). In f' with $f \succeq f'$, the fourth row is $(a, 6-b, 6, 2)$ with $a \in \{0,1\}$ and $0 \leq b \leq 6$. By the h-transform, this row becomes $(a, -3a-b+6, 3a+2b-6, 2-a-b)$. Here, $3a+2b-6$ and $2-a-b$ cannot be nonnegative simultaneously. Hence this f-triangle does not have nonnegative dominated h-triangles.

(2),
(3a)-(3e): The obstructions (2) and (3a)-(3e) have a vertex whose link is isomorphic to (0), hence do not satisfy the property SNNDH.

(4a): The f-triangle is $\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 1 & 5 & 10 & 5 \end{pmatrix}$. In f' with $f \succeq f'$, the fourth row is $(a, 5-b, 10, 5)$ with $a \in \{0,1\}$ and $0 \leq b \leq 5$. By the h-transform, this row becomes

$(a, -3a-b+5, 3a+2b, -a-b)$. It is required that $a=b=0$ for the h-transform to be nonnegative. On the other hand, if $a=b=0$, then the second row of f' is $(1,0)$ whose h-transform is $(1,-1)$, or the third row is $(1,0,0)$ whose h-transform is $(1,-2,1)$. Hence this f-triangle does not have nonnegative dominated h-triangles.

(4b): The f-triangle is $\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & \alpha & \\ 1 & 6 & 12 & 6 \end{pmatrix}$, where $0 \le \alpha \le 3$. In f' with $f \succeq f'$, the fourth row is $(a, 6-b, 12, 6)$ with $a \in \{0,1\}$ and $0 \le b \le 6$. By the h-transform, this row becomes $(a, -3a-b+6, 3a+2b, -a-b)$. It is required that $a=b=0$ for the h-transform to be nonnegative. On the other hand, if $a=b=0$, then the second row of f' is $(1,0)$ whose h-transform is $(1,-1)$, or the third row is $(1,0,\alpha)$ whose h-transform is $(1,-2,\alpha+1)$. Hence this f-triangle does not have nonnegative dominated h-triangles.

(4c): The f-triangle is $\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & \alpha & \\ 1 & 7 & 14 & 7 \end{pmatrix}$, where $0 \le \alpha \le 7$. In f' with $f \succeq f'$, the fourth row is $(a, 7-b, 14, 7)$ with $a \in \{0,1\}$ and $0 \le b \le 7$. By the h-transform, this row becomes $(a, -3a-b+7, 3a+2b, -a-b)$. It is required that $a=b=0$ for the h-transform to be nonnegative. On the other hand, if $a=b=0$, then the second row of f' is $(1,0)$ whose h-transform is $(1,-1)$, or the third row is $(1,0,\alpha)$ whose h-transform is $(1,-2,\alpha+1)$. Hence this f-triangle does not have nonnegative dominated h-triangles.

$\square$

Remark. For hereditary properties of link-preserving properties, we can use "strong obstructions" instead of obstructions ([8, Sec. 4]): a simplicial complex is a strong obstruction to a property $\mathcal{P}$ if Γ does not satisfy $\mathcal{P}$ but $\mathrm{link}_{\Gamma[W]}(H)$ satiafy $\mathcal{P}$ for any $W \subseteq V(\Gamma)$ and $H \in \Gamma$ unless $W = V(\Gamma)$ and $H = \emptyset$. If properties $\mathcal{P}$ and $\mathcal{Q}$ satisfies $\mathcal{P} \Rightarrow \mathcal{Q}$ and both are link-preserving, then hereditary-$\mathcal{P}$ and hereditary-$\mathcal{Q}$ are equivalent if and only if all strong obstructions to $\mathcal{P}$ do not satisfy $\mathcal{Q}$ (Prop. 4.8 of [8]). The proof of Theorem 4.10 is essentially doing this.

Corollary 4.11. ([8]) For dimensions ≤ 2, hereditary-shellability, hereditary-sequentially Cohen-Macaulay and hereditary-partitionability coincide.

For dimensions ≥ 3, not so many things are known currently. We close this section by the following question.

Question. Do hereditary-shellability and hereditary-SNNDH coincide in general? If negative, for what class of simplicial complexes do these two properties coincide?

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