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Chaos and indecomposability of continua

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1 Introduction

This is a joint work with U. Darji. We use recent developments in local entropy theory to prove that positive topological entropy implies the existence of chaos in dynamical systems and complicated structures in the underlying spaces. In 1994, Barge and Diamond proved that if G is a finite graph and $f : G \to G$ is any map with positive topological entropy, then the inverse limit space $\varprojlim(X, f)$ contains an indecomposable continuum. In 2011, Mouron proved that if X is a chainable continuum which admits a homeomorphism f with positive topological entropy, then X contains an indecomposable subcontinuum.

In this article, we generalize the results of Barge, Diamond and Mouron. We show that if X is a G-like continuum for some finite graph G and $f : X \to X$ is a any homeomorphism with positive topological entropy, then X contains an indecomposable continuum. Moreover we obtain that if X is a G-like continuum for some finite tree G and $f : X \to X$ is a any monotone map with positive topological entropy, then X contains an indecomposable continuum. This answers some questions raised by Mouron and generalizes the theorem of Barge and Diamond.

Also, Barge and Diamond in 1994 showed that for piecewise monotone surjections of graphs, the conditions of having positive entropy, containing a horse shoe and the inverse limit space containing an indecomposable subcontinuum are all equivalent. However, one cannot hope to generalize their result to arbitrary maps. A classical example of Henderson shows that there is a map of the interval with no chaotic behavior, in particular with topological entropy zero, whose inverse limit space is the pseudo-arc (= chainable hereditarily indecomposable continuum). Hence the best one can hope for is to obtain results where positive entropy implies indecomposability in the corresponding inverse limit space. On the other hand, it is known that there is a homeomorphism f of the Cantor fan (which is hereditarily decomposable) such that the homeomorphism f has positive topological entropy.

2 Main Theorems

We have the following main theorems.

Theorem 2.1. *Let G be a finite graph, X be a G-like continuum and $h : X \to X$ a homeomorphism with positive entropy. Then X contains an indecomposable continuum.*

Corollary 2.2. *Let G be a finite graph, X be a G-like continuum and $f : X \to X$ any map with positive entropy. Then $\varprojlim(X, f)$ contains an indecomposable continuum.*

Moreover, we obtain

Corollary 2.3. *Let G be a finite tree, X be a G-like continuum and $f : X \to X$ a monotone map with positive entropy. Then X contains an indecomposable continuum.*

For the case of uniform positive entropy, we have the following.

Theorem 2.4. *Let G be a finite graph, X be a G-like continuum and $f : X \to X$ a homeomorphism with uniform positive entropy. Then X is itself indecomposable.*

3 Definitions and notations

A continuum is a compacted connected metric space. We say that a continuum is nondegenerate if it has more than one point. A continuum is indecomposable if it has more than one point and is not the union of two proper subcontinua. Let G be a compact metric space. A continuous mapping g from X onto G is an ϵ-mapping if for every $x \in X$, the diameter of $g^{-1}(x)$ is less than ϵ. A continuum X is G-like if for every $\epsilon > 0$ there is an ϵ-mapping from X onto G. Our focus in this article is on G-like continua where G is a finite graph.

If $f : X \to X$ is a map, then we use $\varprojlim(X, f)$ to denote the inverse limit of X with f as the bonding maps, i.e.,

$$\varprojlim(X, f) = \left\{(x_i) \in X^{\mathbb{N}} : f(x_{i+1}) = x_i\right\}.$$

Define the shift map $\tilde{f} : \varprojlim(X, f) \to: \varprojlim(X, f)$ by

$$\tilde{f}(x_1, x_2, x_3, ...) = (f(x_1), f(x_2), f(x_3), ...) = (f(x_1), x_1, x_2, , ...).$$

Let X be a compact metric space and $\mathcal{U}, \mathcal{V}$ be two covers of X. Then,

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The quantity $N(\mathcal{U})$ denote minimal cardinality of subcover of $\mathcal{U}$. Let $f : X \to X$ be continuous and $\mathcal{U}$ be an open cover of X. Then,

$$h_{top}(\mathcal{U}, f) = \lim_{n\to\infty} \frac{\ln\left[N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \ldots \vee f^{-n+1}(\mathcal{U}))\right]}{n}.$$

The *topological entropy of f*, denoted by $h_{top}(f)$, is simply the supremum of $h_{top}(f, \mathcal{U})$ over all open covers $\mathcal{U}$ of X.

Let X be a compact metric space and $f : X \to X$ be a map. Let $\mathcal{A}$ be a collection of subsets of X. We say that *$\mathcal{A}$ has an independence set with positive density (for brevity, $\mathcal{A}$ has i.s.p.d.)* if there exists a set $I \subset \mathbb{N}$ with positive density such that for all finite set $J \subseteq I$, we have that

$$\cap_{j\in J} f^{-j}(Y_j) \neq \emptyset$$

for all $Y_j \in \mathcal{A}$.

We recall that set $I \subseteq \mathbb{N}$ has positive density if

$$\lim_{n\to\infty} \frac{|I \cap [1,n]|}{n} > 0.$$

We now recall the definition of IE-tuple. Let $(x_1, \ldots, x_n)$ be a sequence of points in X. We say that $(x_1, \ldots, x_n)$ *is a IE-tuple* if whenever $A_1, \ldots, A_n$ are open sets containing $x_1, \ldots, x_n$, respectively, the collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ has an independence set with positive density. In case that $n = 2$, we use the term IE-pair. We use IE_k to denote the set of all IE-tuples of length k. We will use the following facts from the local entropy theory.

Theorem 3.1 (D. Kerr and H. Li). *Let X be a compact metric space and $f : X \to X$ be a map.*

1. *Let $(A_1, \ldots, A_k)$ be a tuple of closed subsets of X which has an independent set of positive density. Then, there is and IE-tuple $(x_1, \ldots, x_k)$ with $x_i \in A_i$ for $1 \leq i \leq k$.*
2. *$h_{top}(f) > 0$ if and only if there is an IE-pair (x_1, x_2) with $x_1 \neq x_2$.*
3. *IE_k is closed and $f \times \ldots \times f$ invariant subset of X^k.*
4. *If $(A_1, \ldots, A_k)$ has i.s.p.d. and, for $1 \leq i \leq k$, $\mathcal{A}_i$ is a finite collection of sets such that $A_i = \cup \mathcal{A}_i$, then there is $A'_i \in \mathcal{A}_i$ such that $(A'_1, \ldots, A'_k)$ has i.s.p.d.*

Blanchard introduced the notion of *uniform positive entropy (u.p.e)*. A map $h : X \to X$ has u.p.e. if and only if every tuple of X^2 is an IE-tuple for f.

Let X be a continuum which is G-like for some graph G, $f : X \to X$, U, V two subsets of X and $\mathcal{G}$ an open cover of X. Let $l > 1$ be odd. We say that a *chain* $\{C_1, \ldots, C_n\} \subseteq \mathcal{G}$ *is a l-zigzag from U to V* if there exists $1 = k_1 < k_2 < \ldots k_{l+1} = n$ such that

- for all i odd, $C_{k_i} \cap U \neq \emptyset$,
- for all i even, $C_{k_i} \cap V \neq \emptyset$, and
- $\{C_{k_i} \cap U : 1 \leq i \leq l+1, i \text{ odd}\} \bigcup \{C_{k_i} \cap V : 1 \leq i \leq l+1, i \text{ even}\}$ has an i.s.p.d.

A pair (x, y) *is a Z-pair for f* if for every open sets U, V, contains x, y and for every > 0 and for all odd $l \in \mathbb{N}$, we have that there is an -cover $\mathcal{G}$ of X whose nerve is G and a **free** chain $\{C_1, \ldots, C_n\} \subseteq \mathcal{G}$, with $x \in C_1, y \in C_n$, which is a l-zigzag from U to V. (We drop the phrase "for f" and say simply Z-pair when the mapping is clear from the context.) We use $Z(X)$ to denote the set of Z-pairs subset of X.

4 Lemmas and Propositions

To prove the main results, we need the following results:

Proposition 4.1. *Let $I \subseteq \mathbb{N}$ be a set with positive density and $n \in \mathbb{N}$. Then, there is a finite set $F \subseteq I$ with $|F| = n$ and a positive density set B such that $F + B \subseteq I$.*

Proposition 4.2. *Let X be a compact metric space and $f : X \to X$. Let $\mathcal{A}$ be a collection which has an i.s.p.d. and $n \in \mathbb{N}$. Then, there is a finite set F with $|F| = n$ such that*

$$\mathcal{A}_F = \{\cap_{i \in F} f^{-1}(Y_i) : Y_i \in \mathcal{A}\}$$

has an i.s.p.d.

In the following proposition, if $\sigma \in \{0,1\}^n$, we write $\sigma = (\sigma(1), \sigma(2), ...\sigma(n))$, where $\sigma(i) \in \{0,1\}$.

Proposition 4.3. *Let $l, n \geq 1$, and $\sigma_1, \ldots, \sigma_{(n+2)(n+1)^{l-1}}$ be distinct elements of $\{0,1\}^n$. Then, there are i, $1 \leq k_1 < k_2 < k_3 < \ldots < k_{2^l} \leq (n+2)(n+1)^{l-1}$ such that $\sigma_{k_j}(i) = 0$ for j odd and $\sigma_{k_j}(i) = 1$ for j even.*

Lemma 4.4. *Let X be a continuum, $f : X \to X$ be a homeomorphism, $\mathcal{A} = \{A_0, A_1\}$ be subsets of X, $F \subseteq \mathbb{N}$ with $|F| = n$. Furthermore, assume that C is a chain consisting of open subsets of X such that each element of C intersects at most one element of $\mathcal{A}_F$ and that there is a subcollection C' of C of cardinality at least $(n+2)(n+1)^{l-1}$ such that $\{L \cap (\cup \mathcal{A}_F) : L \in C'\}$ has an i.s.p.d. Then, there is a subchain $\mathcal{D}$ of C and an $i \in F$ such that $f^i(\mathcal{D})$ is a $(2^l - 1)$-zigzag chain from A_0 to A_1.*

Lemma 4.5. *Let X be a continuum which is G-like for some graph G, $f : X \to X$ a homeomorphism, $\epsilon > 0$ and $l > 1$ odd. If (A_0, A_1) is a disjoint pair of closed sets which has an i.s.p.d., then there is a ϵ-cover $\mathcal{H}$ of X whose nerve is G and a free chain $\mathcal{E} \subseteq \mathcal{H}$ such that $\mathcal{E}$ is an l-zigzag from A_0 to A_1.*

Lemma 4.6. *Let X be a continuum which is G-like for some graph G, $f : X \to X$ a homeomorphism, and (A_0, A_1) a disjoint pair of closed sets which has an i.s.p.d. Then, there is $x \in A_0$ and $y \in A_1$ such that (x, y) is a Z-pair.*

Corollary 4.7. *Let X be a continuum which is G-like for some graph G and $f : X \to X$ a homeomorphism with positive entropy. Then, arbitrarily close to every $(x, y) \in IE_2(X) - \Delta_2(X)$, there is a Z-pair.*

Theorem 4.8. *Let X be a G-like continuum for a tree G, and $f : X \to X$ a homeomorphism with a Z-pair (a, b). Then, every irreducible continuum between a and b is indecomposable. In particular, X contains an indecomposable continuum containing a and b.*

Corollary 4.9. *Let G be a tree and X a G-like continuum. If $f : X \to X$ is a monotone map of X with positive entropy, then X contains an indecomposable continuum.*

Let $\mathcal{G}$ and $\mathcal{H}$ be collections of subsets of X. By $\mathcal{G}[\mathcal{H}]$ we mean the collection $\{g \in \mathcal{G} : \exists h \in \mathcal{H} \text{ with } \overline{h} \subseteq g\}$. Let X be a continuum and $\{\mathcal{G}_n\}$ be a sequence of covers of X. We say that *$\{\mathcal{G}_n\}$ is a defining sequence of X* provided that the following conditions hold:

- $\mathcal{G}_{n+1}$ is a refinement of $\mathcal{G}_n$, i.e., for each $g \in \mathcal{G}_{n+1}$, there is $g' \in \mathcal{G}_n$ such that $\overline{g} \subseteq g'$, and
- $\lim_{n \to} mesh(\mathcal{G}_n) = 0$.

Lemma 4.10. *Let X be a continuum and $\{\mathcal{G}_n\}$ be a defining sequence of X. Furthermore, assume that for each n, there exists a free chain $\mathcal{C}_n \subseteq \mathcal{G}_n$ and disjoint subchains of $\mathcal{D}_n$, and $\mathcal{E}_n$ and such that $\mathcal{C}_n[\mathcal{D}_{n+1}] = \mathcal{C}_n[\mathcal{E}_{n+1}] = \mathcal{C}_n$. Then X contains an indecomposable continuum.*

We consider the following condition (*): Let G be a finite graph, X be G-like, $h : X \to X$ be a homeomorphism and U, V two disjoint nonempty opens subsets of X. Furthermore, assume that $\mathcal{G}$ is an open cover of X, $\mathcal{G}' = \{g_1, g_2, \ldots, g_n\}$ a free chain of $\mathcal{G}$ and $1 < a_1 < b_1 < a_2 < b_2 \ldots < a_6 < b_6 < n$ are such that

- $\overline{g}_{a_i} \subset U$ for all $1 \leq i \leq 6$,
- $\overline{g}_{b_i} \subset V$ for all $1 \leq 6$, and
- $\{g_{a_i}, g_{b_i} : 1 \leq i \leq 6\}$ has an independent set of positive density.

Lemma 4.11. *If the above condition (*) is satisfied, then there exits an open cover $\mathcal{H}$ which is a refinement of $\mathcal{G}$, a free chain $\mathcal{H}' = \{h_1, h_2, \ldots, h_m\}$ of $\mathcal{H}$ and $1 < c_1 < d_1 < c_2 < d_2 \ldots < c_6 < d_6 < m$ such that*

1. *$\overline{h}_{c_i} \subset U$ for all $1 \leq i \leq 6$,*
2. *$\overline{h}_{d_i} \subset V$ for all $1 \leq 6$,*
3. *$\{h_{c_i}, h_{d_i} : 1 \leq i \leq 6\}$ has an independent set of positive density, and*
4. *for one of $\mathcal{C} = \{g_{a_2}, \ldots, g_{b_3}\}$ or $\mathcal{C} = \{g_{a_4}, \ldots, g_{b_6}\}$ the following holds.*

$$\mathcal{C}[\mathcal{A}_i] = \mathcal{C}_i,$$

for all $1 \leq i \leq 6$, where $\mathcal{A}_i$ is the subchain $\{h_{c_i}, \ldots, h_{d_i}\}$ of $\mathcal{H}'$.

Lemma 4.12. *Let G be a finite graph, X be G-like and $h : X \to X$ be a homeomorphism. Suppose that (x, y) is an IE-pair of distinct points and U, V are open sets containing x, y, respectively. Then, X contains an indecomposable continuum which intersects U and V.*

By use of the above results, we can prove the main results.

References

[1] M. Barge, J. Martin, *Chaos, periodicity, and snakelike continua.* Trans. Amer. Math. Soc. **289** (1985), no. 1, 355-365.

[2] M. Barge, B. Diamond, *The dynamics of continuous maps of finite graphs through inverse limits.* Trans. Amer. Math. Soc. **344** (1994), no. 2, 773-790.

[3] F. Blanchard, *Fully positive topological entropy and topological mixing.* Symbolic Dynamics and its Applications, Amer. Math. Soc., Contem. Math., **135** (1992), 95-105.

[4] L.S.Block, W.A.Coppel, *Dynamics in one dimension,* Lecture Notes in Mathematics textbf 1513, Springer 1992.

[5] J. P. Boroński, P. Oprocha, *On indecomposability in chaotic attractors.* Proc. Am. Math. Soc., to appear.

[6] R. Bowen *Topological entropy and axiom A.* 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) pp. 23-41 Amer. Math. Soc., Providence, R.I.

[7] U. Darji and Hisao Kato, *Chaos and Indecomposability,* preprint.

[8] E. Glasner, X. Ye, *Local entropy theory.* Ergodic Theory Dynamal Systems **29** (2009), no. 2, 321-356.

[9] G. W. Henderson, *The pseudo-arc as an inverse limit with one binding map.* Duke Math. J. **31** (1964), 421-425.

[10] W. T. Ingram, *Periodic points for homeomorphisms of hereditarily decomposable chainable continua.* Proc. Amer. Math. Soc. **107** (1989), no. 2, 549-553.

[11] H. Kato, *Topological entropy of monotone maps and confluent maps on regular curves.* Proceedings of the 18th Summer Conference on Topology and its Applications. Topology Proc. **28** (2004), no. 2, 587-593.

[12] Kato, Hisao, *Topological entropy of maps on regular curves.* Topology Appl. **154** (2007), no. 6, 1027-1031.

[13] D. Kerr, H. Li, *Independence in topological and C^*-dynamics.* Math. Ann. **338** (2007), no. 4, 869-926.

[14] C. Kuratowski, Topologie I et II. Reprint of the fourth (Part I) and third (Part II) editions. ゾitions Jacques Gabay, Sceaux, 1992.

[15] J. Krasinkiewicz, P. Minc, *Mappings onto indecomposable continua.* Bull. Acad. Polon. Sci. Math. Astronom. Phys. **25** (1977), no. 7, 675-680.

[16] J. Llibre, M. Misiurewicz, *Horseshoes, entropy and periods for graph maps.* Topology **32** (1993), no. 3, 649-664.

[17] J. Lü, J. Xiong, X. Ye, *The inverse limit space and the dynamics of a graph map.* Topology Appl. **107** (2000), no. 3, 275-295.

[18] J. Lü, J. Xiong, X. Ye, *An equivalent condition for continuous maps of a class of continua to have zero topological entropy.* Tsukuba J. Math. **23** (1999), no. 3, 539-550.

[19] H. Méndez-Lango, *On period three and topological entropy in hereditarily decomposable arc-like continua.* Continuum theory (Denton, TX, 1999), 271279, Lecture Notes in Pure and Appl. Math., 230, Dekker, New York, 2002.

[20] P. Minc, W. R. R. Transue, *Šarkovskiĭ's theorem for hereditarily decomposable chainable continua.* Trans. Amer. Math. Soc. **315** (1989), no. 1, 173-188.

[21] C. Mouron, *Positive entropy homeomorphisms of chainable continua and indecomposable subcontinua.* Proc. Amer. Math. Soc. **139** (2011), no. 8, 2783-2791.

[22] S. Nadler, *Continuum Theory. An introduction.* Monographs and Textbooks in Pure and Applied Mathematics, **158**, Marcel Dekker, Inc., New York, 1992.

[23] G. T. Seidler, *The topological entropy of homeomorphisms on one-dimensional continua.* Proc. Amer. Math. Soc. **108** (1990), no. 4, 1025-1030.

[24] X. Ye, *Topological entropy of the induced maps of the inverse limits with bonding maps.* Topology Appl. **67** (1995), no. 2, 113-118.

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