数理解析研究所講究録
第 1987 巻 2016 年 55-66

Topological structures of hyperspaces of compact / finite sets

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1 Introduction

In this article, we discuss the topological structures of hyperspaces of compact or finite sets in metrizable spaces. In infinite-dimensional topology, it is one of the most important problems to determine the homeomorphism types of hyperspaces. The main purpose of this article is to provide necessary and sufficient conditions on spaces whose hyperspaces of compact or finite sets are homeomorphic to pre-Hilbert spaces. This article is a résumé of the papers [7, 8].

Throughout this article, any space is assumed metrizable and any map is assumed continuous. We let κ denote an infinite cardinal and often consider it as a space endowed with the discrete topology. Let $\mathrm{Comp}(X)$ be the hyperspace of non-empty compact sets in a space X with the Vietoris topology, and let $\mathrm{Fin}(X) \subset \mathrm{Comp}(X)$ be the hyperspace of non-empty finite sets in X with the relative topology. By $\ell_2(\kappa)$ we denote the Hilbert space of density κ, and by $\ell_2^f(\kappa)$ we denote the linear subspace spanned by the canonical orthonormal basis of $\ell_2(\kappa)$. The hyperspace $\mathrm{Comp}(X)$ is a classical object in infinite-dimensional topology and has been studied by many mathematicians. By the efforts of D.W. Curtis, R.M. Schori and J.E. West, the following celebrated theorem was established:

Theorem 1.1. *For a space X, the hyperspace $\mathrm{Comp}(X)$ is homeomorphic to the Hilbert cube if and only if X is non-degenerate, connected, locally connected and compact.*

D.W. Curtis [2, 3] obtained some characterizations of the hyperspace $\mathrm{Comp}(X)$ for a non-compact space X. Especially, he characterized a space X whose hyperspace $\mathrm{Comp}(X)$ is homeomorphic to the separable Hilbert space as follows:

Theorem 1.2. *Let X be a space. The hyperspace $\mathrm{Comp}(X)$ is homeomorphic to $\ell_2(\omega)$ if and only if X is separable, connected, locally connected, topologically complete and nowhere locally compact.*

*This work was supported by JSPS KAKENHI Grant Number 15K17530.

Concerning the hyperspace $\mathrm{Fin}(X)$, D.W. Curtis and N.T. Nhu [5] gave a necessary and sufficient condition on X for $\mathrm{Fin}(X)$ to be homeomorphic to $\ell_2^f(\omega)$ as follows:

Theorem 1.3. *For a space X, the hyperspace* $\mathrm{Fin}(X)$ *is homeomorphic to $\ell_2^f(\omega)$ if and only if X is non-degenerate, connected, locally path-connected, strongly countable-dimensional and σ-compact.*

Recall that a space X is *strongly countable-dimensional* if it is a countable union of finite-dimensional closed subsets. We say that a space X is *σ-(locally)compact* if it is written as a countable union of (locally)compact subsets. By an *X-manifold* we understand a space in which each point has a neighborhood homeomorphic to an open subset of a space X. Recently, K. Mine, K. Sakai and M. Yaguchi [18, 9] proved the following:

Theorem 1.4. *For a connected $\ell_2(\kappa)$-manifold X, the hyperspace* $\mathrm{Comp}(X)$ *is homeomorphic to $\ell_2(\kappa)$.*

Theorem 1.5. *If a space X is a connected $\ell_2^f(\kappa)$-manifold, then the hyperspace* $\mathrm{Fin}(X)$ *is homeomorphic to $\ell_2^f(\kappa)$.*

In this article, we generalize their results as follows:

Theorem A. *Let X be a space. The hyperspace* $\mathrm{Comp}(X)$ *is homeomorphic to $\ell_2(\kappa)$ if and only if X is connected, locally connected, topologically complete, nowhere locally compact, and every non-empty open subset of X is of density κ.*

Theorem B. *For a space X, the hyperspace* $\mathrm{Fin}(X)$ *is homeomorphic to $\ell_2^f(\kappa)$ if and only if X is connected, locally path-connected, strongly countable-dimensional, σ-locally compact and each non-empty open set in X is of density κ.*

In the last section, we will deal with the topological type of pair of hyperspaces. Given spaces X and Y, by (X, Y) we understand Y is a subspace of X. A pair (X, Y) of spaces is homeomorphic to (X', Y') if there is a homeomorphism $f : X \to X'$ such that $f(Y) = Y'$. A subset A of a space X is called *locally non-separating* in X if for any non-empty connected open set U in X, $U \setminus A$ is non-empty and connected. As a corollary of Theorems A and B, the following can be established:

Corollary. *Let X be a connected, locally path-connected, strongly countable-dimensional and σ-locally compact space whose any non-empty open subset is of density κ. Suppose that X admits a locally connected and nowhere locally compact completion $\overline{X}$. Then the pair* $(\mathrm{Comp}(\overline{X}), \mathrm{Fin}(X))$ *of hyperspaces is homeomorphic to $(\ell_2(\kappa), \ell_2^f(\kappa))$ if and only if the remainder $\overline{X} \setminus X$ is locally non-separating in $\overline{X}$.*

2 Characterizations of $\ell_2(\kappa)$ and $\ell_2^f(\kappa)$

In this section, we shall introduce characterizations of pre-Hilbert spaces $\ell_2(\kappa)$ and $\ell_2^f(\kappa)$ used in the proofs of Theorems A and B. For maps $f : X \to Y$ and $g : X \to Y$, and for an open cover $\mathcal{U}$ of Y, f is *$\mathcal{U}$-close* to g if for each point $x \in X$, there exists $U \in \mathcal{U}$ containing $f(x)$ and $g(x)$. A space X has *the countable locally finite approximation property* if the following condition holds:

- For each open cover $\mathcal{U}$ of X, there is a sequence $\{f_n : X \to X\}_{n<\omega}$ of maps such that every f_n is $\mathcal{U}$-close to the identity map on X and the collection $\{f_n(X)\}_{n<\omega}$ is locally finite in X.

It is said that a space X has *the κ-discrete n-cells property*, where n is a non-negative integer, provided that the following condition is satisfied:

- For every open cover $\mathcal{U}$ of X and every map $f : [0,1]^n \times \kappa \to X$, there exists a map $g : [0,1]^n \times \kappa \to X$ such that g is $\mathcal{U}$-close to f and the family $\{g([0,1]^n \times \{\gamma\})\}_{\gamma<\kappa}$ is discrete in X.

This property plays an important role in characterizing non-separable $\ell_2(\kappa)$ and $\ell_2^f(\kappa)$. H. Toruńczyk [14, 15] gave the following elegant characterization to Hilbert spaces (cf. [1, Theorem 3.1]):

Theorem 2.1. *A space X is homeomorphic to $\ell_2(\kappa)$ if and only if the following conditions are satisfied:*

(1) *X is a topologically complete AR of density κ;*

(2) *X has the countable locally finite approximation property;*

(3) *X has the κ-discrete n-cells property for every $n < \omega$.*

A closed subset A of a space X is said to be a *(strong) Z-set* in X if the following condition is satisfied:

- For each open cover $\mathcal{U}$ of X, there is a map $f : X \to X$ such that f exists $\mathcal{U}$-close to the identity map on X and the (closure of)image $f(X)$ misses A.

We define a space X to be *strongly universal* for a class $\mathcal{C}$ if the following condition holds:

- Let $A \in \mathcal{C}$, B be a closed subset of A, and $f : A \to X$ be a map such that the restriction $f|_B$ is an embedding and $f(B)$ is a Z-set in X. For any open cover $\mathcal{U}$ of X, there is an embedding $g : A \to X$ such that g is $\mathcal{U}$-close to f, $g(A)$ is a Z-set in X, and $g|_B = f|_B$.

J. Mogilski [10] characterized the separable pre-Hilbert space $\ell_2^f(\omega)$ as follows:

Theorem 2.2. *A space X is homeomorphic to $\ell_2^f(\omega)$ if and only if the following conditions hold:*

(1) *X is a strongly countable-dimensional, σ-compact AR;*

(2) *X is strongly universal for the class of finite-dimensional compact metrizable spaces;*

(3) *every finite-dimensional compact subset of X is a strong Z-set in X.*

The above theorem was generalized by K. Sakai and M. Yaguchi [12]:

Theorem 2.3. *A space X is homeomorphic to $\ell_2^f(\kappa)$ if and only if the following conditions are satisfied:*

(1) *X is a strongly countable-dimensional, σ-locally compact AR of density κ, and is written as a countable union of strong Z-sets;*

(2) *X is strongly universal for the class of strongly countable-dimensional, locally compact spaces of density $\leq \kappa$.*

Using the κ-discrete n-cells property, the author [6] modified their result as follows:

Theorem 2.4. *A space X is homeomorphic to $\ell_2^f(\kappa)$ if and only if the following conditions hold:*

(1) *X is a strongly countable-dimensional, σ-locally compact AR of density κ;*

(2) *X is strongly universal for the class of finite-dimensional compact metrizable spaces;*

(3) *every finite-dimensional compact subset of X is a strong Z-set in X;*

(4) *X has the κ-discrete n-cells property for every $n < \omega$.*

3 Basic facts on $\mathrm{Comp}(X)$ and $\mathrm{Fin}(X)$

In this section, we shall list some basic properties of the hyperspaces $\mathrm{Comp}(X)$ and $\mathrm{Fin}(X)$. The following propositions are well-known, refer to [11, Theorem 5.12.5 (2)] and [9, Proposition 5.3]:

Proposition 3.1. *A space X is topologically complete if and only if so is the hyperspace* $\mathrm{Comp}(X)$.

Proposition 3.2. *For any space X, X is strongly countable-dimensional and σ-locally compact if and only if so is* $\mathrm{Fin}(X)$.

For the sake of convenience, by $\mathrm{Hyp}(X)$ we mean the hyperspace $\mathrm{Comp}(X)$ or $\mathrm{Fin}(X)$. Concerning the density of $\mathrm{Hyp}(X)$, the following holds, see [18, Corollary 4.2] and [9, Proposition 5.3]:

Proposition 3.3. *The hyperspace* $\mathrm{Hyp}(X)$ *has the same density as a space X.*

We can easily observe the following proposition:

Proposition 3.4. *For a space X, if all non-empty open sets in* $\mathrm{Hyp}(X)$ *are of density $\geq \kappa$, then so are all non-empty open subsets of X.*

On the ANR-property of Comp(X), the following holds [17, 13] (cf. [3, Theorem 1.6]):

Proposition 3.5. *Let X be topologically complete. Then X is locally connected (connected and locally connected) if and only if* Comp(X) *is an ANR (AR).*

Combining Lemmas 2.3 and 3.6 of [5] with the proof of Theorem 2.4 in [5] (cf. [18, Proposition 3.1]), the following can be proven:

Proposition 3.6. *A space X is locally path-connected (connected and locally path-connected) if and only if* Fin(X) *is an ANR (AR).*

4 The countable locally finite approximation property and the κ-discrete n-cells property in Comp(X) and Fin(X)

This section is devoted to detecting the countable locally finite approximation property and the κ-discrete n-cells property in the hyperspaces Comp(X) and Fin(X). For a non-negative integer n, let $\mathbf{S}^n$ be the n-dimensional unit sphere and let $\mathbf{B}^n$ be the n-dimensional unit ball. The following lemma established in [5, Lemma 3.3] is very useful for constructing maps from polyhedra[1] to hyperspaces.

Lemma 4.1. *For each $n \geq 1$, there is a "retraction" $r : \mathbf{B}^{n+1} \to \mathrm{Fin}(\mathbf{S}^n)$ such that $r(x) = \{x\}$ for all $x \in \mathbf{S}^n$.*

Given a simplicial complex K, we denote the polyhedron of K by $|K|$ and the n-skeleton of K by $K^{(n)}$ for each $n < \omega$. We often regard $\sigma \in K$ as a simplicial complex consisting of its faces. Now we prove the following lemma, that is a generalization of the technique used in the proof of [2, Theorem E].

Lemma 4.2. *Let X be a locally path-connected space. Suppose that every non-empty open set in X contains a discrete subset of cardinality $\geq \kappa$. Then the hyperspace* Hyp(X) *satisfies the following:*

- *Let K_γ be a simplicial complex, $\gamma < \kappa$, and K be a discrete union of K_γs. For each open cover $\mathcal{V}$ of* Hyp(X) *and each map $g : |K| = \bigoplus_{\gamma<\kappa} |K_\gamma| \to \mathrm{Hyp}(X)$, there is a $\mathcal{V}$-close map $h : |K| \to \mathrm{Hyp}(X)$ to g such that the family $\{h(|K_\gamma|)\}_{\gamma<\kappa}$ is locally finite in* Hyp(X).

Sketch of proof.

- Replace K with a sufficiently small subdivision.
- For each $n < \omega$, choose a locally finite open cover $\mathcal{V}_n$ of X of mesh $< 1/n$.

[1] Polyhedra are not needed to be metrizable.

- Each non-empty $V \in \mathcal{V}_n$ contains a discrete subset $Z(V) = \{z_V^\gamma\}_{\gamma<\kappa}$ of cardinality κ such that $Z^\gamma(n) \cap Z^\tau(n) = \emptyset$ if $\gamma \neq \tau$, where $Z^\gamma(n) = \{z_V^\gamma \mid \emptyset \neq V \in \mathcal{V}_n\}$, $\gamma < \kappa$.

(1) Construction of $h|_{K^{(0)}}$

- For every $v \in K_\gamma^{(0)}$, there are $n_v \geq 2$ and $z^\gamma(v) \in Z^\gamma(n_v)$ such that
  - $z^\gamma(v)$ is sufficiently close to $g(v)$;
  - for any $u, u' \in K^{(0)}$, $n_u = n_{u'}$ if $g(u)$ is sufficiently close to $g(u')$.
- Let $h(v) = g(v) \cup \{z^\gamma(v)\}$.

(2) Construction of $h|_{|K^{(1)}|}$

- For each $\sigma \in K_\gamma^{(1)} \setminus K_\gamma^{(0)}$ and $\sigma^{(0)} = \{v_1, v_2\}$, define $h(\hat{\sigma}) = g(\hat{\sigma}) \cup \{z^\gamma(v_m) \mid 1 \leq m \leq 2\}$, where $\hat{\sigma}$ is the barycenter of σ.
- Taking an arc $\alpha_m : [0,1] \to X$ from some point of $g(\hat{\sigma})$ to $z^\gamma(v_m)$ of sufficiently small diameter, we define a path $\phi : [0,1] \to \operatorname{Hyp}(X)$ from $g(\hat{\sigma})$ to $h(\hat{\sigma})$ by $\phi(t) = g(\hat{\sigma}) \cup \{\alpha_m(t) \mid 1 \leq m \leq 2\}$.
- For each $m \in \{1,2\}$, define $h : \langle v_m, \hat{\sigma}\rangle \to \operatorname{Hyp}(X)$, where $\langle v_m, \hat{\sigma}\rangle$ is the segment between v_m and $\hat{\sigma}$, as follows:

$$h((1-t)v_m + t\hat{\sigma}) = \begin{cases} g((1-2t)v_m + 2t\hat{\sigma}) \cup \{z^\gamma(v_m)\} & \text{if } 0 \leq t \leq 1/2, \\ \phi(2t-1) \cup \{z^\gamma(v_m)\} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

(3) Construction of h

- Assume that h extends over $|K^{(n)}|$ for some $n < \omega$ such that for every $y \in \sigma \in K^{(n)} \setminus K^{(0)}$, $h(y) = \bigcup_{a \in A} h(a)$ for some $A \in \operatorname{Fin}(|\sigma^{(1)}|)$.
- For each $\sigma \in K^{(n+1)} \setminus K^{(n)}$, there exists a map $r : \sigma \to \operatorname{Fin}(\partial\sigma)$, where $\partial\sigma$ is the boundary of σ, such that $r(y) = \{y\}$ for all $y \in \partial\sigma$ by Lemma 4.1.
- For each $y \in \sigma$, let $h(y) = \bigcup_{y' \in r(y)} h(y')$.
- Replace $h(y)$ with $g(y) \cup h(y)$, that is the desired map.

□

We will verify the countable locally finite approximation property:

Proposition 4.3. *Let X be a locally path-connected and nowhere locally compact space. Then* $\operatorname{Hyp}(X)$ *has the countable locally finite approximation property.*

Sketch of proof.

- Approximate $\operatorname{Hyp}(X) \times \omega$ by a polyhedron $|K| = \bigoplus_{n<\omega} |K_n|$, and apply Lemma 4.2.

□

We shall introduce useful lemmas for recognizing the κ-discrete n-cells property in spaces.

Lemma 4.4 (Lemma 3.1 of [1]). *Let $n < \omega$. A space X has the κ-discrete n-cells property if and only if the following condition holds:*

- *For each open cover $\mathcal{U}$ of X and each map $f : [0,1]^n \times \kappa \to X$, there is a $\mathcal{U}$-close map $g : [0,1]^n \times \kappa \to X$ to f such that $\{g([0,1]^n \times \{\gamma\})\}_{\gamma<\kappa}$ is locally finite in X.*

Lemma 4.5 (Lemma 3.2 of [1]). *For $n < \omega$, a space X with the countable locally finite approximation property has the κ-discrete n-cells property if and only if X has the λ-discrete n-cells property for any $\lambda \leq \kappa$ of uncountable cofinality.*

The next lemma follows from the proof of [1, Lemma 6.2]:

Lemma 4.6. *Suppose that κ has uncountable cofinality. Any space X of density $\geq \kappa$ contains a closed discrete subset of cardinality $\geq \kappa$.*

Now we prove the following:

Proposition 4.7. *Let X be a locally path-connected and nowhere locally compact space. If any non-empty open subset of X is of density $\geq \kappa$, then the hyperspace* $\mathrm{Hyp}(X)$ *has the κ-discrete n-cells property for every $n < \omega$.*

Sketch of proof.

- The hyperspace $\mathrm{Hyp}(X)$ has the countable locally finite approximation property by Proposition 4.3.
- We may assume that κ is of uncountable cofinality due to Lemma 4.5, and hence any non-empty open set in X contains a discrete subset of cardinality $\geq \kappa$ by Lemma 4.6.
- According to Lemma 4.4, it suffices to prove that any map $g : [0,1]^n \times \kappa \to \mathrm{Hyp}(X)$ can be approximated by a map $h : [0,1]^n \times \kappa \to \mathrm{Hyp}(X)$ such that $\{h([0,1]^n \times \{\gamma\})\}_{\gamma<\kappa}$ is locally finite in $\mathrm{Hyp}(X)$.
- Triangulate $[0,1]^n \times \kappa$ into a polyhedron $|K| = \bigoplus_{\gamma<\kappa} |K_\gamma|$, and use Lemma 4.2.

□

5 The strong universality and the compact-strong Z-set property of $\mathrm{Fin}(X)$

This section is devoted to verifying the strong universality of $\mathrm{Fin}(X)$ for the class of finite-dimensional compact metrizable spaces and the strong Z-set property of compact sets in $\mathrm{Fin}(X)$. D.W. Curtis and N.T. Nhu showed the following:

Lemma 5.1 (Lemma 4.2 of [5]). *Let X be a locally path-connected space with finitely many components, at least one of which is non-degenerate, and let $\mathcal{A} \subset \mathrm{Fin}(X)$ be separable. Suppose that each member of $\mathcal{A}$ intersects all components of X and $\mathcal{A}$ has the following expansion property:*

- *For each $F \in \mathrm{Fin}(X)$, if F contains some element of $\mathcal{A}$, then $F \in \mathcal{A}$.*

Then $\mathcal{A}$ is strongly universal for the class of finite-dimensional compact metrizable spaces.

Using this lemma, we show the strong universality of a non-separable hyperspace $\mathrm{Fin}(X)$.

Proposition 5.2. *If X is a non-degenerate, connected, locally path-connected space, then the hyperspace $\mathrm{Fin}(X)$ is strongly universal for the class of finite-dimensional compact metrizable spaces.*

Sketch of proof.

- For any finite-dimensional compact metrizable space A and any map $f : A \to \mathrm{Fin}(X)$, regard A as a closed subset of $[0,1]^n$ for some $n < \omega$, and extend f to a map $\tilde{f} : [0,1]^n \to \mathrm{Fin}(X)$.
- The set $\mathcal{A} = \{F \in \mathrm{Fin}(\bigcup \tilde{f}([0,1]^n)) \mid F$ contains some element of $\tilde{f}([0,1]^n)\}$ is strongly universal for the class of finite-dimensional compact metrizable spaces by Lemma 5.1.

□

The following lemma is useful for detecting strong Z-sets in ANRs.

Lemma 5.3 (Lemma 7.2 of [4]). *Suppose that A is a topologically complete and closed subset of an ANR X. If A is written as a countable union of strong Z-sets in X, then it is a strong Z-set.*

For each $k < \omega$, the set $\mathrm{Fin}^k(X) = \{A \in \mathrm{Fin}(X) \mid A$ is of cardinality $\leq k\}$ is closed in $\mathrm{Fin}(X)$.

Proposition 5.4. *Let X be a non-degenerate, connected and locally path-connected space. Then for every $k < \omega$, $\mathrm{Fin}^k(X)$ is a strong Z-set in $\mathrm{Fin}(X)$.*

Sketch of proof.

- Approximate $\mathrm{Fin}(X)$ by a polyhedron $|K|$ whose simplexes are sufficiently small.
- It remains to show that for each map $g : |K| \to \mathrm{Fin}(X)$, there exists a map $h : |K| \to \mathrm{Fin}(X)$ sufficiently close to g such that the closure of $h(|K|)$ dose not meet $\mathrm{Fin}^k(X)$.

(1) Construction of $h|_{K^{(0)}}$

- Find $k+1$ points $z(v,0), \cdots, z(v,k) \in X$ for each $v \in K^{(0)}$ so that
  - each $z(v,j))$ is sufficiently close to $g(v)$;

  - $z(v,j)$s keep a sufficient distance from each other.
- Let $h(v) = g(v) \cup \{z(v,j) \mid 0 \leq j \leq k\}$.

(2) Construction of h

- Follow the same argument as in Lemma 4.2.

□

Combining Lemma 5.3 with Proposition 5.4, we can obtain the following proposition:

Proposition 5.5. *If X is non-degenerate, connected and locally path-connected space, then every compact set in* $\mathrm{Fin}(X)$ *is a strong Z-set.*

6 Proofs of Theorems A and B

In this section, applying the Characterizing Theorems 2.1 and 2.4, we proves Theorems A and B.

Proof of Theorem A.

- The case that X is separable follows from Theorem 1.2, so we may only consider κ uncountable.

(1) The "only if" part

- According to Propositions 3.1, 3.3 and 3.5, $\mathrm{Comp}(X)$ is a topologically complete AR of density κ.
- The countable locally finite approximation property of $\mathrm{Comp}(X)$ follows from Proposition 4.3.
- By Proposition 4.7, $\mathrm{Comp}(X)$ has the κ-discrete n-cells property for every $n < \omega$.
- Use the Toruńczyk Characterization Theorem 2.1.

(2) The "if" part

- Due to Propositions 3.1 and 3.5, X is connected, locally connected and topologically complete.
- It follows from Proposition 3.4 that all non-empty open subsets of X are of density κ, and hence X is nowhere locally compact.

□

Proof of Theorem B.

- The separable case follows from Theorem 1.3, so we can assume κ to be uncountable.

(1) The "only if" part

- By virtue of Propositions 3.2 and 3.6, $\mathrm{Fin}(X)$ is a strongly countable-dimensional and σ-locally compact AR of density κ.
- Due to Proposition 5.2, $\mathrm{Fin}(X)$ is strongly universal for the class of finite-dimensional compact metrizable spaces.
- According to Proposition 5.5, any finite-dimensional compact subset of $\mathrm{Fin}(X)$ is a strong Z-set in $\mathrm{Fin}(X)$.
- It follows from Proposition 4.7 that $\mathrm{Fin}(X)$ has the κ-discrete n-cells property of $\mathrm{Fin}(X)$ for each $n < \omega$.
- Use Theorem 2.4.

(2) The "if" part

- By Propositions 3.2 and 3.6, X is connected, locally path-connected, strongly countable-dimensional and σ-locally compact.
- Any non-empty open set in X is of density κ by Proposition 3.4.

□

7 Pair of hyperspaces

In this final section, we will discuss the topological structure of pair of hyperspaces. To show Corollary, we will use the following characterization of the pair $(\ell_2(\kappa), \ell_2^f(\kappa))$ [16, 6]:

Theorem 7.1. *A pair (X, Y) of spaces is homeomorphic to $(\ell_2(\kappa), \ell_2^f(\kappa))$ if and only if X is homeomorphic to $\ell_2(\kappa)$, Y is homeomorphic to $\ell_2^f(\kappa)$ and Y is homotopy dense in X.*

A subset A of a space X is defined to be *homotopy dense* in X if there is a homotopy $h : X \times [0, 1] \to X$ such that $h(x, 0) = x$ for all $x \in X$ and $h(X \times (0, 1]) \subset A$. The homotopy density between ANRs is characterized as follows [11, Corollary 7.4.6]:

Lemma 7.2. *Suppose that X and Y are ANRs and Y is dense in X. Then Y is homotopy dense in X if and only if the following condition is satisfied:*

- *For each point $x \in X$ and each neighborhood U of x in X, there exists a neighborhood $V \subset U$ of x such that any map $f : \mathbf{S}^n \to V \cap Y$ can be extended to a map $\tilde{f} : \mathbf{B}^{n+1} \to U \cap Y$ for all $n < \omega$.*

Using this lemma, we shall prove the following proposition:

Proposition 7.3. *Let X be a locally path-connected space that admits a locally connected completion $\overline{X}$. Then* $\mathrm{Fin}(X)$ *is homotopy dense in* $\mathrm{Comp}(\overline{X})$ *if and only if $\overline{X} \setminus X$ is locally non-separating in $\overline{X}$.*

Sketch of proof.

(1) The "only if" part

- Follow the same argument as the proof of implication (ii) $\Rightarrow$ (iii) in [4, Theorem 3.2].

(2) The "if" part

- The hyperspaces $\mathrm{Comp}(\overline{X})$ and $\mathrm{Fin}(X)$ are ANRs by Propositions 3.5 and 3.6.
- For each point $A \in \mathrm{Comp}(\overline{X})$ and each neighborhood $\mathcal{U}$ of A in $\mathrm{Comp}(\overline{X})$, take a finite number of points $a_1, \cdots, a_n \in A$ and connected open neighborhoods U_i of a_i, $1 \leq i \leq n$, so that
  - $A \subset \bigcup_{i=1}^n U_i$;
  - for any $B \in \mathrm{Comp}(\overline{X})$, $B \in \mathcal{U}$ if $B \subset \bigcup_{i=1}^n U_i$ and $B \cap U_i \neq \emptyset$ for all $i \in \{1, \cdots, n\}$.
- The set $\mathcal{V} = \{B \in \mathrm{Comp}(\overline{X}) \mid B \subset \bigcup_{i=1}^n U_i \text{ and } B \cap U_i \neq \emptyset \text{ for every } 1 \leq i \leq n\}$ is a neighborhood of A as in Lemma 7.2.

□

Combining the above proposition with Theorems A, B and 7.1, we can prove Corollary.

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