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Strongly almost disjoint functions, Kurepa trees, and side condition methods

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Abstract

We force a family of strongly almost disjoint functions by finite conditions. Our forcing construction is divided into two stages. The first stage provides a Kurepa tree and forced by side conditions only. The second stage provides a family of strongly almost disjoint functions by a c.c.c. poset that makes use of the Kurepa tree forced in the first stage. This explicates a role of side conditions in side condition methods.

Introduction

Let κ be a regular cardinal with $\kappa \geq \omega_2$. We force a family of strongly almost disjoint functions of a size κ by a two-step iteration. Our notion of forcing is of a form (a proper poset by finite conditions)$*$(a c.c.c. poset by finite conditions). We first force a matrix, that is thought of a structured collection of countable universes of set theory, a la [A-M]. But we force with side conditions only. The matrix in turn entails a Kurepa tree of height ω_1 with at least κ-many cofinal branches. Any Kurepa tree as such entails an indexed family $\langle g_\alpha \mid \alpha < \kappa\rangle$ of almost disjoint functions $g_\alpha : \omega_1 \longrightarrow \omega$. Any family of functions as such entails a ccc poset that forces an indexed family $\langle f_\alpha \mid \alpha < \kappa\rangle$ of strongly almost disjoint functions $f_\alpha : \omega_1 \longrightarrow \omega$. Our construction is based on a remark by Galvin in [Ka, page 163]. There are several related constructions in [Z], [Ko], and [I]. We originally constructed a family of almost disjoint functions directly out of our matrix forced. But the composition of this paper via a Kurepa tree reflects a comment by Y. Yoshinobu.

§1. Forcing a matrix

This section is based on [M]. We first force what we called a matrix in [M].

1.1 Theorem. ([M]) Let κ be a regular cardinal with $\kappa \geq \omega_2$. Then there exists a notion of forcing P that is proper, has the ω_2-c.c. (CH), and that forces a collection $\dot{\mathcal{N}}$ of countable elementary substructures $N \in V$, where V stands for the ground model, of H_κ^V such that

(1) For $N, M \in \dot{\mathcal{N}}$, if $N \cap \omega_1 = M \cap \omega_1$, then there exists a unique isomorphism Φ_{NM} between $(N, \in, \dot{\mathcal{N}} \cap N)$ and $(M, \in, \dot{\mathcal{N}} \cap M)$ and Φ_{NM} is the identity on the intersection $N \cap M$.

(2) For any $N, M \in \dot{\mathcal{N}}$, if $N \cap \omega_1 < M \cap \omega_1$, then there exists $M' \in \dot{\mathcal{N}}$ such that $N \in M'$ and $M' \cap \omega_1 = M \cap \omega_1$.

(3) $\bigcup \dot{\mathcal{N}} = H_\kappa^V$.

Proof. (Outline) Our poset is identical to the very first step P_0 of Aspero-Mota iteration in [A-M]. We define $p \in P$, if p is a finite set of countable elementary substructures of H_κ such that

(1) For $N, M \in p$, if $N \cap \omega_1 = M \cap \omega_1$, then there exists a unique isomorphism Φ_{NM} between $(N, \in, p \cap N)$ and $(M, \in, p \cap M)$ and Φ_{NM} is the identity on the intersection $N \cap M$.

(2) For any $N, M \in p$, if $N \cap \omega_1 < M \cap \omega_1$, then there exists $M' \in p$ such that $N \in M'$ and $M' \cap \omega_1 = M \cap \omega_1$.

For $p, q \in P$, we set $q \leq p$, if $q \supseteq p$. Let G be P-generic over the ground model V and let

$$\dot{\mathcal{N}} = \bigcup G.$$

Then this $\dot{\mathcal{N}}$ works. Notice that for any $N, M \in \dot{\mathcal{N}}$, there exists $M' \in \dot{\mathcal{N}}$ such that $N, M \in M'$. Namely, $\dot{\mathcal{N}}$ is $\in$-directed. This gets entailed, say, by the fact that $\dot{\mathcal{N}}$ is $\in$-cofinal in H_κ^V.

□

We do not expect that this $\dot{\mathcal{N}}$, called a matrix, entails any morass. However, a matrix $\dot{\mathcal{N}}$ entails a Kurepa tree.

1.2 Theorem. ([M]) Any collection $\dot{\mathcal{N}}$ as above entails a Kurepa tree of height ω_1 with at least κ-many branches.

If we have a Kurepa tree of height ω_1 with at least κ-many branches, then we have an indexed family $\langle g_\alpha \mid \alpha < \kappa \rangle$ of almost disjoint functions $g_\alpha : \omega_1 \longrightarrow \omega$. Namely, $E^g_{\alpha\beta} = \{\gamma < \omega_1 \mid g_\alpha(\gamma) = g_\beta(\gamma)\} (= E^g_{\beta\alpha})$ is of a size countable for all $\alpha, \beta < \kappa$ with $\alpha \neq \beta$.

For the convenience of the readers, we reproduce a section of [M] that provides a proof of 1.2 Theorem.

§2. Forming a Kurepa tree

In this section, we assume that we are in the generic extension by P. Hence we have $\dot{\mathcal{N}}$ that satisfies (1), (2), and (3) of 1.1 Theorem. We show that there exists a Kurepa tree of height ω_1 with at least κ-many cofinal paths. Let $I = \{N \cap \omega_1 \mid N \in \dot{\mathcal{N}}\}$.

2.1 Definition. For $i \in I$, let us fix $N_i \in \dot{\mathcal{N}}$ with $N_i \cap \omega_1 = i$. Transitive collase N_i onto $\overline{N_i}$. Let $F_{i\omega_1} = \{(c_N)^{-1} \mid N \in \dot{\mathcal{N}}$ and $N \cap \omega_1 = i\}$. For $i, j \in I$ with $i < j$, let $F_{ij} = \{c_M \circ (c_N)^{-1} \mid N, M \in \dot{\mathcal{N}},\ N \in M,\ N \cap \omega_1 = i$ and $M \cap \omega_1 = j\}$. Here c_N and c_M are the transitive collapses of N and M respectively.

The following is a represetation of $\dot{\mathcal{N}}$. Write $\overline{N_{\omega_1}} = H_\kappa^V$.

2.2 Lemma. (1) For all $i < j$ in $I \cup \{\omega_1\}$ and all $f \in F_{ij}$, $f : \overline{N_i} \longrightarrow \overline{N_j}$ are elementary embeddings.

(2) For all $i < j$ in I, F_{ij} is a countable set.

(3) For all $i < j < k$ in $I \cup \{\omega_1\}$, we have $F_{ik} = F_{jk} \circ F_{ij}$. (pairwise compositions)

(4) For all i_1, i_2 in I and all $f_1 \in F_{i_1\omega_1}$, $f_2 \in F_{i_2\omega_1}$, there exist (g_1, g_2, h, k) such that $i_1, i_2 < k \in I$, $g_1 \in F_{i_1k}$, $g_2 \in F_{i_2k}, h \in F_{k\omega_1}$ and $f_1 = h \circ g_1$, $f_2 = h \circ g_2$.

(5) $\overline{N_{\omega_1}} = \bigcup\{f[\overline{N_i}] \mid i \in I, f \in F_{i\omega_1}\}$, where $f[\overline{N_i}] = \{f(x) \mid x \in \overline{N_i}\}$.

(6) For all $i < j$ in $I \cup \{\omega_1\}$, all $f_1, f_2 \in F_{ij}$, all $\overline{e_1}, \overline{e_2} \in \overline{N_i}$, if $f_1(\overline{e_1}) = f_2(\overline{e_2})$, then $\overline{e_1} = \overline{e_2}$. (tree order)

Proof. (1): Some account for the case $j < \omega_1$. Let $f \in F_{ij}$ and let $f = c_M \circ (c_N)^{-1}$. Since $N \in M$, we have $N \prec M$. Since $c_N : N \longrightarrow \overline{N_i}$ and $c_M : M \longrightarrow \overline{N_j}$, we have $f = c_M \circ (c_N)^{-1} : \overline{N_i} \longrightarrow \overline{N_j}$.

(2): $F_{ij} = \{c_{N_j} \circ (c_N)^{-1} \mid N \in \dot{\mathcal{N}} \cap N_j, N \cap \omega_1 = i\}$ holds and so F_{ij} is countable. Some details follows. Let $f \in F_{ij}$. Take $N', M \in \dot{\mathcal{N}}$ such that $N' \in M$ and $f = c_M \circ (c_{N'})^{-1}$. Since $N_j \cap \omega_1 = j = M \cap \omega_1$, there exists an isomorphism $\phi : M \longrightarrow N_j$. Let $N = \phi(N')$. Then $N \in \dot{\mathcal{N}} \cap N_j$, $N \cap \omega_1 = N' \cap \omega_1 = i$, $c_M = c_{N_j} \circ \phi$ and $c_{N'} = c_N \circ (\phi \lceil N')$. Hence $f = c_{N_j} \circ (c_N)^{-1}$ holds.

(3): Let $i < j < k < \omega_1$ in I. The case $k = \omega_1$ is similar. Let $f = c_M \circ (c_N)^{-1} \in F_{ik}$ with $N \in M$. Take $N' \in \dot{\mathcal{N}}$ such that $N \in N' \in M$ and $N' \cap \omega_1 = j$. Then $c_{N'} \circ (c_N)^{-1} \in F_{ij}$ and $c_M \circ (c_{N'})^{-1} \in F_{jk}$. It is clear that $f = (c_M \circ (c_{N'})^{-1}) \circ (c_{N'} \circ (c_N)^{-1}) \in F_{jk} \circ F_{ij}$. Conversely, let $f \in F_{ij}$ and $g \in F_{jk}$. Then $g = c_{N_k} \circ (c_M)^{-1}$. Since M and N_j are isomorphic, we may assume $f = c_M \circ (c_N)^{-1}$ for some $N \in M \in N_k$. Hence $g \circ f = (c_{N_k} \circ (c_M)^{-1}) \circ (c_M \circ (c_N)^{-1}) = c_{N_k} \circ (c_N)^{-1} \in F_{ik}$.

(4): Let $f_1 = (c_{N_1})^{-1}$ and $f_2 = (c_{N_2})^{-1}$. Since $\dot{\mathcal{N}}$ is $\in$-directed, there exists $N \in \dot{\mathcal{N}}$ such that $N_1, N_2 \in N$. Let $k = N \cap \omega_1$, $h = (c_N)^{-1}$, $g_1 = c_N \circ (c_{N_1})^{-1}$ and $g_2 = c_N \circ (c_{N_2})^{-1}$. Then $h \in F_{k\omega_1}$, $g_1 \in F_{i_1k}$, $g_2 \in F_{i_2k}$ and $f_1 = h \circ g_1$, $f_2 = h \circ g_2$ hold.

(5): Let $e \in H_\kappa^V = \bigcup \dot{\mathcal{N}}$. Then there exists $N \in \dot{\mathcal{N}}$ with in $e \in N$. Hence e is in the range of $(c_N)^{-1} \in F_{i\omega_1}$.

(6): First with $j = \omega_1$. Let $f_1 = (c_{N_1})^{-1}$ and $f_2 = (c_{N_2})^{-1}$ with $N_1 \cap \omega_1 = N_2 \cap \omega_1 = i$. Let $e = f_1(\overline{e_1}) = f_2(\overline{e_2})$. Then $e \in N_1 \cap N_2$. Since two structures $(N_1, \in)$ and $(N_2, \in)$ are isomorphic and the isomorphim $\phi : N_1 \longrightarrow N_2$ is the identity on $N_1 \cap N_2$, we have $\overline{e_1} = c_{N_1}(e) = (c_{N_2} \circ \phi)(e) = c_{N_2}(e) = \overline{e_2}$.

Next $i < j < \omega_1$ in I. Let $f_1(\overline{e_1}) = f_2(\overline{e_2})$. Take any $h \in F_{j\omega_1}$. Then $(h \circ f_1)(\overline{e_1}) = (h \circ f_2)(\overline{e_2})$. Hence we have seen that $\overline{e_1} = \overline{e_2}$.

□

2.3 Definition. Let $T = \{(i, \overline{e}) \mid i \in I \cup \{\omega_1\}, \overline{e} \in \overline{N_i}\}$. For $t_1 = (i_1, \overline{e_1}), t_2 = (i_2, \overline{e_2})$, we set $t_1 <_T t_2$, if $i_1 < i_2$ and there exists $f \in F_{i_1 i_2}$ with $f(\overline{e_1}) = \overline{e_2}$.

2.4 Lemma. (1) $(T, <_T)$ is a tree.

(2) For $e \in \overline{N_{\omega_1}}$, let $i_e \in I$ be the least $i \in I$ such that $e \in N$ for some $N \in \dot{\mathcal{N}}$ with $N \cap \omega_1 = i$. Then for all $i \in I$ with $i \geq i_e$, there exists a unique $\pi_i(e) \in \overline{N_i}$ such that there exists $h \in F_{i\omega_1}$ with $h(\pi_i(e)) = e$. The set $\{(i, \pi_i(e)) \mid i_e \leq i \in I\} \cup \{(\omega_1, e)\}$ forms a chain in $(T, <_T)$.

(3) For different $e_1, e_2 \in \overline{N_{\omega_1}}$, $\{\pi_i(e_1) \mid i \geq i_{e_1} \text{ in } I\}$ and $\{\pi_i(e_2) \mid i \geq i_{e_2} \text{ in } I\}$ split at some point.

Proof. (1): (irreflexive) $(i, \overline{e}) <_T (i, \overline{e})$ does not hold, as $i < i$ does not hold.

(transitive) Let $(i_1, \overline{e_1}) <_T (i_2, \overline{e_2}) <_T (i_3, \overline{e_3})$. Then $i_1 < i_2 < i_3$, $f(\overline{e_1}) = \overline{e_2}$, $g(\overline{e_2}) = \overline{e_3}$. Hence $i_1 < i_3$ and $(g \circ f)(\overline{e_1}) = \overline{e_3}$.

(comparable below a node) Let $(i_1, \overline{e_1}), (i_2, \overline{e_2}) <_T (i, \overline{e})$. We have $f_1(\overline{e_1}) = \overline{e} = f_2(\overline{e_2})$. Let $i_1 = i_2$, then we know $\overline{e_1} = \overline{e_2}$. Two nodes are identical in this case. Let $i_1 < i_2$. Then $f_1 = h \circ g$ with $g \in F_{i_1 i_2}$ and $h \in F_{i_2 i}$. Then $h(g(\overline{e_1})) = f_2(\overline{e_2})$. Hence $g(\overline{e_1}) = \overline{e_2}$. Therefore $(i_1, \overline{e_1}) <_T (i_2, \overline{e_2})$. The remaining case is similar.

(linear order below any node is well-ordered) Since $(i_1, \overline{e_1}) <_T (i_2, \overline{e_2})$ entails $i_1 < i_2$, the linear order below any node is well-ordered.

(2): Let $c_N(e) = \pi_{i_e}(e)$. Then for any $i > i_e$ in I, we have $f_i \in F_{i_e i}$ and $h_i \in F_{i\omega_1}$ such that $(c_N)^{-1} = h_i \circ f_i$. Hence let $\pi_i(e) = f_i(\pi_{i_e}(e))$. Then $h_i(\pi_i(e)) = e$ and so $(i, \pi_i(e)) <_T (\omega_1, e)$. Hence if $i_e \leq i_1 < i_2$ in I, we have $(i_1, \pi_{i_1}(e)) <_T (i_2, \pi_{i_2}(e))$.

(3): Take $N \in \dot{\mathcal{N}}$ with $e_1, e_2 \in N$. Let $i_{e_1 e_2} = N \cap \omega_1$. Then for any $i \in I$ with $i \geq i_{e_1 e_2}$, we see that $\pi_i(e_1)$ and $\pi_i(e_2)$ are different.

□

2.5 Theorem. There exists a Kurepa tree of height ω_1 with at least κ-many paths.

Proof. Since $\overline{N_{\omega_1}} = \{f(\overline{e}) \mid i \in I, f \in F_{i\omega_1}, \overline{e} \in \overline{N_i}\}$ and $\{(i, \overline{e}) \mid i \in I, \overline{e} \in \overline{N_i}\}$ is of a size ω_1, there exists $i_0 \in I$ and $\overline{e_0} \in \overline{N_{i_0}}$ such that $K = \{f(\overline{e_0}) \mid f \in F_{i_0\omega_1}\}$ is of a size κ. We may call $root = (i_0, \overline{e_0})$. Then the subtree $(\{(i, \pi_i(e)) \mid i_0 \leq i \in I, e \in K\}, <_T)$ with the single *root* works.

□

Notice that the Kurepa tree we constructed may not be normal (at some limit level, there may exist two nodes with the same cofinal path below them).

§3. A c.c.c. poset

Throughout this section, we fix an indexed family $\langle g_\alpha \mid \alpha < \kappa \rangle$ of almost disjoint functions $g_\alpha : \omega_1 \longrightarrow \omega$ with a regular cardinal $\kappa \geq \omega_2$. Namely, $E^g_{\alpha\beta} = \{\gamma < \omega_1 \mid g_\alpha(\gamma) = g_\beta(\gamma)\}$ is of a size countable for all $\alpha, \beta < \kappa$ with $\alpha \neq \beta$. We want to force an indexed family $\langle f_\alpha \mid \alpha < \kappa \rangle$ of strongly almost disjoint functions $f_\alpha : \omega_1 \longrightarrow \omega$ by finite conditions. Namely, $E^f_{\alpha\beta} = \{\gamma < \omega_1 \mid f_\alpha(\gamma) = f_\beta(\gamma)\}$ is finite for all $\alpha, \beta < \kappa$ with $\alpha \neq \beta$. We are going to have a c.c.c. poset P by making use of $E^g_{\alpha\beta}$ in such a way that $E^f_{\alpha\beta} \subseteq E^g_{\alpha\beta}$.

3.1 Definition. Let $p \in P$, if

(1) $p : a^p \times b^p \longrightarrow \omega$, where a^p is a finite subset of κ and b^p is a finite subset of ω_1.

(2) $E^p_{\alpha\beta} \subseteq E^g_{\alpha\beta}$ for all $\alpha, \beta \in a^p$ with $\alpha \neq \beta$.

For $p, q \in P$, we set $q \leq p$, if

(1) $q \supseteq p$.

(2) If $\gamma \in b^q \setminus b^p$, then $p(\cdot, \gamma) : a^p \longrightarrow \omega$ is one-to-one. Namely, for any $\alpha, \beta \in a^p$ with $\alpha \neq \beta$, we demand $p(\alpha, \gamma) \neq p(\beta, \gamma)$.

3.2 Lemma. (1) For any $p \in P$ and $\gamma < \omega_1$, there exists $q \in P$ such that $q \leq p$ and $\gamma \in b^q$.

(2) For any $p \in P$ and $\alpha < \kappa$, there exists $q \in P$ such that $q \leq p$ and $\alpha \in a^q$.

Proof. For (1): We may assume that $\gamma \notin b^p$. Let $v : a^p \times \{\gamma\} \longrightarrow \omega$ be any one-to-one map. Let $q : a^p \times (b^p \cup \{\gamma\}) \longrightarrow \omega$ be a map such that p and q agree on $a^p \times b^p$ and q and v agree on $a^p \times \{\gamma\}$. Namely, $q = p \cup v$. Then $q \in P$ and $q \leq p$ hold. In particular, we have $E^q_{\alpha\beta} = E^p_{\alpha\beta} \subseteq E^g_{\alpha\beta}$ for all $\alpha, \beta \in a^q = a^p$ with $\alpha \neq \beta$.

For (2): We may assume that $\alpha \notin a^p$. Let $h : \{\alpha\} \times b^p \longrightarrow \omega$ be a map such that the images $h[\{\alpha\} \times b^p] = \{h(\alpha, \gamma) \mid \gamma \in b^p\}$ and $p[a^p \times b^p] = \{p(\beta, \gamma) \mid \beta \in$

$a^p, \gamma \in b^p\}$ are disjoint. Let $q : (a^p \cup \{\alpha\}) \times b^p \longrightarrow \omega$ be a map such that p and q agree on $a^p \times b^p$ and q and h agree on $\{\alpha\} \times b^p$. Namely, $q = p \cup h$. Then $q \in P$ and $q \leq p$ hold. In particular, for any $\beta \in a^p$, we have $E^q_{\beta\alpha} = \emptyset \subseteq E^g_{\beta\alpha}$.

□

3.3 Lemma. P has the c.c.c.

Proof. Let $\langle p_k \mid k < \omega_1 \rangle$ be an indexed family of conditions of P. By the Δ-system argument and counting the number of isomorphism types that is just at most countable, we may find a pair $p = p_i$ and $q = p_j$ with $i \neq j$ such that there exist a pair of isomorphisms $e_1 : (a^p, <) \longrightarrow (a^q, <)$ and $e_2 : (b^p, <) \longrightarrow (b^q, <)$ such that

(1) e_1 on the intersection $a^p \cap a^q$ is the identity on $a^p \cap a^q$.

(2) e_2 on the intersection $b^p \cap b^q$ is the identity on $b^p \cap b^q$.

(3) $g_\alpha(\gamma) = g_{e_1(\alpha)}(e_2(\gamma))$ for all $\alpha \in a^p$ and $\gamma \in b^p$.

(4) $p(\alpha, \gamma) = q(e_1(\alpha), e_2(\gamma))$ for all $\alpha \in a^p$ and $\gamma \in b^p$.

(5) Let us denote

$$\Delta_a = a^p \cap a^q, \ \Delta_b = b^p \cap b^q,$$
$$t^p_a = a^p \setminus \Delta_a, \ t^q_a = a^q \setminus \Delta_a,$$
$$t^p_b = b^p \setminus \Delta_b, \ t^q_b = b^q \setminus \Delta_b.$$

Then we have four disjoint unions;

$$a^p = \Delta_a \cup t^p_a, \ b^p = \Delta_b \cup t^p_b,$$
$$a^q = \Delta_a \cup t^q_a, \ b^q = \Delta_b \cup t^q_b.$$

Now we may demand two additional pairwise disjointness;

$$(\bigcup\{E^g_{\alpha\beta} \mid \alpha, \beta \in \Delta_a, \alpha \neq \beta\}) \cap t^p_b = \emptyset.$$

$$(\bigcup\{E^g_{\alpha\beta} \mid \alpha, \beta \in \Delta_a, \alpha \neq \beta\}) \cap t^q_b = \emptyset.$$

This is possible, since there are ω_1-many disjoint possible candidates $b^{(p_k)} \setminus \Delta_b$, while $\bigcup\{E^g_{\alpha\beta} \mid \alpha, \beta \in \Delta_a, \alpha \neq \beta\}$ is a countable set. Notice that p and q agree on $\Delta_a \times \Delta_b$.

Claim 1. Let us consider p on $\Delta_a \times t^p_b$. For $\alpha, \beta \in \Delta_a$ with $\alpha \neq \beta$ and $\gamma \in t^p_b$, we have $p(\alpha, \gamma) \neq p(\beta, \gamma)$.

Proof. Since $E^g_{\alpha\beta} \cap t^p_b = \emptyset$ and $E^p_{\alpha\beta} \subseteq E^g_{\alpha\beta}$, we conclude that $p(\alpha, \gamma) \neq p(\beta, \gamma)$.

□

Claim 2. Let us consider q on $\Delta_a \times t_b^q$. For $\alpha, \beta \in \Delta_a$ with $\alpha \neq \beta$ and $\gamma \in t_b^q$, we have $q(\alpha, \gamma) \neq q(\beta, \gamma)$.

Proof. Since $E_{\alpha\beta}^g \cap t_b^q = \emptyset$ and $E_{\alpha\beta}^q \subseteq E_{\alpha\beta}^g$, we conclude that $q(\alpha, \gamma) \neq q(\beta, \gamma)$. □

By Claim 1 and Claim 2, we may fix two maps $V : t_a^q \times t_b^p \longrightarrow \omega$ and $W : t_a^p \times t_b^q \longrightarrow \omega$ such that

(1) Three sets $p[a^p \times b^p] \cup q[a^q \times b^q]$, $V[t_a^q \times t_b^p]$, and $W[t_a^p \times t_b^q]$ are pairwise disjoint finite subsets of ω.
(2) For any $\gamma \in t_b^p$, V on $t_a^q \times \{\gamma\}$ is one-to-one.
(3) For any $\gamma \in t_b^q$, W on $t_a^p \times \{\gamma\}$ is one-to-one.

Let

$$r = p \cup q \cup V \cap W.$$

Notice that

$$\begin{aligned}(a^p \cup a^q) \times (b^p \cup b^q) &= (\Delta_a \cup t_a^p \cup t_a^q) \times (\Delta_b \cup t_b^p \cup t_b^q) \\ &= \mathrm{dom}(p) \cup \mathrm{dom}(q) \cup \mathrm{dom}(V) \cup \mathrm{dom}(W)\end{aligned}$$

and three sets $\mathrm{dom}(p) \cup \mathrm{dom}(q)$, $\mathrm{dom}(V)$, and $\mathrm{dom}(W)$ are disjoint. Hence $r : (a^p \cup a^q) \times (b^p \cup b^q) \longrightarrow \omega$ is a map such that $r \supset p, q$. We also assured that

(4) r on $a^q \times \{\gamma\}$ is one-to-one for all $\gamma \in t_b^p = b^r \setminus b^q$.
(5) r on $a^p \times \{\gamma\}$ is one-to-one for all $\gamma \in t_b^q = b^r \setminus b^p$.

It remains to show that $r \in P$. To show this, we argue in 21 cases.

Let $\alpha, \beta \in a^r = a^p \cup a^q$ with $\alpha \neq \beta$. We need to show $E_{\alpha\beta}^r \subseteq E_{\alpha\beta}^g$. Let $\gamma \in b^r = b^p \cup b^q$. Suppose $r(\alpha, \gamma) = r(\beta, \gamma)$. We want to show $g_\alpha(\gamma) = g_\beta(\gamma)$.

Case 1. $\alpha, \beta \in \Delta_a$:

Subcase 1.1. $\gamma \in \Delta_b$: Since $p(\alpha, \gamma) = r(\alpha, \gamma) = r(\beta, \gamma) = p(\beta, \gamma)$, we get $g_\alpha(\gamma) = g_\beta(\gamma)$.

Subcase 1.2. $\gamma \in t_b^p$: Similar.

Subcase 1.3. $\gamma \in t_b^q$: Similar.

Case 2. $\alpha, \beta \in t_a^p$:

Subcase 2.1. $\gamma \in \Delta_b$: Since $p(\alpha, \gamma) = r(\alpha, \gamma) = r(\beta, \gamma) = p(\beta, \gamma)$, we get $g_\alpha(\gamma) = g_\beta(\gamma)$.

Subcase 2.2. $\gamma \in t_b^p$: Similar.

Subcase 2.3. $\gamma \in t_b^q$: Since $r(\alpha,\gamma) = W(\alpha,\gamma) \neq W(\beta,\gamma) = r(\beta,\gamma)$. This case does not occur.

Case 3. $\alpha, \beta \in t_a^q$:

Subcase 3.1. $\gamma \in \Delta_b$: Since $q(\alpha,\gamma) = r(\alpha,\gamma) = r(\beta,\gamma) = q(\beta,\gamma)$, we get $g_\alpha(\gamma) = g_\beta(\gamma)$.

Subcase 3.2. $\gamma \in t_b^p$: Since $r(\alpha,\gamma) = V(\alpha,\gamma) \neq V(\beta,\gamma) = r(\beta,\gamma)$. This case does not occur.

Subcase 3.3. $\gamma \in t_b^q$: Since $q(\alpha,\gamma) = r(\alpha,\gamma) = r(\beta,\gamma) = q(\beta,\gamma)$, we get $g_\alpha(\gamma) = g_\beta(\gamma)$.

Case 4. $\alpha \in \Delta_a$ and $\beta \in t_a^p$:

Subcase 4.1. $\gamma \in \Delta_b$: Since $p(\alpha,\gamma) = r(\alpha,\gamma) = r(\beta,\gamma) = p(\beta,\gamma)$, we get $g_\alpha(\gamma) = g_\beta(\gamma)$.

Subcase 4.2. $\gamma \in t_b^p$: Similar.

Subcase 4.3. $\gamma \in t_b^q$: Since $r(\alpha,\gamma) = q(\alpha,\gamma) \neq W(\beta,\gamma) = r(\beta,\gamma)$, this case does not occur.

Case 5. $\alpha \in \Delta_a$ and $\beta \in t_a^q$:

Subcase 5.1. $\gamma \in \Delta_b$: Since $q(\alpha,\gamma) = r(\alpha,\gamma) = r(\beta,\gamma) = q(\beta,\gamma)$, we get $g_\alpha(\gamma) = g_\beta(\gamma)$.

Subcase 5.2. $\gamma \in t_b^p$: Since $r(\alpha,\gamma) = p(\alpha,\gamma) \neq V(\beta,\gamma) = r(\beta,\gamma)$, this case does not occur.

Subcase 5.3. $\gamma \in t_b^q$: Since $q(\alpha,\gamma) = r(\alpha,\gamma) = r(\beta,\gamma) = q(\beta,\gamma)$, we get $g_\alpha(\gamma) = g_\beta(\gamma)$.

Case 6. $\alpha \in t_a^p$, $\beta \in t_a^q$ and $\beta \neq e_1(\alpha)$:

Subcase 6.1. $\gamma \in \Delta_b$: Since $q(e_1(\alpha),\gamma) = q\big(e_1(\alpha), e_1(\gamma)\big) = p(\alpha,\gamma) = r(\alpha,\gamma) = r(\beta,\gamma) = q(\beta,\gamma)$, we get $g_{e_1(\alpha)}(\gamma) = g_\beta(\gamma)$. But $g_\alpha(\gamma) = g_{e_1(\alpha)}(\gamma)$. Hence $g_\alpha(\gamma) = g_\beta(\gamma)$.

Subcase 6.2. $\gamma \in t_b^p$: Since $r(\alpha,\gamma) = p(\alpha,\gamma) \neq V(\beta,\gamma) = r(\beta,\gamma)$, this case does not occur.

Subcase 6.3. $\gamma \in t_b^q$: Since $r(\alpha,\gamma) = W(\alpha,\gamma) \neq q(\beta,\gamma) = r(\beta,\gamma)$, this case does not occur.

Case 7. $\alpha \in t_a^p$, $\beta \in t_a^q$, and $\beta = e_1(\alpha)$:

Subcase 7.1. $\gamma \in \Delta_b$: Simply, we have $g_\beta(\gamma) = g_{e_1(\alpha)}(e_2(\gamma)) = g_\alpha(\gamma)$.

Subcase 7.2. $\gamma \in t_b^p$: Since $r(\alpha,\gamma) = p(\alpha,\gamma) \neq V(\beta,\gamma) = r(\beta,\gamma)$, this case does not occur.

Subcase 7.3. $\gamma \in t_b^q$: Since $r(\alpha,\gamma) = W(\alpha,\gamma) \neq q(\beta,\gamma) = r(\beta,\gamma)$, this case does not occur.

This completes the proof.

□

Therefore, we established the following.

3.4 Theorem. Let κ be a regular cardinal with $\kappa \geq \omega_2$. Let $\langle g_\alpha \mid \alpha < \kappa \rangle$ be an indexed family of almost disjoint functions $g_\alpha : \omega_1 \longrightarrow \omega$. Then there exists a c.c.c. poset that forces an indexed family $\langle f_\alpha \mid \alpha < \kappa \rangle$ of strongly almost disjoint functions $f_\alpha : \omega_1 \longrightarrow \omega$ such that for all $\alpha, \beta < \kappa$ with $\alpha \neq \beta$, the finite sets $E^f_{\alpha\beta}$ satisfy $E^f_{\alpha\beta} \subseteq E^g_{\alpha\beta}$, where $E^f_{\alpha\beta} = \{\gamma < \omega_1 \mid f_\alpha(\gamma) = f_\beta(\gamma)\}$ and $E^g_{\alpha\beta} = \{\gamma < \omega_1 \mid g_\alpha(\gamma) = g_\beta(\gamma)\}$.

3.5 Theorem. Let κ be a regular cardinal with $\kappa \geq \omega_2$. Then there exists a notion of forcing that consists of finite conditions, is proper, has the ω_2-c.c. (CH), and that forces an indexed family $\langle f_\alpha \mid \alpha < \kappa \rangle$ of strongly almost disjoint functions $f_\alpha : \omega_1 \longrightarrow \omega$.

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