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SUPERCOMPACT CARDINALS IN ZF

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1. Introduction

We study supercompact cardinals in the context of ZF. Throughout this note, our base theory is ZF, so we do not assume the axiom of choice.

Definition 1.1 (Woodin, Definition 132 in [1]). Let κ be an uncountable cardinal.

(1) κ is *inaccessible* if for every $x \in V_\kappa$, there is no cofinal map from x into κ (that is, $f``x$ is bounded in κ).

(2) κ is *supercompact* if for every $\alpha > \kappa$, there is $\beta > \alpha$, a transitive set N, and an elementary embedding $j : V_\beta \to N$ such that:

(a) The critical point of j is κ and $\alpha < j(\kappa)$.

(b) ${}^{V_\alpha}N \subseteq N$.

It is easy to see that every inaccessible cardinal is regular, and every supercompact cardinal is inaccessible.

Theorem 1.2 (Woodin, Theorem 227 in [1]). *Suppose λ is a singular cardinal and a limit of supercompact cardinals. Then λ^+ is regular, and the non-stationary ideal over λ^+ is λ^+-complete. Here an ideal I over the set A and an cardinal κ, I is* κ-complete *if for every $\alpha < \kappa$ and every sequence $\langle x_i : i < \alpha \rangle$ of I-measure zero sets, we have $\bigcup_{i<\alpha} X_i \in I$.*

Woodin's proof used a forcing method. In this note, we will give a direct and simple proof of this theorem.

2. Proofs

First we prove the following useful lemma, which can be seen as a Löwenheim-Skolem theorem in the context of ZF.

Lemma 2.1. *Let κ be a supercompact cardinal. Then for every $\alpha > \kappa$ and $x \in V_\alpha$, there is a set $M \prec V_\alpha$ such that:*

(1) $x \in M$ and $M \cap \kappa \in \kappa$.

(2) $V_{M\cap\kappa} \subseteq M$.

(3) If $\overline{M}$ is the transitive collapse of M, then $\overline{M} \in V_\kappa$.

Proof. Fix $\alpha > \kappa$ and $x \in V_\alpha$. Since κ is supercompact, there is $\beta > \alpha$, a transitive set N and an elementary embedding $j : V_\beta \to N$ such that

(1) The critical point of j is κ and $\alpha < j(\kappa)$.
(2) ${}^{V_\alpha}N \subseteq N$.

First we see that $j(a) = a$ for every $a \in V_\kappa$. We prove this by induction on the rank of sets. Suppose $\alpha < \kappa$, and $j(a) = a$ for every $a \in V_\kappa$ with rank $< \alpha$. Fix $a \in V_\kappa$ with rank α. We know $\text{rank}(a) = \alpha < \kappa$, thus $\text{rank}(j(a)) = \text{rank}(a)$. $j(b) = b$ for every $b \in a$, so we know $a \subseteq j(a)$. Pick $b \in j(a)$. $\text{rank}(j(a)) = \alpha$, hence we have $\text{rank}(b) < \alpha$, and $j(b) = b$ by the induction hypothesis. Then $j(b) = b \in j(a)$, so $b \in a$.

Since ${}^{V_\alpha}N \subseteq N$, we have that $j``V_\alpha \in N$. Moreover $j``V_\alpha \cap j(\kappa) = \kappa$. Since $j(a) = a$ for every $a \in V_\kappa$, we have $V_\kappa \subseteq j``V_\alpha$. We also know that $j(x) \in j``V_\alpha$ and the transitive collapse of $j``V_\alpha$ is just $V_\alpha$. By the elementarity of $j$, $j``V_\alpha$ is an elementary submodel of $j(V_\alpha)$. $\alpha < j(\kappa)$, hence N satisfies the following statement:

> There is a set $M \prec j(V_\alpha)$ such that $M \cap j(\kappa) \in j(\kappa)$, $V_{M\cap j(\kappa)} \subseteq M$, $j(x) \in M$, and the transitive collapse of M is of the form V_γ for some $\gamma < j(\kappa)$.

By the elementarity of j, V_β satisfies the following:

> There is a set $M \prec V_\alpha$ such that $M \cap \kappa \in \kappa$, $V_{M\cap\kappa} \subseteq M$, $x \in M$, and the transitive collapse of M is of the form V_γ for some $\gamma < \kappa$.

Clearly this M is as required. □

Now the theorem follows from the propositions below.

Proposition 2.2. *Suppose κ is supercompact. Then for every cardinal $\lambda \geq \kappa$, we have that* $\text{cf}(\lambda^+) \geq \kappa$.

Proof. Suppose to the contrary that $\text{cf}(\lambda^+) = \mu < \kappa$. Fix a large limit ordinal $\alpha > \lambda^+$. By Lemma 2.1, we can find $M \prec V_\alpha$ such that:

(1) $\{\mu, \kappa, \lambda, \lambda^+\} \in M$ and $M \cap \kappa \in \kappa$.
(2) If $\overline{M}$ is the transitive collapse of M, then $\overline{M} \in V_\kappa$.

Note that $\mu \subseteq M$ since $\mu < \kappa$ and $M \cap \kappa \in \kappa$. Moreover, since $\mu = \text{cf}(\lambda^+) < \kappa$ and $M \prec V_\alpha$, there is a cofinal map $f \in M$ from μ into λ^+, hence we have that $\sup(M \cap \lambda^+) = \sup(f``\mu) = \lambda^+$.

Let $\overline{M}$ be the transitive collapse of M, and $\pi : \overline{M} \to M$ the inverse map of the collapsing map.

Define $h : M \times \lambda \to \lambda^+$ as follows:

(1) For $\langle x, \eta \rangle \in M \times \lambda$, if x is a surjection from λ onto some $\xi < \lambda^+$, then $h(x, \eta) = x(\eta)$.
(2) Otherwise, $h(x, \eta) = 0$.

For each $\xi \in M \cap \lambda^+$, since $M \prec V_\alpha$, there is a surjection $f \in M$ from λ onto ξ. Thus h is a surjection from $M \times \lambda$ onto λ^+. Fix $\eta < \lambda$, and let $h_\eta : \overline{M} \to \lambda^+$ be the function defined by $h_\eta(y) = h(\pi(y), \eta)$. So h_η is a map from $\overline{M}$ into λ^+.

Let $X_\eta = h_\eta``\overline{M}$. Since κ is inaccessible and $\overline{M} \in V_\kappa$, we have that X_η has cardinality $< \kappa$, otherwise we can take a cofinal map from $\overline{M}$ into κ. We know $\lambda^+ = \bigcup_{\eta<\lambda} X_\eta$, hence we can define a map g from $\lambda \times \kappa$ onto λ^+ such that $g(\eta, \gamma)$ is the γ-th element of X_η. Since $|\lambda \times \kappa| = \lambda$ in ZF, we have that $|\lambda^+| = \lambda$, this is a contradiction. □

Proposition 2.3. *Suppose κ is supercompact. Let $\lambda \geq \kappa$ be a cardinal.*

(1) If $\mathrm{cf}(\lambda) \geq \kappa$ then the non-stationary ideal over λ is at least κ-complete.
(2) The non-stationary ideal over λ^+ is at least κ-complete.

Proof. (2) is immediate from (1) and Proposition 2.2.

For (1), fix a cardinal $\mu < \kappa$ and $\langle X_\xi : \xi < \mu \rangle$ measure-one sets of the non-stationary ideal over λ. We will find a club C in λ with $C \subseteq \bigcap_{\xi<\mu} X_\xi$.

By Lemma 2.1, we can find a large $\alpha > \lambda^+$ and $M \prec V_\alpha$ such that:

(1) $\{\mu, \kappa, \lambda, \langle X_\xi : \xi < \mu \rangle\} \in M$ and $M \cap \kappa \in \kappa$.
(2) If $\overline{M}$ is the transitive collapse of M, then $\overline{M} \in V_\kappa$.

We know that $X_\xi \in M$ for every $\xi < \mu$. Put $C = \bigcap\{D \in M : D \text{ is a club in } \lambda\}$. For each $\xi < \mu$, there is a club $D \in M$ in λ with $D \subseteq X_\xi$. Thus we have that $C \subseteq \bigcap_{\xi<\mu} X_\xi$. We see that C is a club in λ. Closedness is clear. Hence it is enough to see that C is unbounded in λ.

Fix $\gamma < \lambda$. We will show that C has an element greater than γ. By Lemma 2.1 again, we can find a large $\alpha' > \alpha$ and $M' \prec V_{\alpha'}$ such that:

(1) $\{\kappa, \lambda, M, C, \gamma\} \in M'$ and $M' \cap \kappa \in \kappa$.
(2) $V_{M' \cap \kappa} \subseteq M'$.
(3) If $\overline{M'}$ is the transitive collapse of M', then $\overline{M'} \in V_\kappa$.

We know that $M \subseteq M'$; let $\overline{M}$ be the transitive collapse of M. We have $\overline{M} \in V_\kappa \cap M'$, hence $\overline{M} \in V_{M' \cap \kappa}$, and $\overline{M} \subseteq V_{M' \cap \kappa} \subseteq M'$. If $\pi : \overline{M} \to M$ is the inverse map of the transitive collapsing map, then $\pi \in M'$, hence $M = \pi``\overline{M} \subseteq M'$.

Since $\mathrm{cf}(\lambda) \geq \kappa$, we have that $\gamma < \sup(M' \cap \lambda) < \lambda$; If $\sup(M' \cap \lambda) = \lambda$, there is a cofinal map from $M' \cap \mathrm{cf}(\lambda)$ into λ. Hence we can take a cofinal map from the transitive collapse $\overline{M'}$ into λ. Since $\mathrm{cf}(\lambda) \geq \kappa$, we can also take a cofinal map from $\overline{M'}$ into κ, this contradicts that $\overline{M'} \in V_\kappa$ and κ is inaccessible.

We see that $\sup(M' \cap \lambda) \in D$ for every club $D \in M$ in λ, then $\gamma < \sup(M' \cap \lambda) \in \bigcap\{D \in M : D \text{ is a club}\} = C$, as required. Fix a club $D \in M$. We have $D \in M'$. By the elementarity of M', $M' \cap D$ is unbounded in $\sup(M' \cap \lambda)$. Since D is a club in λ and $\sup(M' \cap \lambda) < \lambda$, we have that $\sup(M' \cap \lambda) = \sup(M' \cap D) \in D$. □

Corollary 2.4. *Suppose λ is a cardinal and a limit of supercompact cardinals.*

(1) $\mathrm{cf}(\lambda^+) \geq \lambda$, *and the non-stationary ideal over λ^+ is at least λ-complete.*
(2) If λ is singular, then λ^+ is regular and the non-stationary ideal over λ^+ is λ^+-complete.

(3) If λ is regular, then the non-stationary ideal over λ is λ-complete.

Note 2.5. We can strengthen Propositions 2.2 and 2.3 as follows: suppose κ is supercompact, and $\lambda \geq \kappa$ a cardinal.

(1) For every $x \in V_\kappa$, there is no cofinal map from x into λ^+.
(2) If $\mathrm{cf}(\lambda) \geq \kappa$, then for every $x \in V_\kappa$ and every sequence $\langle X_a : a \in x \rangle$ of non-stationary sets in λ, we have that $\bigcup_{a \in x} X_a$ is non-stationary.

References

[1] H. Woodin, *Suitable extender models I.* Journal of Mathematical Logic, Vol. 10, No 1&2, 101–339 (2010).

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