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Solvability of the initial value problem to a model system for water waves

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1 Introduction

The water wave problem is mathematically formulated as a free boundary problem for an irrotational flow of an inviscid and incompressible fluid under the gravitational field. The basic equations for water waves are complicated due to the nonlinearity of the equations together with the presence of an unknown free surface. Therefore, until now many approximate equations have been proposed and analyzed to understand natural phenomena for water waves. Famous examples of such approximate equations are the shallow water equations, the Green–Naghdi equations, Boussinesq type equations, the Korteweg–de Vries equation, the Kadomtsev–Petviashvili equation, the Benjamin–Bona–Mahony equation, the Camassa–Holm equation, the Benjamin–Ono equations, and so on. All of them are derived from the water wave problem under the shallowness assumption of the water waves, which means that the mean depth of the water is sufficiently small compared to the typical wavelength of the water surface.

On the other hand, in coastal engineering some model equations were derived without using any shallowness assumption of the water waves. It is well-known that the water wave problem has a variational structure. In fact, J. C. Luke [7] gave a Lagrangian in terms of the velocity potential Φ and the surface variation η. His Lagrangian has the form

$$\mathscr{L}(\Phi,\eta)=\int_{b(x)}^{h+\eta(x,t)}\left(\Phi_t(x,z,t)+\frac{1}{2}|\nabla_{x,z}\Phi(x,z,t)|^2+g(z-h)\right)\mathrm{d}z \tag{1.1}$$

and the action function is

$$\mathscr{J}(\Phi,\eta)=\int_{t_0}^{t_1}\int_{\Omega}\mathscr{L}(\Phi,\eta)\,\mathrm{d}x\mathrm{d}t,$$

where g is the gravitational constant, h is the mean depth of the water, b represents the bottom topography, and Ω is an appropriate region in $\mathbf{R}^n$. In view of Bernoulli's law

$$\Phi_t+\frac{1}{2}|\nabla_{x,z}\Phi|^2+\frac{1}{\rho}(p-p_0)+g(z-h)\equiv 0, \tag{1.2}$$

we see that Luke's Lagrangian is essentially the integral of the pressure p in the vertical direction of the water region. J. C. Luke showed that the corresponding Euler–Lagrange equation is exactly the basic equations for water waves. M. Isobe [2, 3] and T. Kakinuma [4, 5, 6] used this variational structure of the water waves to derive approximate model equations. They approximated the velocity potential in Luke's Lagrangian as

$$\Phi(x,z,t) \simeq \sum_{k=0}^{K} \Psi_k(z;b)\phi^k(x,t),$$

where $\{\Psi_k\}$ is an appropriate function system, and derived an approximate Lagrangian for $(\eta, \phi^0, \phi^1, \ldots, \phi^K)$. Their model equations are the corresponding Euler–Lagrange equations.

There are several choices of the function system $\{\Psi_k\}$. As was shown by J. Boussinesq [1], in the case of the flat bottom the velocity potential Φ can be expanded in a Taylor series with respect to the vertical spatial variable as

$$\Phi(x,z,t) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}(-\Delta)^k \phi_0(x,t),$$

where ϕ_0 is the trace of the velocity potential Φ on the bottom. Therefore, one of the choice of the approximation was given by

$$\Phi(x,z,t) \simeq \sum_{k=0}^{K} (z-b(x))^{2k}\phi^k(x,t). \tag{1.3}$$

In the case $K=0$, that is, if we use the approximation $\Phi(x,z,t) \simeq \phi^0(x,t)$ in Luke's Lagrangian, then the Lagrangian (1.1) is approximated by

$$\begin{aligned}\mathscr{L}(\phi^0,\eta) &= \int_{b(x)}^{h+\eta(x,t)} \left(\phi_t^0 + \frac{1}{2}|\nabla\phi^0|^2 + g(z-h)\right)\mathrm{d}z \\ &= \left(\phi_t^0 + \frac{1}{2}|\nabla\phi^0|^2\right)(h+\eta-b) + \frac{1}{2}g(\eta^2-(b-h)^2).\end{aligned}$$

The corresponding Euler–Lagrange equations are exactly the shallow water equations.

$$\begin{cases} \eta_t + \nabla\cdot((h+\eta-b)\nabla\phi) = 0, \\ \phi_t + \dfrac{1}{2}|\nabla\phi|^2 + g\eta = 0. \end{cases}$$

In the case $K=1$, that is, if we use the approximation

$$\Phi(x,z,t) \simeq \phi^0(x,t) + (z-b(x))^2\phi^1(x,t) \tag{1.4}$$

in Luke's Lagrangian, then the Lagrangian (1.1) is approximated by

$$
\begin{aligned}
\mathscr{L}(\phi^0,\phi^1,\eta) = H\phi_t^0 &+ \frac{1}{3}H^3\phi_t^1 \\
&+ \frac{1}{2}H|\nabla\phi^0|^2 + \frac{1}{10}H^5|\nabla\phi^1|^2 + \frac{2}{3}H^3(1+|\nabla b|^2)(\phi^1)^2 \\
&+ \frac{1}{3}H^3\nabla\phi^0\cdot\nabla\phi^1 - H^2\phi^1\nabla b\cdot\nabla\phi^0 - \frac{1}{2}H^4\phi^1\nabla b\cdot\nabla\phi^1 \\
&+ \frac{1}{2}g\big(\eta^2 - (b-h)^2\big),
\end{aligned}
$$

where $H = H(x,t) = h + \eta(x,t) - b(x)$ is the depth at the point x at time t. The corresponding Euler–Lagrange equations have the form

$$
(1.5)\qquad
\begin{cases}
\eta_t + \nabla\cdot\Big(H\nabla\phi^0 + \frac{1}{3}H^3\nabla\phi^1 - H^2\phi^1\nabla b\Big) = 0, \\
H^2\eta_t + \nabla\cdot\Big(\frac{1}{3}H^3\nabla\phi^0 + \frac{1}{5}H^5\nabla\phi^1 - \frac{1}{2}H^4\phi^1\nabla b\Big) \\
\quad +H^2\nabla b\cdot\nabla\phi^0 + \frac{1}{2}H^4\nabla b\cdot\nabla\phi^1 - \frac{4}{3}H^3(1+|\nabla b|^2)\phi^1 = 0, \\
\phi_t^0 + H^2\phi_t^1 + g\eta + \frac{1}{2}|\nabla\phi^0|^2 + \frac{1}{2}H^4|\nabla\phi^1|^2 \\
\quad +H^2\nabla\phi^0\cdot\nabla\phi^1 - 2H\phi^1\nabla b\cdot\nabla\phi^0 - 2H^3\phi^1\nabla b\cdot\nabla\phi^1 \\
\quad +2H^2(1+|\nabla b|^2)(\phi^1)^2 = 0.
\end{cases}
$$

This is one of the model proposed by M. Isobe and T. Kakinuma. In this communication, we report the solvability of the initial value problem for this Isobe–Kakinuma model under the initial conditions

$$
(1.6)\qquad (\eta,\phi^0,\phi^1) = (\eta_0,\phi_0^0,\phi_0^1) \quad \text{at} \quad t=0.
$$

2 Basic properties of the model

The linearized equations of the Isobe–Kakinuma model (1.5) around the trivial flow are

$$
\begin{cases}
\eta_t + h\Delta\phi^0 + \dfrac{h^3}{3}\Delta\phi^1 = 0, \\
\eta_t + \dfrac{h}{3}\Delta\phi^0 + \dfrac{h^3}{5}\Delta\phi^1 - \dfrac{4}{3}h\phi^1 = 0, \\
\phi_t^0 + h^2\phi_t^1 + g\eta = 0.
\end{cases}
$$

This system has a non-trivial solution of the form $\eta(x,t) = \eta_0 e^{i(\xi\cdot x - \omega t)}$ if and only if the wave vector $\xi \in \mathbf{R}^n$ and the angular frequency $\omega \in \mathbf{C}$ satisfy the relation

$$
(2.1)\qquad (6h^2|\xi|^2 + 15)\omega^2 - gh|\xi|^2(h^2|\xi|^2 + 15) = 0.
$$

This is the linear dispersion relation for the Isobe–Kakinuma model (1.5), so that the phase speed $c_{IK} = \frac{\omega}{|\xi|}$ is given by

$$c_{IK}(\xi) = \pm\sqrt{gh\frac{1+\frac{1}{15}h^2|\xi|^2}{1+\frac{2}{5}h^2|\xi|^2}}.$$

We will compare the dispersion relation to those of well-known model equations. The linear dispersion relations of the shallow water (SW) equations, the Korteweg–de Vries (KdV) equation, the Benjamin–Bona–Mahony (BBM) equation, the Green–Naghdi (GN) equations, and the full water wave (WW) equations and the corresponding phase speeds c_{SW}, c_{KdV}, c_{BBM}, c_{GN}, and c_{WW} are given by

$$\begin{array}{lll}
\text{(SW)} & \omega^2 - gh|\xi|^2 = 0, & c_{SW} = \pm\sqrt{gh}, \\[2ex]
\text{(KdV)} & \pm\omega - \sqrt{gh}\xi + \dfrac{1}{6}\sqrt{gh}h^2\xi^3 = 0, & c_{KdV}(\xi) = \pm\sqrt{gh}(1 - \dfrac{1}{6}h^2\xi^2), \\[2ex]
\text{(BBM)} & \pm\Big(1 + \dfrac{1}{6}h^2\xi^2\Big)\omega - \sqrt{gh}\xi = 0, & c_{BBM}(\xi) = \pm\dfrac{\sqrt{gh}}{1+\frac{1}{6}h^2\xi^2}, \\[2ex]
\text{(GN)} & \Big(1 + \dfrac{1}{3}h^2|\xi|^2\Big)\omega^2 - gh|\xi|^2 = 0, & c_{GN}(\xi) = \pm\sqrt{\dfrac{gh}{1+\frac{1}{3}h^2|\xi|^2}}, \\[2ex]
\text{(WW)} & \omega^2 - g|\xi|\tanh(h|\xi|) = 0, & c_{WW}(\xi) = \pm\sqrt{\dfrac{g\tanh(h|\xi|)}{|\xi|}}.
\end{array}$$

Dispersion curves for these equations are depicted in Figure 1.

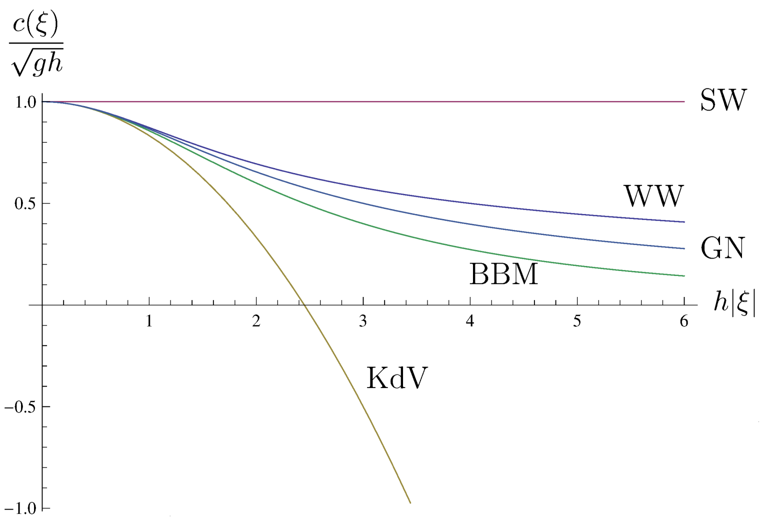

Figure 1: Dispersion curves for SW, KdV, BBM, GN, and WW equations

Among these model equations, the Green–Naghdi equations give the best approximation in the shallow water regime $h|\xi| \ll 1$. Now, let us compare the dispersion curves for the Green–Naghdi equations and the Isobe–Kakimuma (IK) model.

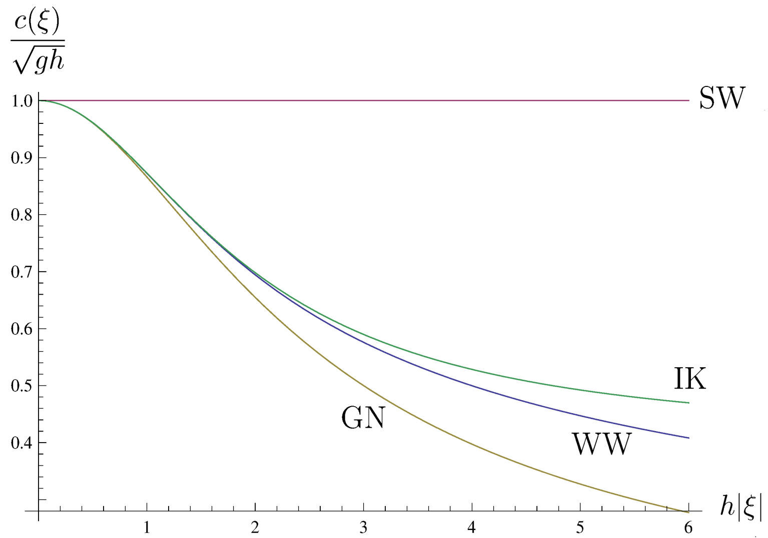

Figure 2: Dispersion curves for SW, GN, IK, and WW equations

In view of Figure 2, we see that the Isobe–Kakinuma model gives a much better approximation than the Green–Naghdi equations in the shallow water regime. Mathematically, we can characterize a relation of these approximations as follows.

$$\text{(KdV)} \quad c_{KdV}(\xi) = \pm\sqrt{gh}\left(1 - \frac{1}{6}h^2\xi^2\right) = \text{Taylor approximation of } c_{WW}(\xi),$$

$$\text{(BBM)} \quad c_{BBM}(\xi) = \pm\frac{\sqrt{gh}}{1+\frac{1}{6}h^2\xi^2} = [0/2] \text{ Padé approximant of } c_{WW}(\xi),$$

$$\text{(GN)} \quad \left(c_{GN}(\xi)\right)^2 = \frac{gh}{1+\frac{1}{3}h^2|\xi|^2} = [0/2] \text{ Padé approximant of } \left(c_{WW}(\xi)\right)^2,$$

$$\text{(IK)} \quad \left(c_{IK}(\xi)\right)^2 = gh\frac{1+\frac{1}{15}h^2|\xi|^2}{1+\frac{2}{5}h^2|\xi|^2} = [2/2] \text{ Padé approximant of } \left(c_{WW}(\xi)\right)^2.$$

Therefore, the Isobe–Kakinuma model (1.5) gives a very good approximation in the shallow water regime at least in the linear level.

It is well known that the full water wave problem has a conserved energy $E_{WW}(t)$, which is the sum of the kinetic and potential energies and given by

$$E_{WW}(t) = \frac{\rho}{2}\int_{\mathbf{R}^n}\left\{\int_{b(x)}^{h+\eta(x,t)} |\nabla_{x,z}\Phi(x,z,t)|^2 \mathrm{d}z + g(\eta(x,t))^2\right\}\mathrm{d}x.$$

Putting the approximation (1.4) in this energy, we obtain formally the energy function

$E_{IK}(t)$ for the Isobe–Kakinuma model (1.5)

$$E_{IK}(t) = \frac{\rho}{2} \int_{\mathbf{R}^n} \left\{ \int_{b(x)}^{h+\eta(x,t)} \left| \nabla_{x,z} \left(\phi^0(x,t) + (z - b(x))^2 \phi^1(x,t) \right) \right|^2 \mathrm{d}z + g\big(\eta(x,t)\big)^2 \right\} \mathrm{d}x.$$

It is easy to show that this is a conserved quantity for smooth solutions to the Isobe–Kakinuma model (1.5).

The Isobe–Kakinuma model (1.5) is written in the matrix form as

$$\begin{pmatrix} 1 & 0 & 0 \\ H^2 & 0 & 0 \\ 0 & 1 & H^2 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \eta \\ \phi^0 \\ \phi^1 \end{pmatrix} + \{\text{spatial derivatives}\} = \mathbf{0}.$$

Since the coefficient matrix always has the zero eigenvalue, the hypersurface $t = 0$ in the space-time $\mathbf{R}^n \times \mathbf{R}$ is characteristic for the model, so that the initial value problem (1.5) and (1.6) is not solvable in general. In fact, if the problem has a solution (η, ϕ^0, ϕ^1), then by eliminating the time derivative η_t from the first two equations in (1.5) we see that the solution has to satisfy the relation

$$\begin{aligned} & H^2 \nabla \cdot \Big(H \nabla \phi^0 + \frac{1}{3} H^3 \nabla \phi^1 - H^2 \phi^1 \nabla b \Big) \\ & = \nabla \cdot \Big(\frac{1}{3} H^3 \nabla \phi^0 + \frac{1}{5} H^5 \nabla \phi^1 - \frac{1}{2} H^4 \phi^1 \nabla b \Big) \\ & \quad + H^2 \nabla b \cdot \nabla \phi^0 + \frac{1}{2} H^4 \nabla b \cdot \nabla \phi^1 - \frac{4}{3} H^3 (1 + |\nabla b|^2) \phi^1. \end{aligned} \tag{2.2}$$

Therefore, as a necessary condition the initial date $(\eta_0, \phi^0_0, \phi^1_0)$ and the bottom topography b have to satisfy the relation (2.2) for the existence of the solution.

3 Main result

Before giving our main result we note a generalized Rayleigh–Taylor sign condition for the full water wave problem. It is known that the well-posedness of the initial value problem for the full water wave equations may be broken unless the following sign condition is satisfied.

$$-\frac{\partial p}{\partial N} \geq c_0 > 0 \quad \text{on the water surface,}$$

where p is the pressure and N is the unit outward normal to the water surface. Since p is constant on the water surface, this condition is equivalent to the condition

$$-\frac{\partial p}{\partial z} \geq c_0 > 0 \quad \text{on the water surface.}$$

Now, we define a function $a = a(x,t)$ by

$$\begin{aligned} a := & \, g + 2H\phi^1_t + 2H^3 |\nabla \phi^1|^2 + 2H \nabla \phi^0 \cdot \nabla \phi^1 - 2\phi^1 \nabla b \cdot \nabla \phi^0 \\ & - 6H^2 \phi^1 \nabla b \cdot \nabla \phi^1 + 4H(1 + |\nabla b|^2)(\phi^1)^2. \end{aligned} \tag{3.1}$$

Then, in view of Bernoulli's law (1.2) and our approximation (1.4), we have $-\frac{1}{\rho}\partial_z p = g + \partial_z\Phi_t + \nabla_X\partial_z\Phi \cdot \nabla_X\Phi = a$ on the water surface. Therefore, it is natural to assume that this function a is positive definite at the initial time $t = 0$. We also remark that we can express the term $\phi_t^1(x, 0)$ in $a(x, 0)$ in terms of the initial data and b although the hypersurface $t = 0$ is characteristic. Let H^m be the Sobolev space of order m on $\mathbf{R}^n$ equipped with a norm $\|\cdot\|_m$. The following is our main theorem in this communication.

Theorem 3.1 ([8]). *Let g, h, c_0, M_0 be positive constants and m an integer such that $m > \frac{n}{2}+1$. There exists a time $T > 0$ such that if the initial data $(\eta_0, \phi_0^0, \phi_0^1)$ and b satisfy the relation* (2.2) *and*

$$\begin{cases} \|\eta_0\|_m + \|\nabla\phi_0^0\|_m + \|\phi_0^1\|_{m+1} + \|b\|_{W^{m+2,\infty}} \le M_0, \\ h + \eta_0(x) - b(x) \ge c_0, \quad a(x, 0) \ge c_0 \qquad \text{for} \quad x \in \mathbf{R}^n, \end{cases} \tag{3.2}$$

then the initial value problem (1.5) *and* (1.6) *has a unique solution (η, ϕ^0, ϕ^1) satisfying*

$$\eta, \nabla\phi^0 \in C([0, T]; H^m), \quad \phi^1 \in C([0, T]; H^{m+1}).$$

The idea to prove this theorem is so simple. We transform the Isobe–Kakinuma model (1.5) to a system of equations for which the hypersurface $t = 0$ is noncharacteristic by using the necessary condition (2.2). In fact, differentiating the necessary condition (2.2) with respect to time t and using the first (or the second) equation in (1.5) to eliminate η_t we obtain

$$\begin{aligned} &H^2\nabla \cdot \Big(H\nabla\phi_t^0 + \frac{1}{3}H^3\nabla\phi_t^1 - H^2\phi_t^1\nabla b\Big) \\ &= \nabla \cdot \Big(\frac{1}{3}H^3\nabla\phi_t^0 + \frac{1}{5}H^5\nabla\phi_t^1 - \frac{1}{2}H^4\phi_t^1\nabla b\Big) \\ &\quad + H^2\nabla b \cdot \nabla\phi_t^0 + \frac{1}{2}H^4\nabla b \cdot \nabla\phi_t^1 - \frac{4}{3}H^3(1 + |\nabla b|^2)\phi_t^1 + \{\text{spatial derivatives}\}. \end{aligned}$$

This together with the third equation in (1.5) gives evolution equations for ϕ^0 and ϕ^1. We superimpose the first and the second equation in (1.5) to derive an evolution equation for η in order that the resulting system of equations has a good symmetric structure. In such a way we can derive the following system of equations.

$$\begin{cases} a_0\phi_t^0 - \nabla \cdot (4H\nabla\phi_t^0) \\ \quad -\nabla \cdot \big((4H\boldsymbol{u} \cdot \nabla)\nabla\phi^0\big) + \nabla \cdot (aH\nabla\eta) = f_1, \\ a_1\phi_t^1 - \nabla \cdot (\frac{4}{5}H^5\nabla\phi_t^1) \\ \quad -\nabla \cdot \big((\frac{4}{5}H^5\boldsymbol{u} \cdot \nabla)\nabla\phi^1\big) - \nabla \cdot (aH^3\nabla\eta) = f_2, \\ 9a\eta_t + 9a\boldsymbol{u} \cdot \nabla\eta - \nabla \cdot (aH\nabla\phi^0) + \nabla \cdot (aH^3\nabla\phi^1) = f_3, \end{cases} \tag{3.3}$$

where a is defined by (3.1), f_1, f_2, f_3 are collections of lower order terms and do not include any time derivatives, and $a_0, a_1, \boldsymbol{u}$ are given by

$$\begin{cases} a_0 := \dfrac{15}{2}H^{-1}\left\{\dfrac{4}{3}(1+|\nabla b|^2)+2\nabla b\cdot\nabla H+\dfrac{8}{3}|\nabla H|^2+\dfrac{2}{3}H\Delta H-\dfrac{1}{2}H\Delta b\right\}, \\ a_1 := \dfrac{3}{2}H^3\left\{\dfrac{4}{3}(1+|\nabla b|^2)+2\nabla b\cdot\nabla H-\dfrac{4}{3}|\nabla H|^2-\dfrac{4}{3}H\Delta H-\dfrac{1}{2}H\Delta b\right\}, \\ \boldsymbol{u} := \nabla\phi^0+H^2\nabla\phi^1-2H\phi^1\nabla b. \end{cases}$$

We can rewrite (3.3) in the matrix form as

$$\begin{aligned} &\begin{pmatrix} a_0-\nabla\cdot 4H^5\nabla & 0 & 0 \\ 0 & a_1-\nabla\cdot\frac{4}{5}H\nabla & 0 \\ 0 & 0 & a \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\phi^0\\ \phi^1\\ \eta\end{pmatrix} \\ &+\begin{pmatrix} -\nabla\cdot((4H\boldsymbol{u}\cdot\nabla)\nabla & 0 & \nabla\cdot aH\nabla \\ 0 & -\nabla\cdot((\frac{4}{5}H^5\boldsymbol{u}\cdot\nabla)\nabla & -\nabla\cdot aH^3\nabla \\ -\nabla\cdot aH\nabla & \nabla\cdot a_2H^3\nabla & 9a\boldsymbol{u}\cdot\nabla \end{pmatrix}\begin{pmatrix}\phi^0\\ \phi^1\\ \eta\end{pmatrix} = F. \end{aligned}$$

It is easy to see that the matrix operator in the second term in the left hand side is skew-symmetric in L^2 modulo lower order terms, so that once we show the positivity of the functions a_0 and a_1 we see that the matrix operator in the first term behaves a symmetrizer. Once we find a symmetrizer, we can define a mathematical energy function and derive an energy estimate, which leads to the existence of the solution for the initial value problem to the reduced system (3.3). To this end, we show the following key lemma.

Lemma 3.1 *Suppose that* $0 < c_0 \le H(x) \le c_1$ *and* $\nabla b \in L^\infty(\mathbf{R}^n)$. *There exists a positive constant* $C = C(c_0, c_1)$ *depending only on* c_0 *and* c_1 *such that we have*

$$\begin{cases} ((a_0-\nabla\cdot 4H^5\nabla)\psi^0,\psi^0)_{L^2} \ge C^{-1}\|\psi^0\|_1^2, \\ ((a_1-\nabla\cdot\frac{4}{5}H\nabla)\psi^1,\psi^1)_{L^2} \ge C^{-1}\|\psi^1\|_1^2. \end{cases}$$

Thanks to this lemma, under physically reasonable conditions on the initial data together with the sign condition, we can prove the solvability of the initial value problem for the reduced system (3.3) and (1.6). Here we do not need the necessary condition (2.2). Finally, we have to show that the solution satisfies the original Isobe–Kakinuma model (1.5) under the condition (2.2). We refer to [8] for the details.

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