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STABLE MAPS AND BRANCHED SHADOWS OF 3-MANIFOLDS

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1. Introduction

In this note, we define the notion of stable map complexity for a compact orientable 3-manifold bounded by (possibly empty) tori counting the minimal number of singular fibers of codimension 2 of stable maps into the real plane, and prove that this number equals the minimal number of vertices of its branched shadows. As a consequence, we give a complete characterization of hyperbolic links in the 3-sphere whose exteriors have stable map complexity 1 in terms of Dehn surgeries. We also provide relation between the stable map complexity, shadow complexity, and hyperbolic volumes. This note is adapted from the talk at the Camp-style Seminar "Topology, Geometry and Algebra of low-dimensiional manifolds (2015)" held in Numazu. We refer the readers to [11] for the details. Throughout the note, we will work in the smooth category unless otherwise mentioned.

2. Preliminaries

2.1. Shadows and branched shadows of 3-manifolds. A compact polyhedron P is said to be *almost-special* if each point of P has a neighborhood homeomorphic to one of the five local models shown in Figure 1: A point of P having a neighborhood shaped on

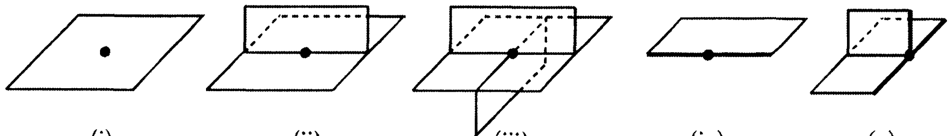

FIGURE 1. The local models of an almost-special polyhedron.

the model (iii) is called a *true vertex* of P, and we denote the set of true vertices of P by $V(P)$. The set of points of P having neighborhoods on the models (ii), (iii) or (v) is called the *singular set* of P, and we denote it by $S(P)$. The set of points having neighborhoods shaped on the models (iv) or (v) is called the *boundary* of P, and we denote it by ∂P. A point having a neighborhood shaped on the model (v) is called a *boundary-vertex* of P, and we denote the set of boundary-vertices of P by $BV(P)$. Throughout the note, we set $c(P) = |V(P)| + |BV(P)|$. The polyhedron P is said to be *closed* if $\partial P = \emptyset$. A component of $P \setminus S(P)$ is called a *region*.

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Let P be an almost-special polyhedron. A *coloring* of ∂P is a map from the set of components of ∂P to $\{i, e, f\}$. Then with respect to the coloring, ∂P decomposes into three peaces $\partial_i P$, $\partial_e P$ and $\partial_f P$. An almost-special polyhedron is said to be *boundary-decorated* if it is equipped with a coloring of ∂P. If $\partial_f(P) = \emptyset$, P is said to be *proper*.

Definition 2.1. Let M be a compact orientable 3-manifold and L a (possibly empty) link in M. A boundary-decorated almost-special polyhedron P properly embedded in a compact oriented smooth 4-manifold W is called a *shadow* of (M, L) if

- W collapses onto P after equipping the natural PL structure on W;
- P is locally flat, that is, each point p of P has a neighborhood $\mathrm{Nbd}(p; P)$ that lies in a 3-dimensional submanifold of W; and
- $(M, L) = (\partial W \setminus \mathrm{IntNbd}(\partial_e P; \partial W), \partial_i P)$.

When $L = \emptyset$, we say that P is a shadow of M for simplicity.

In [22, 23], Turaev proved that any pair of a compact orientable 3-manifold with no spherical boundary components and a (possibly empty) link in it has a shadow. In [6, 8], the *shadow complexity* of (M, L), denoted by $\mathrm{sc}(M, L)$, was defined to be the minimal number of true and boundary-vertices in any of its shadows.

A *branched polyhedron* is defined to be an almost-special polyhedron P equipped with an orientation of each of its regions so as to satisfy the following condition:

- the orientations on each component of $S(P) \setminus V(P)$ induced by the three germs of regions do not coincide (See Figure 2).

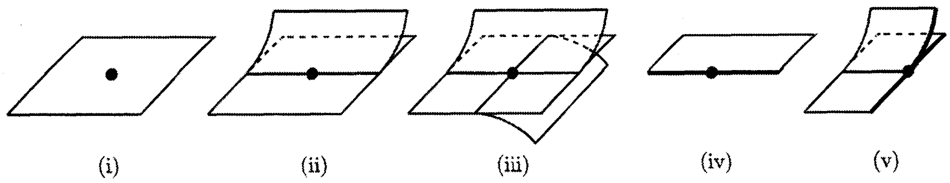

FIGURE 2. The local models of a branched polyhedron.

We refer the reader to Benedetti-Petronio [4] for general properties of branched polyhedra.

Definition 2.2. Let M be a compact orientable 3-manifold and L a (possibly empty) link in M. A shadow P of (M, L) equipped with a branching is called a *branched shadow* of (M, L).

In [6, Theorem 3.1.7] and [7, Proposition 3.4], Costantino showed that any pair of a compact orientable 3-manifold with no spherical boundary components and a (possibly empty) link in it has a branched shadow.

Definition 2.3. Let M be a compact orientable 3-manifold and L a (possibly empty) link in M. The *branched shadow complexity* of (M, L), denoted by $\mathrm{bsc}(M, L)$, is the minimal number of true and boundary-vertices in any of its branched shadows. A branched shadow P of (M, L) is said to be *minimal* if it satisfies $c(P) = \mathrm{bsc}(M, L)$, that is, it contains the least possible number of true and boundary-vertices.

A *gleam* on an almost-special polyhedron P is a coloring of all the interior regions of P with half integers satisfying a certain condition. We call an almost-special polyhedron P equipped with gleams a *shadowed polyhedron*. In [22, 23], Turaev showed the following:

(1) If an almost-special polyhedron P is embedded in a compact oriented smooth 4-manifold W is a shadow of ∂W, then there exists a canonical coloring of the interior regions of P with half integers, that is, we have the canonical gleam on P.
(2) (*Turaev's reconstruction*) From a shadowed polyhedron P, we can reconstruct a compact oriented smooth 4-manifold W and an embedding $P \hookrightarrow W$ in a unique way (up to diffeomorphism) so that $P \subset W$ is a shadow of ∂W and the canonical gleam on P given by the embedding $P \hookrightarrow W$ coincides with the prefixed gleam on P.

We note that, in this correspondence, the gleam of a region of a shadow is the generalization of the Euler number of closed surfaces embedded in oriented 4-manifolds.

2.2. **Stable maps and their Stein factorizations.** Let M be a closed orientable 3-manifold. Let f be a smooth map of M into $\mathbb{R}^2$. We denote by $S(f)$ the set of singular points of f, that is, $S(f) = \{p \in M \mid \text{rank } df_p < 2\}$. A map f of M into $\mathbb{R}$ is said to be *stable* if there exists an open neighborhood of f in $C^\infty(M, \mathbb{R}^2)$ such that for any map g in this neighborhood there exist diffeomorphisms $\Phi : M \to M$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $g = \varphi \circ f \circ \Phi^{-1}$. Here $C^\infty(M, \mathbb{R}^2)$ is the set of smooth maps of M into $\mathbb{R}^2$ with the Whitney C^∞ topology. If f is stable, there exist local coordinates centered at p and $f(p)$ such that f is locally described in one of the following way:

(1) $(u, x, y) \mapsto (u, x)$;
(2) $(u, x, y) \mapsto (u, x^2 + y^2)$;
(3) $(u, x, y) \mapsto (u, x^2 - y^2)$;
(4) $(u, x, y) \mapsto (u, y^2 + ux - x^3)$.

In the cases of (1), (2), (3), and (4), p is called a *regular point*, a *definite fold point*, an *indefinite fold point* and a *cusp point*, respectively. Further, we require that

(5) $f^{-1} \circ f(p) \cap S(f) = \{p\}$ for a cusp point p;
(6) restriction of f to $S(f) \setminus \{\text{cusp points}\}$ is an immersion with only normal crossings.

Conversely, if a smooth map satisfies the above conditions, then it is a stable map. The stable maps form an open dense set in the space $C^\infty(M, \mathbb{R}^2)$.

Let M be a compact orientable 3-manifold with (possibly empty) boundary consisting of tori. A smooth map f of M into $\mathbb{R}^2$ is called an *S-map* if

(1) the restriction of f to $\text{Int}\, M$ is a stable map (here a stable map means that, as in the case where M is closed, there exists an open neighborhood of f in $C^\infty(\text{Int}\, M, \mathbb{R}^2)$ such that for any map g in this neighborhood there exist diffeomorphisms $\Phi : M \to M$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $g = \varphi \circ f \circ \Phi^{-1}$);
(2) for each $p \in \partial M$ there exist local coordinates (u, x, y) centered at p, where ∂M corresponds to $\{x = 0\}$, and local coordinates of $f(p)$ such that f is locally described as $(u, x, y) \mapsto (u, x)$.

As in Saeki [20], we denote by $S_0(f)$ and $C(f)$ the sets of definite fold and cusp points, respectively, of the restriction of f to $\text{Int}\, M$.

Let f be an S-map of a compact orientable 3-manifold M with (possibly empty) boundary consisting of tori into $\mathbb{R}^2$. We say that two points p_1 and p_2 are *equivalent* if they are contained in the same component of the fibers of f. We denote by W_f the quotient space of M with respect to the equivalence relation and by q_f the quotient map. We define the map $\bar{f} : W_f \to N$ so that $f = \bar{f} \circ q_f$. The quotient space W_f, or the composition $\bar{f} \circ q_f$ is called the *Stein factorization* of f. The Stein factorization W_f is homeomorphic to a polyhedron, that is, the underlying space of a finite 2-dimensional simplicial complex. By Kushner-Levine-Porto [13] and Levine [16], the local models of the Stein factorization W_f can be summarized as in Figure 3. In the case of Figure 3 (iv) ((v), respectively), the

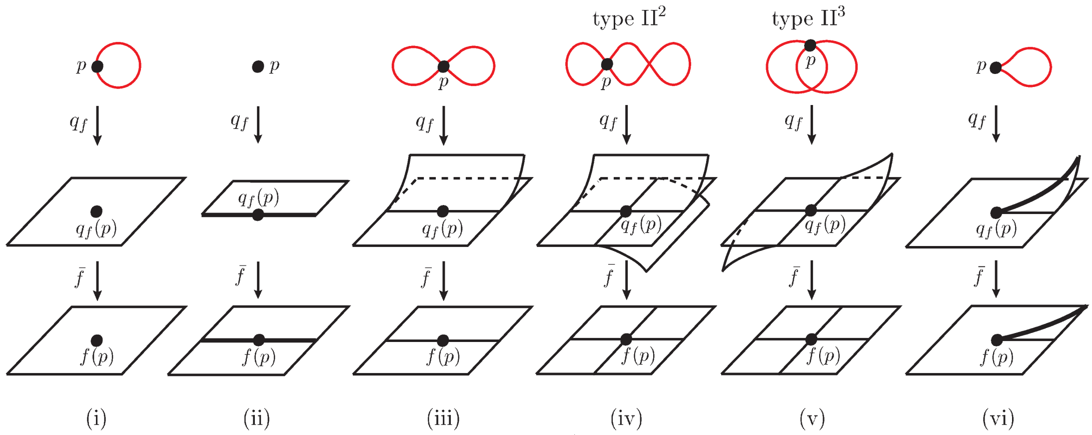

FIGURE 3. The local models of a stable map and its Stein factorization for: (i) a regular point; (ii) a definite fold point; (iii)-(v) indefinite fold points; (vi) a cusp point.

singular fiber $q_f^{-1} \circ q_f(p)$ is said to be of type II^2 (II^3, respectively) (cf. Saeki [21]). We denote by $\mathrm{II}^2(f)$ and $\mathrm{II}^3(f)$ the sets of singular fibers of types II^2 and II^3, respectively, of f.

Stable maps are defined without any condition of the dimensions of both source and target manifolds. They are especially used for obtaining topological information of the source manifold from the types of their singularities and singular fibers. A typical example is the usage of critical points of a Morse function, which is nothing else but a stable map of a manifold into the real line. Here we provide a history of the study of stable maps of 3-manifolds into $\mathbb{R}^2$:

(1) (Levine [15]) The cusp points of a stable map of a closed 3-manifold into $\mathbb{R}^2$ can be eliminated by a homotopical deformation. This implies that every closed 3-manifold admits a stable map into $\mathbb{R}^2$ without cusp points.
(2) (Burlet-de Rham [5]) If a closed 3-manifold M admits a stable map into $\mathbb{R}^2$ with only definite fold points, then M is either the 3-sphere or connected sums of $S^2 \times S^1$.
(3) (Saeki [20]) A closed 3-manifold admits a stable map with neither non-simple crossings nor cusp points if and only if M is a graph manifold. Here we recall that a compact orientable 3-manifold is called a *graph manifold* if we can cut it off by embedded tori into S^1-bundles over surfaces. (This is a generalization of (2) above.)
(4) (Costantino-Thurston [8], Gromov [10]) For a stable map f from a closed 3-manifold M into $\mathbb{R}^2$, the following holds:

$$||M|| \leqslant 10 \left(\#\mathrm{II}^2(f) + \#\mathrm{II}^3(f)\right).$$

Here $||M||$ is the Gromov norm of M. (This is a generalization of the "only if" part of (3) above.)

Definition 2.4. Let M be a compact, orientable 3-manifold with (possibly empty) boundary consisting of tori and L a (possibly empty) link in M. Let $f : M \to \mathbb{R}^2$ be an S-map. We say that f is an *S-map of* (M, L) (or simply *of* L) if $S_0(f) \supset L$. An S-map f of (M, L) is said to be *proper* if $S_0(f) = L$. When M is a closed 3-manifold, we call f a stable map of (M, L).

3. Branched shadow complexity and stable map complexity

The following is one of our main theorems.

Theorem 3.1. *Let M be a compact, orientable 3-manifold with (possibly empty) boundary consisting of tori and L a (possibly empty) link in M. Then we have* $\mathrm{bsc}(M, L) = \mathrm{smc}(M, L)$.

The inequality $\mathrm{bsc}(M, L) \leqslant \mathrm{smc}(M, L)$ follows essentially from Costantino-Thurston [8, Theorem 4.2]. In fact, the Stein factorization W_f of a given S-map of (M, L) is already "almost" a branched shadow of (M, L). However, the local model of W_f shown on the left hand side of Figure 4 is not allowed as a local model of a branched shadow. This model corresponds to a type II^3 singular fiber of f (recall Figure 3). Replacing each of these parts of W_f with the one shown on the right hand side of Figure 4, we obtain a branched shadow of (M, L).

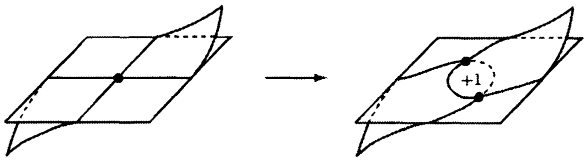

FIGURE 4. Local replacement of W_f.

The proof of $\mathrm{bsc}(M, L) \geqslant \mathrm{smc}(M, L)$ is much more complicated. We can show that, given a minimal branched shadow P of (M, L), one can actually construct an S-map so that each vertex of P corresponds to a type II^2 singular fiber, and no other types II^2 or II^3 singular fibers are created.

Remark 3.2. Let (M, L) be as in Theorem 3.1, and let $f : (M, L) \to \mathbb{R}^2$ be an S-map. If $\mathrm{II}^3(f) = \emptyset$, the Stein factorization W_f is exactly a branched shadow of (M, L). Suppose that $\mathrm{II}^3(f) \neq \emptyset$, and let $N_1, N_2, \ldots, N_n$ be closed neighborhoods of the singular fibers of type II^3. As we have seen in Figure 3, each N_i is a genus 3 handlebody. By the proof of Theorem 3.1, we may construct a map $g_i : N_i \to \mathbb{R}^2$ using the shadowed branched polyhedron depicted in Figure 4. Exchanging f with g_i inside N_i for each $i \in \{1, 2, \ldots, n\}$, we get a new S-map $(M, L) \to \mathbb{R}^2$ having no singular fibers of type II^3.

Theorem 3.1 and an easy combinatorial argument allow us to obtain the subadditivity of the stable map complexities under connected sums and torus sums. Further, we have the following.

Corollary 3.3. *Let M be a compact, orientable 3-manifold with (possibly empty) boundary consisting of tori and L a ink in M. Then we have* $\mathrm{smc}(M) \leqslant \mathrm{smc}(M, L) = \mathrm{smc}(E(L))$. *Here $E(L)$ is the exterior of the link L.*

A (possibly empty) link in a compact orientable 3-manifold is called a *graph link* if its exterior is a graph manifold. The following proposition is a direct consequence of Saeki [20], Costantino-Thurston [8, Proposition 3.31] and Theorem 3.1.

Proposition 3.4. *Let M be a compact, orientable 3-manifold with (possibly empty) boundary consisting of tori and L a (possibly empty) link in M. Then the following conditions are equivalent:*
(1) $\mathrm{sc}(M, L) = 0$, (2) $\mathrm{bsc}(M, L) = 0$, (3) *L is a graph link.*

4. Stable maps of links

Let L be a link in S^3, and D_L be its diagram on a disk D. It is easy to see that the mapping cylinder

$$P^*_{D_L} = ((L \times [0,1]) \sqcup D)/(x,0) \sim \pi(x)$$

is a non-proper shadow of (S^3, L). Fixing an orientation of L, we may equip a branching of $P^*_{D_L}$ in a natural way. Suppose that the region R of $P^*_{D_L}$ touching ∂D is an annulus and the orientations of the arcs of $R \cap D_L$ induced by L are compatible. Then by collapsing the branched polyhedron $P^*_{D_L}$ from ∂D, we obtain a proper shadow P_{D_L} of (S^3, L). By the argument of Theorem 3.1, we can actually construct a stable map of (S^3, L) whose Stein factorization is homeomorphic to P_{D_L}.

Example 4.1. The left-hand side in Figure 5 shows a diagram D_K of the figure-eight knot K. The right-hand side in the figure illustrates the branched polyhedron P_{D_K} constructed from D_K. Then we have a proper stable map $f : (S^3, K) \to \mathbb{R}^2$ with Stein factorization $(S^3, K) \xrightarrow{q_f} P_{D_K} \xrightarrow{\bar{f}} \mathbb{R}^2$ such that $|\mathrm{II}^2(f)| = 2$, $\mathrm{II}^3(f) = \emptyset$ and $C(f) = \emptyset$. The configuration of the preimages of the points $x_1, x_2, \ldots, x_6$ in P_{D_K} is shown in the figure.

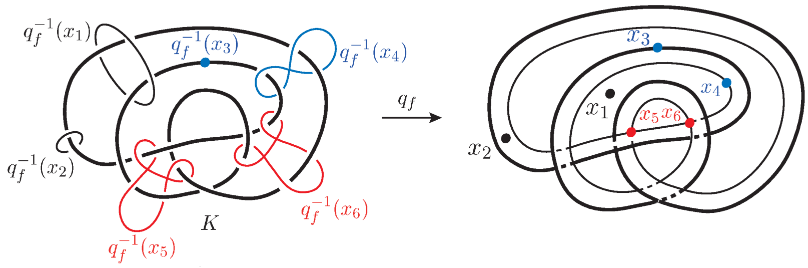

FIGURE 5. Regular and singular fibers of a stable map of the figure-eight knot. $q_f^{-1}(x_1)$ and $q_f^{-1}(x_2)$ are regular fibers of f. $q_f^{-1}(x_3)$ is a singular fiber of type I^0. $q_f^{-1}(x_4)$ is a singular fiber of type I^1. $q_f^{-1}(x_5)$ and $q_f^{-1}(x_6)$ are singular fibers of type II^2.

The following is a consequence of Corollary 3.3 and the above observation.

Corollary 4.2. *Let M be a closed orientable 3-manifold obtained from S^3 by surgery along a non-trivial link $L \subset S^3$. Then there exists a stable map $f : M \to \mathbb{R}^2$ without cusp points such that $|\mathrm{II}^2(f)| \leqslant \mathrm{cr}(L) - 2$ and $\mathrm{II}^3(f) = \emptyset$.*

Remark 4.3. A result similar to Corollary 4.2 is obtained in Kalmár-Stipsicz [12, Theorem 1.2]. The numbers of singular fibers of types II^2 and II^3, and cusp points in our Corollary 4.2 are less than theirs (in particular, $C(f) = \emptyset$ in our result).

5. Links with branched shadow complexity 1

Developing the technique obtained in the previous sections, we can provide the complete list of hyperbolic links in S^3 with $\mathrm{smc}(S^3, L) = 1$.

Theorem 5.1. *Let L be a hyperbolic link in S^3. Then $\mathrm{smc}(S^3, L) = 1$ if and only if the exterior of L is diffeomorphic to a 3-manifold obtained by Dehn filling the exterior of one of the six links L_1, $L_2, \ldots, L_6$ in S^3 along some of (possibly none of) boundary tori, where L_1, $L_2, \ldots, L_6$ are illustrated in Figure 6.*

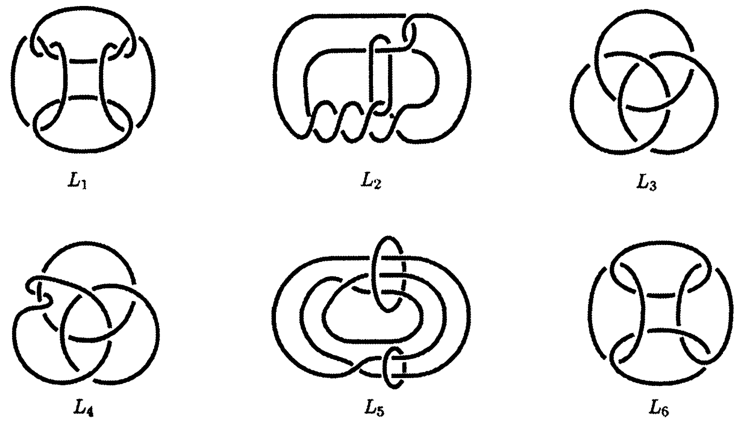

Figure 6. The links $L_1, L_2, \ldots, L_6$ in S^3.

Each of the links L_1, $L_2, \ldots, L_6$ of this theorem is a hyperbolic link having the volume $2V_{\mathrm{oct}}$, where $V_{\mathrm{oct}} = 3.66\ldots$ is the volume of the ideal regular octahedron. See Costantino-Thurston [8, Proposition 3.33]. We note that by Agol-Storm-Thurston [3, Theorem 9.1], the link L_1 is a minimal volume hyperbolic link that contains a meridional incompressible planar surface. See also Agol [1, Example 3.3]. In [24] Yoshida proved that the complement of L_6 is the minimal volume orientable hyperbolic 3-manifold with 4 cusps.

The idea of proof of Theorem 5.1 is as follows. We first list up the possible shapes of the neighborhood of the singular sets of the branched polyhedra having a single vertex. Every branched shadow P of a hyperbolic link L with $\mathrm{bsc}(S^3, L) = 1$ is obtained by attaching a piece of polyhedra (called a *tower*) shown in Figure 7. This comes from the assumption that the exterior of L does not admit an essential torus. Then we can see

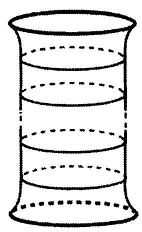

Figure 7. A tower.

that attaching a tower to a branched shadow corresponds to Dehn filling the corresponding 3-manifold. The resulting shadow must be simply-connected, because, in general, the natural projection from a closed 3-manifold onto its shadow induces a surjective homomorphism from their fundamental groups. (Note that a shadow of (S^3, L) is also a shadow of S^3.) The correspondence between links in S^3 and the neighborhoods of the singular sets of branched polyhedra having a single vertex are shown in Figure 8. Here,

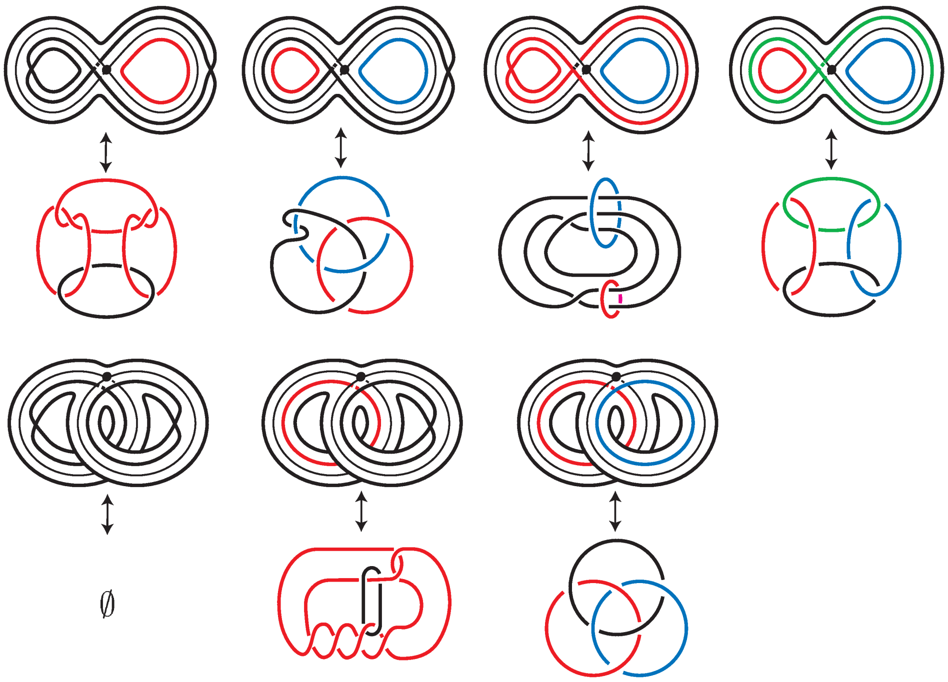

FIGURE 8. The correspondence between links in S^3 and the neighborhoods of the singular sets of branched polyhedra having a single vertex.

there is no links corresponding to the branched polyhedron illustrated on the bottom left side of Figure 8, because in what way we attach towers to it, the polyhedron cannot be simply-connected.

The next corollary follows from Theorems 3.1 and 5.1.

Corollary 5.2. *Let L be a hyperbolic link in S^3. Then there exists a stable map $f : (S^3, L) \to \mathbb{R}^2$ without cusp points such that $|\mathrm{II}^2(f)| = 1$ and $\mathrm{II}^3(f) = \emptyset$ if and only if the exterior of L is diffeomorphic to a 3-manifold obtained by Dehn filling the exterior of one of the six links $L_1, L_2, \ldots, L_6$ in Theorem 5.1 along some of (possibly none of) boundary tori.*

Example 5.3. For the figure-eight knot K, there exists a stable map with a unique singular fiber of type II^2 as shown in Figure 9, and no singular fibers of type II^2. Since the only links of the stable map complexity 0 are graph links by Proposition 3.4, we have $\mathrm{smc}(S^3, K) = 1$.

6. STABLE MAPS AND HYPERBOLIC VOLUME

Throughout the section, we consider a particular type of polyhedron, a *special polyhedron.* An almost-special polyhedron P is said to be *special* if there is no loop without

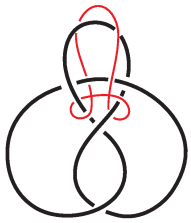

FIGURE 9. A configuration of the singular fiber of type II^2 in the figure-eight knot complement.

vertices in $S(P)$ and each region of P is a disk. We note that in this case, $S(P)$ is connected and P is closed. We call a shadow of a 3-manifold that is a special polyhedron a *special shadow* of M. Remark that every closed orientable 3-manifold admits a branched, special shadow by the moves described in Turaev [23] and Costantino [6].

Let M be a closed orientable 3-manifold with special shadow P and $\pi : M \to P$ be the projection induced by the collapsing $W \searrow P$, where $M = \partial W$. Set $M_{S(P)} = \pi^{-1}(\mathrm{Nbd}(S(P); P))$. Costantino-Thurston [8] showed that $M_{S(P)}$ admits a complete, finite volume hyperbolic structure realized by gluing $2c(P)$ copies of a regular ideal octahedron. Thus, in particular, we have $\mathrm{vol}(M_{S(P)}) = 2c(P)V_{\mathrm{oct}}$. Since each region of a special polyhedron is a disk, M is obtained from $M_{S(P)}$ by attaching solid tori, i.e., by Dehn fillings. In particular, by the 6-Theorem of Agol [2] and Lackenby [14] and the Geometrization Theorem of Perelman [17, 18, 19], if all slope lengths of the Dehn fillings are more than 6 then M admits a complete finite volume hyperbolic structure. Since the hyperbolic structure of $M_{S(P)}$ is explicitly given by the ideal octahedra, the slope lengths of Dehn fillings can be calculated in terms of the combinatorial structure of the special polyhedron P and the gleams on its regions.

Let P be a shadowed, special polyhedron. For each region R of P, set $\mathrm{sl}(R) = \sqrt{(2g)^2 + k^2}$, where $g \in \frac{1}{2}\mathbb{Z}$ is the gleam on R and k is an integer counting how many times the boundary of the closure of R passes through the vertices of P. We can show that $\mathrm{sl}(R)$ is nothing but the slope length of the Dehn filling for the corresponding boundary torus when we obtain M from the hyperbolic manifold $M_{S(P)}$. We set $\mathrm{sl}(P) = \min_R \mathrm{sl}(R)$, where R varies over all regions of P.

Proposition 6.1. *Let M be a closed orientable 3-manifold. Let P be a branched, special shadow of M. If $\mathrm{sl}(P) > 2\pi$, then we have*

$$2\,\mathrm{smc}(M)V_{\mathrm{oct}}\left(1-\left(\frac{2\pi}{\mathrm{sl}(P)}\right)^2\right)^{3/2} \leqslant 2\,c(P)V_{\mathrm{oct}}\left(1-\left(\frac{2\pi}{\mathrm{sl}(P)}\right)^2\right)^{3/2} \leqslant \mathrm{vol}(M) < 2\,\mathrm{smc}(M)V_{\mathrm{oct}}.$$

In fact, the first inequality follows essentially from Theorem 3.1 and Futer-Kalfagianni-Purcell [9, Theorem 1.1]. Here we recall that $\mathrm{sl}(R)$ is nothing but the slope length of the Dehn filling for the corresponding boundary torus when we obtain M from the hyperbolic manifold $M_{S(P)}$. The second inequality is a consequence of Costantino-Thurston [8, Theorem 3.37] and Theorem 3.1.

From Proposition 6.1 we have the following result that concerns the coincidence of shadow complexities, branched shadow complexities and stable map complexities.

Theorem 6.2. *Let M be a closed orientable 3-manifold, and let P be a branched, special shadow of M. If* $\mathrm{sl}(P) > 2\pi\sqrt{2c(P)}$, *then we have* $\mathrm{sc}(M) = \mathrm{bsc}(M) = \mathrm{smc}(M) = c(P)$.

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