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A REPORT ON EXPERIMENTS ON RANDOM BRAID

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1. Introduction

We report a part of authors computer experiments on random braids. Here by a *random braid*, we mean a braid which is obtained by taking random walks on braid groups. Let B_n denote the braid group of degree n, or the n-strand braid group. We fix the standard generating set $S_n := \{\sigma_1, \ldots, \sigma_{n-1}\}$ for B_n, where σ_i is obtained by crossing i-th string over $i+1$-th string. Recall that the braid groups can be regarded as the mapping class groups of punctured discs. Random walks on mapping class groups have been studied by several authors. Among them, one typical motivation is to study what kind of properties hold with asymptotic probability one. It is also interesting to consider the "speed" of convergence of the probability for a given property. A random walk is called *simple* if it is determined by the uniform measure on a symmetric generating set. In this report, we only consider the simple random walk associated to the symmetric generating set $S_n \cup S_n^{-1}$. Namely, we define a probability measure $\mu : B_n \to [0,1]$ by

$$\mu(x) = \begin{cases} \dfrac{1}{2n-2} & \text{if } x \in S_n \cup S_n^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Let $B_n^{\mathbb{N}}$ denote the space of sample paths. We denote by ω_i the i-th coordinate of $\omega \in B_n^{\mathbb{N}}$. A subset $[x_1, x_2, \ldots, x_n] := \{\omega \in B_n^{\mathbb{N}} \mid \omega_i = x_i\}$ is called a *cylinder set.* Any probability measure μ on B_n induces the probability measure $\mathbb{P}$ on $B_n^{\mathbb{N}}$ so that $\mathbb{P}([x_1, x_2, \ldots, x_n]) := \mu(x_1)\mu(x_1^{-1}x_2)\cdots\mu(x_{n-1}^{-1}x_n)$. We consider the probability measure $\mathbb{P}$ that is determined by the symmetric measure μ given above.

By the well-known Nielsen-Thurston classification, a braid is either periodic, reducible, or pseudo-Anosov. Among the three types of braids, pseudo-Anosov is the most "generic" one in many senses. In particular, for the simple random walk on B_n, Maher [7] proved (original statement is much more general, but we here only state for the simple random walk) the probability that we get non pseudo-Anosov braids decays exponentially.

Theorem 1.1 ([7]). *There exist $K > 0$ and $c < 1$ such that,*

$$\mathbb{P}(\omega_n \textit{ is pseudo-Anosov}) > 1 - Kc^n.$$

However it is in general difficult to compute the constants K and c in the statement. The purpose of this paper is to report a computer experiment toward the question;

Question 1.2. For a given n, how many steps do we need to have pseudo-Anosov elements by the simple random walk on B_n.

This question is somewhat vague. Hence we first formalize it in terms of *threshold phenomena* in §2. Then in §3, we explain how to see if given braids are pseudo-Anosov or not. We use two methods, Thurston's gluing equations and strict angle structures.

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Advantage and disadvantage of those methods in our context is also discussed in §3. Finally in §4, we show results of computer experiments.

2. Threshold phenomena

A threshold phenomena, or a phase transition is first observed by Erdös-Rényi [4] for random graphs. In this section, we formalize it in the context of braid group B_n. Let P be a property of braids. We say that a function $F_P : \mathbb{N} \to \mathbb{N}$ is a threshold function if

$$\lim_{n\to\infty} \mathbb{P}(\omega_{N(n)} \text{ is } P) = \begin{cases} 0 & \text{if } \lim\limits_{n\to\infty} N(n)/F_P(n) = 0, \\ 1 & \text{if } \lim\limits_{n\to\infty} N(n)/F_P(n) = +\infty, \end{cases}$$

We want to consider the threshold phenomena for the property being pseudo-Anosov. The existence of this function is known to be a difficult problem. For example, even for well-known 3-SAT problem, the existence of such a function is open. Therefore it is worth having computer experiments for this question. In this report, we focus on the second property, namely we will give an experiment regarding to a function $PA(n)$ such that for $\lim_{n\to\infty} N(n)/PA(n) = +\infty$, we have $\lim_{n\to\infty} \mathbb{P}(\omega_{N(n)}$ is pseudo-Anosov$) = 1$. Obviously $PA(n) > n - 1$ since for a braid with word length less than $n-1$, there must be a string unchanged. This implies the braid is reducible.

3. Detecting pseudo-Anosov

There are several methods to see if given braids are pseudo-Anosov or not by computer. One of the recent progress is due to Bell, who developed a computer program called *flipper* [1]. By flipper, we can decide Nielsen-Thurston type of braids rigorously. However, although flipper is implemented very carefully and quick in certain sense, it is not enough for our purpose. This is because to have a good picture for the threshold phenomena, we need to deal with very long words in the braid group of high degree. Since our goal is to have reasonable computer experiments, we only consider approximate computation. Here we use the work of Thurston which says; a braid is pseudo-Anosov if and only if its mapping torus is hyperbolic. We now explain two well known methods to check if a given manifold is hyperbolic by computer.

3.1. **Thurston's gluing equation.** For a given manifold M, Thurston found a system of complex polynomial equations whose solution with positive imaginary part corresponds to the hyperbolic structure on M. Thurston's gluing equation can be approximately solved by *SnapPea*, or *SnapPy* [3] developed by Culler-Dunfield-Weeks et. al. Here by *approximately solved*, we mean that SnapPea uses floating point arithmetic for the computation and the result is not rigorous. (We can make it rigorous by using interval arithmetic [6].) But this is good enough for our purpose. Indeed, as far as the author knows, SnapPea have never given false positive. Note that SnapPea tries to find hyperbolic structures, and hence its failure (even with verified version in [6]) would never imply non-hyperbolicity of the manifolds. Nevertheless, SnapPea can give certain information for the function $PA(n)$.

3.2. **Strict angle structure.** One another way to compute hyperbolicity is to use strict angle structure. Roughly speaking strict angle structure is obtained by solving a "linear part" of Thurston's gluing equation. For Thurston's gluing equation and strict angle structure, see for example [5]. Although a strict angle structure does not directly correspond to the hyperbolic structure, Casson proved that if a manifold M admits a strict angle structure, then M is hyperbolic (see e.g. [5, Theorem 1.1]). Burton-Budney-Pettersson et.al developed a computer program called *Regina* [2] which computes strict angle structures (if any) of a given manifold. To have a strict angle structure, we must find a positive solution of a linear equation of type $Ax = b$ where A is a singular matrix. Hence unlike the solution of Thurston's gluing equation, the space of strict angle structures may have positive dimension. For our purpose it suffices to find one strict angle structure. Regina tries to find strict angle structures by a linear programing method. This can give a rigorous certificate of the hyperbolicity of a given manifold. However even though the method Regina uses is sophisticated, it is not fast enough for our purpose. Hence we again resign to approximate solution. There are several numerical and iterative methods to solve linear equations. However unlike the case of Thurston's gluing equation, simple applications of those methods do not work well. (Recall that SnapPea uses Newton's method to solve Thurston's gluing equation. It works well since the solution if any, is unique by Mostow-Prasad rigidity.) This is because the equations for strict angle structures may have both positive and non-positive solution, and it often happens that by iterative methods, solutions converge to non-positive solutions even if there exist positive solutions. To overcome this situation, we used randomly generated numbers. Since detailed explanation of this method is beyond the scope of this report, we give a rough idea of the method. An iterative method gives a function F so that the sequence $\{x_n\}$ given by $x_{n+1} := F(x_n)$ converges to a solution. A naive idea to use randomness is to give randomly an initial value x_0 and try iterative methods. In our case this idea didn't work well, in most cases even after reasonably many trials, we only get non-positive solution for manifolds which are quite likely to be hyperbolic (e.g. manifolds which SnapPea thinks they are hyperbolic). Then next possible way is to use randomness at each step of iterative methods. Namely at each step we "push" randomly x_n to the positive direction; $x'_n := F(x_n) + r_n$ where r_n is some randomly chosen positive vector. Remember that we are looking for a positive solution. After carefully setting when and what random positive vectors we add, we can implement a reasonable program to check if a given manifold admits strict angle structure. As is the case of SnapPea, our program uses approximate computation and is not rigorous. However we have not seen any false positive with this program as well.

3.3. **Advantage and disadvantage.** As we have seen in this section, strict angle structure seems to have disadvantages to the solution of Thurston's gluing equation; even though equations are linear, it is harder to solve, and it does not correspond to the hyperbolic structure of the manifold. However the last disadvantage, not corresponding to the hyperbolic structure, does become an *advantage* for our purpose for the following reason. As pointed out by Rivin [8, §5.4], mapping tori obtained from random walks on mapping class groups have injectivity radius converging to 0 as the number of steps. Hence mapping tori of random braids are expected to have very small injectivity radius. Since the solution of Thurston's gluing equation does correspond to the hyperbolic structure, if the injectivity radius is too small for computer to treat, we have less hope to compute the solution by computer. However since strict angle structure is irrelevant to the hyperbolic structure, even for the manifold with small injectivity radius, it may be possible to compute strict angle structure by computer. This observation will be made clear in the result of the experiment.

4. Experimental result

The result of the experiments is summarized in the Figure 1. With authors PC, it took around three days to get the data for B_{90} by SnapPea, hence data for B_{100} by SnapPea is not given. Figure 1 shows the number of steps we needed to have more than 90% of pseudo-Anosov braids. More precisely, for a given n, we first generated 100 random braids (by using python's random function which is based on the Mersenne twister) of length k. Then check if given braids are pseudo-Anosov by using SnapPea or strict angle structure. Here when we use SnapPea, we count a solution of type "all tetrahedra positively oriented" and "contains negatively oriented tetrahedra" as a hyperbolic solution. For each method, we gradually increase k and record when number of pseudo-Anosov braids is greater than or equal to 90 among 100 randomly generated braids. As we have pointed out in §3, both methods can miss pseudo-Anosov braids. It seems that for "large" manifolds, SnapPea misses more than using strict angle structures. (Mapping tori of braids represented by length 1800 word in B_{90} are decomposed into about 2000 tetrahedra).

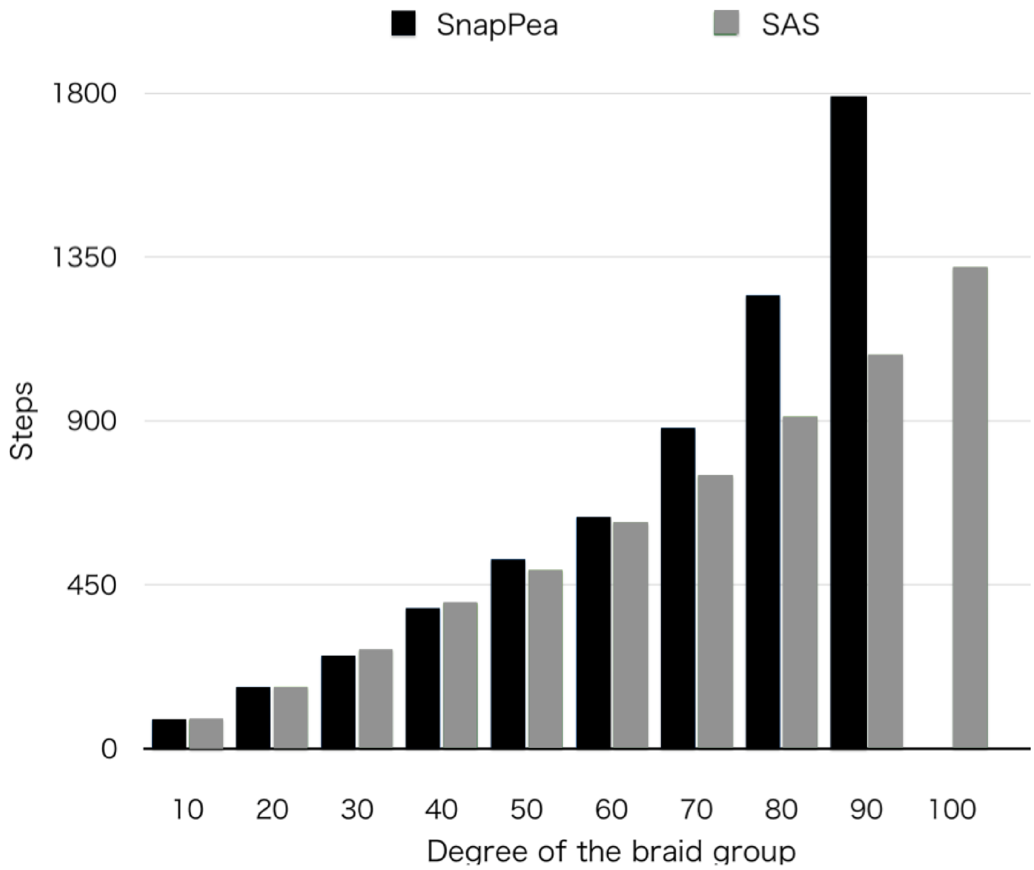

FIGURE 1. Experimental result, SAS stands for strict angle structure.

By the data that we get by SnapPea, it seems quite likely that the function $PA(n)$ is an exponential function of n. Furthermore by computing the growth ratio, even the data from strict angle structure suggest that $PA(n)$ is exponential. Therefore we conclude this report by asking following question.

Question 4.1. Does $PA : \mathbb{N} \to \mathbb{N}$ grow exponentially?

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