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ON EQUIVARIANT PERTURBATIVE INVARIANTS IN 3-DIMENSION BY MORSE THEORY

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1. INTRODUCTION

Aound 1992, Axelrod–Singer and Kontsevich independently developed the method to obtain (mathematical) topological invariants of 3-manifolds by perturbative expansion of Witten's path integral (Chern–Simons perturbation theory, [1, 5]). The invariant is a series of terms corresponding to Feynman diagrams such that each term is given by integration over the configuration space of a 3-manifold. This is known to be very strong, for example, the expansion around the trivial connection dominates all $\mathbb{Q}$-valued Ohtsuki finite type invariants for integral homology 3-spheres ([7]). In this note, we explain about our attempt to construct 'equivariant invariant' of 3-manifolds with the first Betti number 1.

Around 2008, Ohtsuki constructed an equivariant refinement of the LMO invariant[1] for 3-manifolds with the first Betti number 1 ([12, 13]), which pioneered a new direction of perturbative invariants of 3-manifolds. Inspired by Ohtsuki's work, Lescop constructed an equivariant refinement of Chern–Simons perturbation theory for 3-manifolds with the first Betti number 1 for the 2-loop graphs by using a method similar to Marché ([9, 11]). Lescop's construction is as follows.

Let M be a closed 3-manifold with $H_1(M) = \mathbb{Z}$. The *equivariant configuration space* $\mathrm{Conf}_{K_2}(M)$ is defined as the set of tuples (x_1, x_2, γ), $x_1, x_2 \in M$, satisfying the following conditions.

(1) $x_1 \neq x_2$.
(2) γ is the relative bordism class of paths $c : [0, 1] \to M$ that go from x_1 to x_2.

The natural map $\mathrm{Conf}_{K_2}(M) \to \mathrm{Conf}_2(M) = M \times M \setminus \Delta_M$ that forgets γ is an infinite cyclic covering. Instead of removing the diagonal Δ_M in the definition of $\mathrm{Conf}_2(M)$, consider the blowing-up along Δ_M, namely replacing Δ_M with its normal sphere bundle, to obtain a compactification $\overline{\mathrm{Conf}}_2(M)$ of $\mathrm{Conf}_2(M)$. Similarly, by blowing-up along the preimage of Δ_M in the space of tuples (x_1, x_2, γ) satisfying only (2) above, we obtain the 'closure' $\overline{\mathrm{Conf}}_{K_2}(M)$ of $\mathrm{Conf}_{K_2}(M)$.

Lescop defined an invariant of 3-manifolds M with $b_1(M) = 1$ by an equivariant intersection theory in $\overline{\mathrm{Conf}}_{K_2}(M)$. The principal term of it is given by the equivariant triple intersection $\langle Q, Q, Q\rangle_{\mathbb{Z}}$ for a fundamental 4-chain

$$Q \in C_4(\overline{\mathrm{Conf}}_{K_2}(M); \mathbb{Q}) \otimes_{\mathbb{Q}[t,t^{-1}]} \mathbb{Q}(t)$$

which satisfies a certain boundary condition (*equivariant propagator*)[2].

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[1]LMO invariant is defined combinatorially by using Kontsevich's link invariant and is known to be universal among finite type invariants of homology 3-spheres

[2]The Poincaré dual of $[Q] \in H_4(\overline{\mathrm{Conf}}_2(M), \partial\overline{\mathrm{Conf}}_2(M); \mathbb{Q}(t))$ generates $H^2(\overline{\mathrm{Conf}}_2(M); \mathbb{Q}(t)) \cong \mathbb{Q}(t)$. The meaning of the word 'propagator' here differs from the usual one.

Lescop proved the existence of an equivariant propagator by means of homology theoretic arguments. We developed a notion of 'Z-paths' (we previously called 'AL-paths') in a surface bundle M over S^1 and gave an explicit equivariant propagator by the natural map from the moduli space of Z-paths to configuration space ([16]). By using the equivariant propagator, we construct an invariant of fiberwise Morse functions on M ([17]). The construction of the invariant can be applied to a construction of a perturbative isotopy invariant of knots in M, which is useful for the study of finite type invariants of knots in M ([18]).

2. Moduli space of Z-paths

We define the moduli space of Z-paths and its 'closure'.[3]

2.1. Z-path. Let M be an oriented closed 3-manifold. Assume that M admits a structure of an oriented fiber bundle $\kappa : M \to S^1$. We say that a C^∞ map $f : M \to \mathbb{R}$ is a *fiberwise Morse function* if the restriction $f_s = f|_{\kappa^{-1}(s)} : \kappa^{-1}(s) \to \mathbb{R}$ is Morse for each $s \in S^1$ (known to exist for every κ). The totality of the critical points of f_s, $s \in S^1$, forms a 1-submanifold of M (closed braid) and we call each component of the 1-submanifold a *critical locus*. Let ξ be the gradient of f along the fibers, namely, the one whose restriction to each fiber over $s \in S^1$ is $\operatorname{grad} f_s$. Let $\Sigma(\xi)$ denote the union of all critical loci of ξ. For a critical locus p of a fiberwise Morse function f, the *descending/ascending manifold* are defined respectively by

$$\mathscr{D}_p(\xi) = \{x \in M \mid \lim_{t \to -\infty} \Phi^t_{-\xi}(x) \in p\}$$
$$\mathscr{A}_p(\xi) = \{x \in M \mid \lim_{t \to \infty} \Phi^t_{-\xi}(x) \in p\}$$

where $\Phi^t_{-\xi} : M \to M$ is the flow of $-\xi$.

Let $\widetilde{\kappa} : \widetilde{M} \to \mathbb{R}$ be the pullback of κ by the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$.

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\kappa}} & \mathbb{R} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ M & \xrightarrow{\kappa} & S^1 \end{array}$$

The induced map $\pi : \widetilde{M} \to M$ on the total space is an infinite cyclic covering. The function $\widetilde{f} = f \circ \pi : \widetilde{M} \to \mathbb{R}$ is a fiberwise Morse function (for a fiber bundle over $\mathbb{R}$). Let $\widetilde{\xi}$ denote the gradient for $\widetilde{f}$ along the fibers. By replacing ξ with $\widetilde{\xi}$, the critical locus, its descending/ascending manifolds are defined similarly.

We say that an embedding $\sigma : [\mu, \nu] \to \widetilde{M}$ is *horizontal* if $\operatorname{Im} \sigma$ is included in a single fiber of $\widetilde{\kappa}$ and say that it is *vertical* if $\operatorname{Im} \sigma$ is included in a single critical locus of $\widetilde{f}$. A horizontal (resp. vertical) embedding $\sigma : [\mu, \nu] \to \widetilde{M}$ is *descending* if $\widetilde{f}(\sigma(\mu)) \geq \widetilde{f}(\sigma(\nu))$ (resp. $\widetilde{\kappa}(\sigma(\mu)) \geq \widetilde{\kappa}(\sigma(\nu))$). A horizontal embedding $\sigma : [\mu, \nu] \to \widetilde{M}$ is a *flow-line* of $\widetilde{\xi}$ if for each $t \in (\mu, \nu)$, $d\sigma_t(\frac{\partial}{\partial t})$ is a positive multiple of $(-\widetilde{\xi})_{\sigma(t)}$.

Definition 2.1. Let $x, y \in \widetilde{M}$ be such that $\widetilde{\kappa}(x) \geq \widetilde{\kappa}(y)$. A *Z-path* from x to y is a sequence $\gamma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ satisfying the following six conditions.

[3] The definition in this note differs slightly from that of [16].

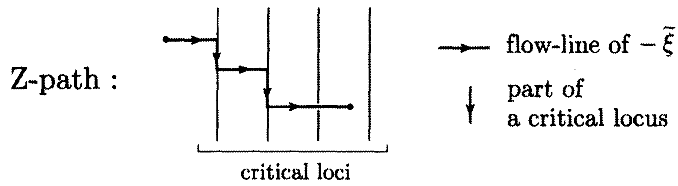


FIGURE 1. Z-path

(1) For each i, σ_i is an embedding $[\mu_i, \nu_i] \to \widetilde{M}$ (μ_i, ν_i are real numbers such that $\mu_i \leq \nu_i$) and it is either horizontal or vertical.
(2) For each i, σ_i is descending.
(3) If σ_i is horizontal, then σ_i is a flow-line of $\widetilde{\xi}$. If it is vertical, then $\mu_i < \nu_i$.
(4) $\sigma_1(\mu_1) = x$, $\sigma_n(\nu_n) = y$.
(5) $\sigma_i(\nu_i) = \sigma_{i+1}(\mu_{i+1})$ for $1 \leq i < n$.
(6) If σ_i is horizontal (resp. vertical) and if $i < n$, then σ_{i+1} is vertical (resp. horizontal).

We say that two Z-paths are *equivalent* if they are related by piecewise reparametrizations. We call a sequence of paths of the form $\pi \circ \gamma = (\pi \circ \sigma_1, \ldots, \pi \circ \sigma_n)$ for a Z-path γ in $\widetilde{M}$ a Z-path in M.

Let $\mathscr{M}_2^{\mathrm{Z}}(\widetilde{\xi})$ be the set of all equivalence classes of Z-paths in $\widetilde{M}$. This has a natural structure of a noncompact manifold with corners. Let t denote the covering translation of the covering $\pi : \widetilde{M} \to M$ that induces the translation $x \mapsto x - 1$ in $\mathbb{R}$. This induces diagonal $\mathbb{Z}$-actions $\gamma \mapsto t^n\gamma$, $(x, y) \mapsto (t^n x, t^n y)$ on $\mathscr{M}_2^{\mathrm{Z}}(\widetilde{\xi})$ and $\widetilde{M} \times \widetilde{M}$. We denote the quotient spaces $\mathscr{M}_2^{\mathrm{Z}}(\widetilde{\xi})/\mathbb{Z}$ and $(\widetilde{M} \times \widetilde{M})/\mathbb{Z}$ respectively by $\mathscr{M}_2^{\mathrm{Z}}(\widetilde{\xi})_{\mathbb{Z}}$ and $\widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$. We consider another $\mathbb{Z}$-action on the quotient spaces, denoted by t^n by abuse of notation, as

$$t([x \times y]) = [x \times ty].$$

For a gradient ξ along the fiber for a fiberwise Morse function f, let $\hat{\xi}$ denote the nonsingular vector field $\xi + \operatorname{grad} \kappa$ on M. Let $s_{\hat{\xi}} : M \to ST(M)$ (ST denotes the unit tangent bundle) be the section given by $-\hat{\xi}/\|\hat{\xi}\|$.

Theorem 2.2 ([16]). *Let Σ be an oriented connected closed surface and let M be the mapping torus of an orientation preserving diffeomorphism $\varphi : \Sigma \to \Sigma$. Let $\widetilde{\Delta}_M \subset \widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$ be the preimage of the diagonal Δ_M of $M \times M$.*

(1) *There is a natural 'closure' $\overline{\mathscr{M}}_2^{\mathrm{Z}}(\widetilde{\xi})_{\mathbb{Z}}$ of $\mathscr{M}_2^{\mathrm{Z}}(\widetilde{\xi})_{\mathbb{Z}}$ that is a countable union of compact manifolds with corners.*
(2) *Suppose that κ induces an isomorphism $H_1(M)/\mathrm{Torsion} \cong H_1(S^1)$. Let $\bar{b} : \overline{\mathscr{M}}_2^{\mathrm{Z}}(\widetilde{\xi})_{\mathbb{Z}} \to \widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$ be the map that assigns the endpoints. Let $B\ell_{\bar{b}^{-1}(\widetilde{\Delta}_M)}(\overline{\mathscr{M}}_2^{\mathrm{Z}}(\widetilde{\xi})_{\mathbb{Z}})$ be the blow-up of $\overline{\mathscr{M}}_2^{\mathrm{Z}}(\widetilde{\xi})_{\mathbb{Z}}$ along $\bar{b}^{-1}(\widetilde{\Delta}_M)$. Then $\bar{b}$ induces a map*

$$B\ell_{\bar{b}^{-1}(\widetilde{\Delta}_M)}(\overline{\mathscr{M}}_2^{\mathrm{Z}}(\widetilde{\xi})_{\mathbb{Z}}) \to \overline{\mathrm{Conf}}_{K_2}(M)$$

that represents a 4-dimensional $\mathbb{Q}(t)$-chain $Q(\widetilde{\xi})$ of $\overline{\mathrm{Conf}}_{K_2}(M)$. Moreover, the following identity in $H_3(\partial\overline{\mathrm{Conf}}_{K_2}(M);\mathbb{Q})\otimes_{\mathbb{Q}[t,t^{-1}]}\mathbb{Q}(t)$ holds.

$$[\partial Q(\widetilde{\xi})] = [s_{\widetilde{\xi}}(M)] + \frac{t\zeta'_\varphi}{\zeta_\varphi}[ST(M)|_K],$$

where ζ_φ is the Lefschetz zeta function for φ and K is a knot such that $\kappa_*([K])$ is the positive generator of $H_1(S^1)$. Furthermore, there is a product $P(t)$ of cyclotomic polynomials such that $P(t)\Delta(M)Q(\widetilde{\xi})$ is a $\mathbb{Q}[t,t^{-1}]$-chain ($\Delta(M)$ is the Alexander polynomial of M).

2.2. **Closure of the moduli space of Z-paths.** We define the space $\mathscr{M}_2(\widetilde{\xi})$ of horizontal paths in $\widetilde{M}$ by

$$\mathscr{M}_2(\widetilde{\xi}) = \{(x,y)\in \widetilde{M}\times\widetilde{M}\,;\, \widetilde{\kappa}(x)=\widetilde{\kappa}(y),\ y=\Phi^t_{-\widetilde{\xi}}(x) \text{ for some } t>0\}.$$

Let $b:\mathscr{M}_2(\widetilde{\xi})\to\widetilde{M}\times\widetilde{M}$ denote the inclusion map. For a continuous parameter $s\in S$ such as real numbers, we denote the sum $\bigcup_{s\in S} V_s$ by $\int_{s\in S} V_s$ and if the parameter is at most countable, then we denote it by $\sum_{s\in S} V_s$ or $V_{s_1}+V_{s_2}+\cdots$ etc.

For a generic $\widetilde{\xi}$, the intersection $\mathscr{D}_p(\widetilde{\xi})\cap\mathscr{A}_q(\widetilde{\xi})$ is transversal and hence is a smooth manifold. There is a free $\mathbb{R}$-action on $\mathscr{D}_p(\widetilde{\xi})\cap\mathscr{A}_q(\widetilde{\xi})$ by $x\mapsto\Phi^T_{-\widetilde{\xi}}(x)$ ($T\in\mathbb{R}$). We put

$$\mathscr{M}_{pq}(\widetilde{\xi}) = (\mathscr{D}_p(\widetilde{\xi})\cap\mathscr{A}_q(\widetilde{\xi}))/\mathbb{R}.$$

Proposition 2.3. *There is a natural closure $\overline{\mathscr{M}}_2(\widetilde{\xi})$ of $\mathscr{M}_2(\widetilde{\xi})$ and the extension $\bar{b}:\overline{\mathscr{M}}_2(\widetilde{\xi})\to\widetilde{M}\times\widetilde{M}$ of b such that for a generic $\widetilde{\xi}$ the following hold ($\Delta_S\subset S\times S$ denotes the diagonal for any set S).*

1. *$\overline{\mathscr{M}}_2(\widetilde{\xi})-\bar{b}^{-1}(\Delta_{\widetilde{M}})$ is a manifold with corners.*
2. *$\bar{b}$ induces a diffeomorphism $\mathrm{Int}\,\overline{\mathscr{M}}_2(\widetilde{\xi})\to\mathscr{M}_2(\widetilde{\xi})$.*
3. *The codimension r stratum of $\overline{\mathscr{M}}_2(\widetilde{\xi})-\bar{b}^{-1}(\Delta_{\widetilde{M}})$ corresponds to broken flow-lines that are broken r times at critical points. The codimension r stratum of $\overline{\mathscr{M}}_2(\widetilde{\xi})-\bar{b}^{-1}(\Delta_{\widetilde{M}})$ for $r\geq 1$ is canonically diffeomorphic to*

$$\begin{cases} \displaystyle\int_{s\in\mathbb{R}}\sum_{q_1\in\Sigma(\widetilde{\xi})}\mathscr{A}_{q_1}(\widetilde{\xi}_s)\times\mathscr{D}_{q_1}(\widetilde{\xi}_s)-\sum_{q_1\in\Sigma(\widetilde{\xi})}\Delta_{q_1} & (r=1)\\ \displaystyle\int_{s\in\mathbb{R}}\sum_{\substack{q_1,\ldots,q_r\in\Sigma(\widetilde{\xi})\\ q_1,\ldots,q_r \text{ distinct}}}\mathscr{A}_{q_1}(\widetilde{\xi}_s)\times\mathscr{M}_{q_1q_2}(\widetilde{\xi}_s)\times\cdots\times\mathscr{M}_{q_{r-1}q_r}(\widetilde{\xi}_s)\times\mathscr{D}_{q_r}(\widetilde{\xi}_s) & (r\geq 2)\end{cases}$$

The formula for the codimension r stratum ($r\geq 2$) in Proposition 2.3 can be rewritten as follows.

$$\int_{s\in\mathbb{R}} X(s)\times\underbrace{\Omega(s)\times\cdots\times\Omega(s)}_{r-1}\times{}^tY(s).$$

Here, if $\Sigma(\widetilde{\xi}) = \{p_1, p_2, \ldots, p_N\}$, then

$$X(s) = (\mathscr{A}_{p_1}(\widetilde{\xi}_s)\ \ \mathscr{A}_{p_2}(\widetilde{\xi}_s)\ \ \cdots\ \ \mathscr{A}_{p_N}(\widetilde{\xi}_s)),$$
$$Y(s) = (\mathscr{D}_{p_1}(\widetilde{\xi}_s)\ \ \mathscr{D}_{p_2}(\widetilde{\xi}_s)\ \ \cdots\ \ \mathscr{D}_{p_N}(\widetilde{\xi}_s)),$$
$$\Omega(s) = \begin{pmatrix} \emptyset & \mathscr{M}_{p_1p_2}(\widetilde{\xi}_s) & \mathscr{M}_{p_1p_3}(\widetilde{\xi}_s) & \cdots & \mathscr{M}_{p_1p_N}(\widetilde{\xi}_s) \\ \mathscr{M}_{p_2p_1}(\widetilde{\xi}_s) & \emptyset & \mathscr{M}_{p_2p_3}(\widetilde{\xi}_s) & \cdots & \mathscr{M}_{p_2p_N}(\widetilde{\xi}_s) \\ \mathscr{M}_{p_3p_1}(\widetilde{\xi}_s) & \mathscr{M}_{p_3p_1}(\widetilde{\xi}_s) & \emptyset & & \mathscr{M}_{p_3p_N}(\widetilde{\xi}_s) \\ \vdots & \vdots & & \ddots & \vdots \\ \mathscr{M}_{p_Np_1}(\widetilde{\xi}_s) & \mathscr{M}_{p_Np_2}(\widetilde{\xi}_s) & \mathscr{M}_{p_Np_3}(\widetilde{\xi}_s) & \cdots & \emptyset \end{pmatrix}$$

and the direct product of matrices is defined by replacing multiplications and sums with direct products and disjoint unions, respectively.

Proposition 2.4. *Let p be a critical locus of $\widetilde{\xi}$ and let $\overline{\mathscr{D}}_p(\widetilde{\xi}) = \bar{b}^{-1}(p \times \widetilde{M})$, $\overline{\mathscr{A}}_p(\widetilde{\xi}) = \bar{b}^{-1}(\widetilde{M} \times p)$. For a generic $\widetilde{\xi}$, the following are satisfied.*

1. *$\overline{\mathscr{D}}_p(\widetilde{\xi})$ (resp. $\overline{\mathscr{A}}_p(\widetilde{\xi})$) is a manifold with corners.*
2. *$\bar{b}$ induces a diffeomorphism $\mathrm{Int}\,\overline{\mathscr{D}}_p(\widetilde{\xi}) \to \mathscr{D}_p(\widetilde{\xi})$ (resp. $\mathrm{Int}\,\overline{\mathscr{A}}_p(\widetilde{\xi}) \to \mathscr{A}_p(\widetilde{\xi})$).*
3. *The codimension r stratum of ${}^t\overline{Y} = (\overline{\mathscr{D}}_{p_1}(\widetilde{\xi})\ \ \overline{\mathscr{D}}_{p_2}(\widetilde{\xi})\ \ \cdots\ \ \overline{\mathscr{D}}_{p_N}(\widetilde{\xi}))$ (resp. $\overline{X} = (\overline{\mathscr{A}}_{p_1}(\widetilde{\xi})\ \ \overline{\mathscr{A}}_{p_2}(\widetilde{\xi})\ \ \cdots\ \ \overline{\mathscr{A}}_{p_N}(\widetilde{\xi}))$) for $r \geq 1$ is canonically diffeomorphic to*

$$\int_{s\in\mathbb{R}} \underbrace{\Omega(s) \times \cdots \times \Omega(s)}_{r} \times {}^tY(s) \quad (\textit{resp.}\ \int_{s\in\mathbb{R}} X(s) \times \underbrace{\Omega(s) \times \cdots \times \Omega(s)}_{r})$$

Proposition 2.5. *Let p, q be critical loci of $\widetilde{\xi}$ and let $\overline{\mathscr{M}}_{pq}(\widetilde{\xi}) = \bar{b}^{-1}(p \times q)$. For a generic $\widetilde{\xi}$, the following fold.*

1. *$\overline{\mathscr{M}}_{pq}(\widetilde{\xi})$ is a manifold with corners.*
2. *There is a natural diffeomorphism $\mathrm{Int}\,\overline{\mathscr{M}}_{pq}(\widetilde{\xi}) \to \mathscr{M}_{pq}(\widetilde{\xi})$.*
3. *The codimension r stratum of $\overline{\Omega} = ((1 - \delta_{ij})\overline{\mathscr{M}}_{p_ip_j}(\widetilde{\xi}))$ for $r \geq 1$ is canonically diffeomorphic to*

$$\int_{s\in\mathbb{R}} \underbrace{\Omega(s) \times \cdots \times \Omega(s)}_{r+1}.$$

A *fiberwise space over a space B* is a pair of a space E and a continuous map $\phi : E \to B$. A *fiber* over a point $s \in B$ is $E(s) = \phi^{-1}(s)$ ([2]). For two fiberwise spaces $E_1 = (E_1, \phi_1)$ and $E_2 = (E_2, \phi_2)$ over B, a *fiberwise product* $E_1 \times_B E_2$ is defined as the following subspace of $E_1 \times E_2$:

$$E_1 \times_B E_2 = \int_{s\in B} E_1(s) \times E_2(s).$$

Namely, $E_1 \times_B E_2$ is the pullback of $E_1 \xrightarrow{\phi_1} B \xleftarrow{\phi_2} E_2$.

For a sequence $A_i = (A_i, \phi_i)$, $\phi_i : A_i \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ of fiberwise spaces over $\mathbb{R}$, we define its iterated integrals as

$$\begin{aligned}\int_{\mathbb{R}} A_1 A_2 \cdots A_n &= \int_{s_1 > s_2 > \cdots > s_n} A_1(s_1) \times A_2(s_2) \times \cdots \times A_n(s_n) \\ &= (\phi_1 \times \cdots \times \phi_n)^{-1}(\{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1 > \cdots > s_n\}), \\ \overline{\int_{\mathbb{R}}} A_1 A_2 \cdots A_n &= \int_{s_1 \geq s_2 \geq \cdots \geq s_n} A_1(s_1) \times A_2(s_2) \times \cdots \times A_n(s_n) \\ &= (\phi_1 \times \cdots \times \phi_n)^{-1}(\{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1 \geq \cdots \geq s_n\})\end{aligned}$$

For a matrix $P = (A_{ij})$ of fiberwise spaces over $\mathbb{R}$, we define a fiber of $s \in \mathbb{R}$ by $P(s) = (A_{ij}(s))$. Then iterated integrals for matrices of fiberwise spaces over $\mathbb{R}$ can be defined by similar formulas as above.

We define matrices X, Y, Ω of fiberwise spaces over $\mathbb{R}$ by

$$\begin{aligned}&X = (\mathscr{A}_{p_1}(\widetilde{\xi}) \;\; \mathscr{A}_{p_2}(\widetilde{\xi}) \;\; \cdots \;\; \mathscr{A}_{p_N}(\widetilde{\xi})), \quad Y = (\mathscr{D}_{p_1}(\widetilde{\xi}) \;\; \mathscr{D}_{p_2}(\widetilde{\xi}) \;\; \cdots \;\; \mathscr{D}_{p_N}(\widetilde{\xi})), \\ &\Omega = ((1 - \delta_{ij})\mathscr{M}_{p_i p_j}(\widetilde{\xi}))_{1 \leq i,j \leq N}.\end{aligned}$$

Then the space of Z-paths in $\widetilde{M}$ is rewritten by means of the iterated integrals as follows.

$$\mathscr{M}_2^{Z}(\widetilde{\xi}) = \mathscr{M}_2(\widetilde{\xi}) + \int_{\mathbb{R}} X\, {}^tY + \int_{\mathbb{R}} X\Omega\, {}^tY + \int_{\mathbb{R}} X\Omega\Omega\, {}^tY + \cdots.$$

We would like to define the 'closure' of this space.

Lemma 2.6. *For a generic $\widetilde{\xi}$, the space $\overline{\int_{\mathbb{R}}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n} {}^t\overline{Y}$ is the disjoint union of finitely many manifolds with corners, and the closure of its codimension 1 stratum is given by the following formula.*

$$\begin{aligned}&\overline{\int_{\mathbb{R}}} (\partial \overline{X}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n} {}^t\overline{Y} + \sum_{i=1}^{n} \overline{\int_{\mathbb{R}}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{i-1} (\partial \overline{\Omega}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-i} {}^t\overline{Y} + \overline{\int_{\mathbb{R}}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n} (\partial {}^t\overline{Y}) \\ &+ \overline{\int_{\mathbb{R}}} (\overline{X} \times_{\mathbb{R}} \overline{\Omega}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-1} {}^t\overline{Y} + \sum_{i=1}^{n-1} \overline{\int_{\mathbb{R}}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{i-1} (\overline{\Omega} \times_{\mathbb{R}} \overline{\Omega}) \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-i-1} {}^t\overline{Y} + \overline{\int_{\mathbb{R}}} \overline{X} \underbrace{\overline{\Omega} \cdots \overline{\Omega}}_{n-1} (\overline{\Omega} \times_{\mathbb{R}} {}^t\overline{Y}).\end{aligned}$$

For $n \geq 0$, let S_n (resp. T_n) denote the first line (resp. the second line) of the formula in Lemma 2.6.

Lemma 2.7. *There is a natural stratification preserving diffeomorphisms*

$$\begin{aligned}&\partial \overline{X} \cong \overline{X} \times_{\mathbb{R}} \overline{\Omega}, \quad \partial\, {}^t\overline{Y} \cong \overline{\Omega} \times_{\mathbb{R}} {}^t\overline{Y}, \\ &\partial \overline{\Omega} \cong \overline{\Omega} \times_{\mathbb{R}} \overline{\Omega}, \quad \partial \overline{\mathscr{M}}_2(\widetilde{\xi}) \cong \Delta_{\widetilde{M}} + \overline{X} \times_{\mathbb{R}} {}^t\overline{Y}.\end{aligned}$$

These induce, for $n \geq 0$, a stratification preserving diffeomorphism

$$S_n \cong T_{n+1}.$$

Let $S_{-1} \subset \partial \overline{\mathscr{M}}_2(\widetilde{\xi})$ be the face that corresponds to $\overline{X} \times_{\mathbb{R}} {}^t\overline{Y}$ by the diffeomorphism of Lemma 2.7.

Definition 2.8.

$$\overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi}) = \left[\overline{\mathscr{M}}_2(\tilde{\xi}) + \overline{\int_{\mathbb{R}} X\,{}^tY} + \overline{\int_{\mathbb{R}} X\Omega\,{}^tY} + \overline{\int_{\mathbb{R}} X\Omega\Omega\,{}^tY} + \cdots\right]\Big/\!\sim$$

Here, for each $n \geq 0$, we identify S_{n-1} with T_n by the diffeomorphism of Lemma 2.7. $\mathbb{Z}$ acts on $\overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})$ by $(x_1, x_2, \ldots, x_n) \mapsto (tx_1, tx_2, \ldots, tx_n)$. We put

$$\overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})_{\mathbb{Z}} = \overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})/\mathbb{Z}.$$

Outline of the proof of Theorem 2.2. By fixing orientations on the manifold pieces in the stratified space $\overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})_{\mathbb{Z}}$, the map $\bar{b} : \overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})_{\mathbb{Z}} \to \widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$ represents a $\mathbb{Q}(t)$-chain of $\widetilde{M} \times_{\mathbb{Z}} \widetilde{M}$. (The proof that the coefficients are rational functions is an analogue of the proof of the rationality of Novikov complexes by Pajitnov ([14, 15]).) By Lemmas 2.6, 2.7 and by checking the orientations on the gluing parts, it turns out that the boundary of $\overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})_{\mathbb{Z}}$ concentrates on the lift $\widetilde{\Delta}_M$ of the diagonal Δ_M. Hence the boundary of $B\ell_{\bar{b}^{-1}(\widetilde{\Delta}_M)}(\overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})_{\mathbb{Z}})$ consists of Z-paths with endpoints agree (in $\widetilde{M}$) and of closed Z-paths (in M). One sees that in the homology class of the boundary of $B\ell_{\bar{b}^{-1}(\widetilde{\Delta}_M)}(\overline{\mathscr{M}}^{\mathbb{Z}}_2(\tilde{\xi})_{\mathbb{Z}})$, the part for closed Z-paths corresponds to the logarithmic derivative of the Lefschetz zeta function.

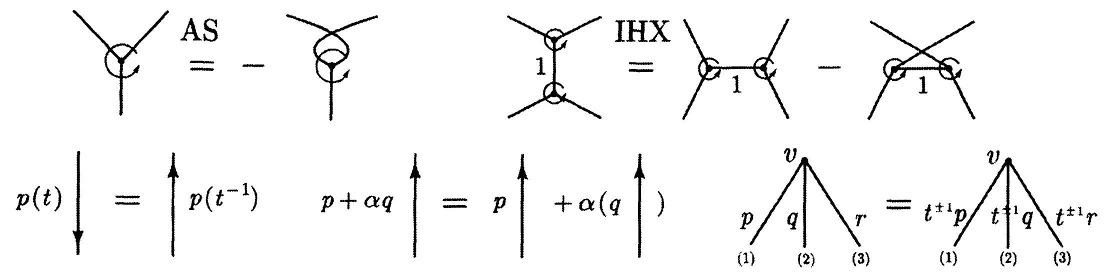
3. Perturbation theory for $\widehat{\Lambda}$-coefficients

Put $\Lambda = \mathbb{Q}[t, t^{-1}]$, $\widehat{\Lambda} = \mathbb{Q}(t)$. Since a recipe for the perturbation theory for Lie algebra local coefficient systems is given by Axelrod–Singer, Kontsevich ([1, 5]), it is expected that one can obtain a perturbative invariant with $\widehat{\Lambda}$-coefficients if there is an appropriate propagator with $\widehat{\Lambda}$-coefficients. The $\mathbb{Q}(t)$-chain given by the moduli space of Z-paths can be considered as an appropriate equivariant propagator and we use it.

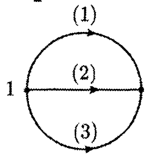
3.1. **$\widehat{\Lambda}$-colored graph.** We call a finite connected graph with edges oriented a graph. A *vertex-orientation* of a graph is an assignment of cyclic order of edges incident to each vertex. For a vertex-oriented graph Γ, a *$\widehat{\Lambda}$-coloring* of Γ is a mapping ϕ : Edges$(\Gamma) \to \widehat{\Lambda}$.

Definition 3.1 (Garoufalidis-Rozansky [4]).

$$\mathscr{A}_n(\widehat{\Lambda}) = \frac{\mathrm{span}_{\mathbb{Q}}\{\Gamma : \text{3-valent, } 2n \text{ vertices, } \widehat{\Lambda}\text{-colored vertex-oriented graphs}\}}{\text{AS, IHX, Orientation reversal, Linearity, Holonomy}}$$



Orientation Reversal | Linearity | Holonomy

3.2. Equivariant configuration space and equivariant intersection. Let $\kappa : M \to S^1$ be a fiber bundle and let $\Theta =$ 

. We define M^Θ as

$$\begin{aligned} M^\Theta := \{(x_1, x_2; \gamma_1, \gamma_2, \gamma_3) \mid & \, x_1, x_2 \in M, \\ & \gamma_i : \text{homotopy class of } c_i : [0,1] \to S^1 \text{ such that} \\ & c_i(0) = \kappa(x_1),\ c_i(1) = \kappa(x_2)\}. \end{aligned}$$

When $H_1(M) = \mathbb{Z}$, the homotopy class γ_i of c_i is the same thing as the relative bordism class of the lift $\widetilde{c}_i : [0,1] \to M$ of c_i. The equivariant configuration space $\overline{\mathrm{Conf}}_\Theta(M)$ for Θ is defined by

$$\overline{\mathrm{Conf}}_\Theta(M) := B\ell(M^\Theta, \text{preimage of } \Delta_M),$$

where $B\ell(X, A)$ is the blow-up of a (real) manifold X along a submanifold A. The projection $\overline{\mathrm{Conf}}_\Theta(M) \to \overline{\mathrm{Conf}}_2(M)$ is a $\mathbb{Z}^3$-covering and we have $\pi_0(\overline{\mathrm{Conf}}_\Theta(M)) \approx H^1(\Theta; \mathbb{Z}) = [\Theta, S^1]$.

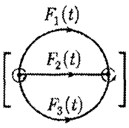
By extending the intersections of chains by $\widehat{\Lambda}$-linearity, we define the multilinear form

$$Q_1 \otimes Q_2 \otimes Q_3 \mapsto \langle Q_1, Q_2, Q_3 \rangle_\Theta \in C_0(\overline{\mathrm{Conf}}_\Theta(M); \mathbb{Q}) \otimes_{\Lambda^{\otimes 3}} \widehat{\Lambda}^{\otimes 3}$$

($\Lambda^{\otimes 3} = \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$, $\widehat{\Lambda}^{\otimes 3} = \mathbb{Q}(t_1) \otimes_\mathbb{Q} \mathbb{Q}(t_2) \otimes_\mathbb{Q} \mathbb{Q}(t_3)$) for 'generic' 4-dimensional $\widehat{\Lambda}$-chains Q_1, Q_2, Q_3 in $\overline{\mathrm{Conf}}_{K_2}(M)$.

We define the trace $\mathrm{Tr}_\Theta : \widehat{\Lambda}^{\otimes 3} \to \mathscr{A}_1(\widehat{\Lambda})$ for $\widehat{\Lambda}$-colored graphs by

$$\mathrm{Tr}_\Theta(F_1(t_1) \otimes F_2(t_2) \otimes F_3(t_3)) = \left[\ \right].$$



This induces the following map.

$$\mathrm{Tr}_\Theta : H_0(\overline{\mathrm{Conf}}_\Theta(M); \mathbb{Q}) \otimes_{\Lambda^{\otimes 3}} \widehat{\Lambda}^{\otimes 3} \to H_0(\overline{\mathrm{Conf}}_2(M); \mathbb{Q}) \otimes_\mathbb{Q} \mathscr{A}_1(\widehat{\Lambda}) = \mathscr{A}_1(\widehat{\Lambda}).$$

Similarly, for a 3-valent graph Γ with $2n$ vertices ($3n$ edges), we obtain

$$\begin{aligned} & Q_1 \otimes Q_2 \otimes \cdots \otimes Q_{3n} \quad (Q_i\text{: codimension 2 } \widehat{\Lambda}\text{-chain in } \overline{\mathrm{Conf}}_{K_2}(M) \text{ or in } M) \\ \mapsto \ & \langle Q_1, Q_2, \ldots, Q_{3n} \rangle_\Gamma \in C_0(\overline{\mathrm{Conf}}_\Gamma(M); \mathbb{Q}) \otimes_{\Lambda^{\otimes 3n}} \widehat{\Lambda}^{\otimes 3n} \\ \mapsto \ & \mathrm{Tr}_\Gamma \langle Q_1, Q_2, \ldots, Q_{3n} \rangle_\Gamma \in H_0(\overline{\mathrm{Conf}}_{2n}(M); \mathbb{Q}) \otimes_\mathbb{Q} \mathscr{A}_n(\widehat{\Lambda}) = \mathscr{A}_n(\widehat{\Lambda}). \end{aligned}$$

Definition 3.2. Let $\kappa_i : M \to S^1$ be an oriented surface bundle such that $\kappa_i \simeq \kappa$. Let $f_i : M \to \mathbb{R}$ be an oriented [4]fiberwise Morse function (w.r.t. κ_i), let ξ_i be the gradient for κ_i along the fibers ($i = 1, 2, \ldots, 3n$). We define Z_n as follows.

$$Z_n := \sum_\Gamma \mathrm{Tr}_\Gamma \langle Q(\widetilde{\xi}_1), Q(\widetilde{\xi}_2), \ldots, Q(\widetilde{\xi}_{3n}) \rangle_\Gamma \in \mathscr{A}_n(\widehat{\Lambda}).$$

The sum is over all (labeled) 3-valent graphs with $2n$ vertices.

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[4]Namely, $\mathscr{D}_p(\xi_i)$ is oriented for each critical locus p.

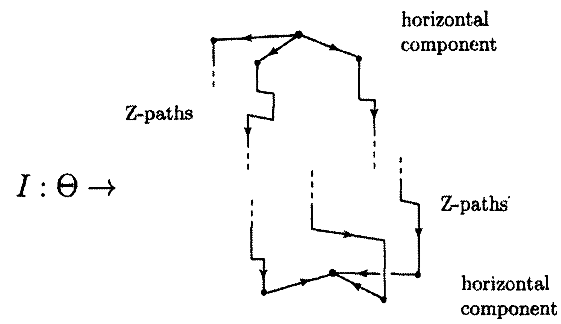

FIGURE 2. Z-graph

Theorem 3.3 ([17]).

$$\widehat{Z}_n = Z_n - Z_n^{\text{anomaly}}(\vec{\rho}_W) \in \mathscr{A}_n(\widehat{\Lambda})$$

is an invariant of $(M, \mathfrak{s}, [\kappa], [f])$. *($Z_n^{\text{anomaly}}(\vec{\rho}_W)$ is a term obtained by counting affine graphs in a rank 3 vector bundle over some compact 4-manifold W such that $\partial W = M$. Here,*

1. *$\mathfrak{s}$ is a spin structure on M.*
2. *$[\kappa] \in H^1(M)$ is the homotopy class of κ.*
3. *$[f]$ is the 'concordance class' of an oriented fiberwise Morse function $f : M \to \mathbb{R}$. (Oriented fiberwise Morse functions f_0 and f_1 are* concordant *if there is a generic homotopy $F : M \times [0,1] \to \mathbb{R}$ between f_0 and f_1 such that for each birth-death locus, its projection to $S^1 \times [0,1]$ is a simple closed curve and is not nullhomotopic.)*

To get an invariant of $(M, [\kappa])$, one must show that $\widehat{Z}_n$ does not depend on the choice of concordance class of oriented fiberwise Morse functions, namely, that $\widehat{Z}_n$ is invariant under a generic homotopy of oriented fiberwise Morse functions. However, as suggested by the definition of concordance, the topology of the moduli space of Z-paths may change if there is a birth-death locus whose projection on $S^1 \times [0,1]$ is nullhomotopic. We guess that the restriction of the homotopy to concordances might be too strong.

Though, this is sufficient to study finite type isotopy invariants of knots in a 3-manifold ([18]). Thanks to the definition of $\widehat{Z}_n$ by Z-paths, Theorem 3.3 can be proved by a standard argument (constructing a cobordism between moduli spaces on the endpoints) without difficulty.

3.3. **Z-graph.** Now we explain that Z_n can be defined by counting certain graphs. In the following, we only consider the graph $\Gamma = \Theta$ for simplicity.

Definition 3.4. Put $\Sigma = \kappa^{-1}(0)$. For $(a_1, a_2, a_3) \in \mathbb{Z}^3$, we define

$$\mathscr{M}^{\mathrm{Z}}_{\Theta(a_1,a_2,a_3)}(\Sigma; \xi_1, \xi_2, \xi_3)$$

as the set of maps $I : \Theta \to M$ such that

1. i-th edge is a Z-path for ξ_i.
2. $\#(i\text{-th edge of } I) \cap \Sigma = a_i$ (count with signs)

We call such a map $I : \Theta \to M$ a *Z-graph*.

This definition is an analogue of the flow-graphs considered in Fukaya's Morse homotopy theory [3]. The following lemma can be proved by a transversality argument as in [3].

Lemma 3.5. *For a generic κ_i, ξ_i $(i = 1,2,3)$, the moduli space $\mathscr{M}^{\mathbb{Z}}_{\Theta(a_1,a_2,a_3)}(\Sigma;\xi_1,\xi_2,\xi_3)$ is a compact oriented 0-dimensional manifold ($\forall(a_1,a_2,a_3) \in \mathbb{Z}^3$)*

Proposition 3.6. *Choose κ_i, ξ_i $(i = 1,2,3)$ generically as in the Lemma. Put*

$$F_\Theta := \sum_{(a_1,a_2,a_3)\in\mathbb{Z}^3} \#\mathscr{M}^{\mathbb{Z}}_{\Theta(a_1,a_2,a_3)}(\Sigma;\xi_1,\xi_2,\xi_3)\, t_1^{a_1}t_2^{a_2}t_3^{a_3}.$$

Then there exist a polynomial $P(t_1,t_2,t_3) \in \mathbb{Q}[t_1^{\pm1},t_2^{\pm1},t_3^{\pm1}]$ and a product $C(t) \in \Lambda$ of cyclotomic polynomials such that

$$\begin{aligned} F_\Theta &= \frac{P(t_1,t_2,t_3)}{C(t_1)C(t_2)C(t_3)\Delta(t_1)\Delta(t_2)\Delta(t_3)} \\ &= \langle Q(\widetilde{\xi}_1), Q(\widetilde{\xi}_2), Q(\widetilde{\xi}_3)\rangle_\Theta \end{aligned}$$

holds. ($\Delta(t)$ is the Alexander polynomial of M)

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