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The Orbit Decompositions of Multiple Flag Manifolds of $SL(3,\mathbb{C})$ under the Diagonal Action

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Abstract

It is shown by Kobayashi-Oshima([3]) that the multiplicities of irreducible representations occurring in the space of functions on a homogeneous space $X=G/H$ are finite if and only if H has an open orbit on the flag manifold G/P_{G} where P_{G} is a minimal parabolic subgroup P_{G} of G. A homogeneous space G_{H} satisfying such conditions is named a real spherical variety by Kobayashi. Furthermore, Brion, Vinberg, Kimelfeld, Bien, Matsūki, and Kobayashi proved that the existence of open H-orbits on G/P_{G} is equivalent to the finiteness of H-orbits on G/P_{G} until 90's.

On the other hand, it may happen that H has some open orbits and infinitely many orbits on G/P simultaneously if P is a (non minimal) general parabolic subgroup. To research on this phenomenon, we want to observe the simplest case of these. In this note, we give a description of infinitely many orbits on 4-tuple flag varieties of special linear groups under the diagonal action. The key idea is considering a new decomposition defined combinatorially which is slightly rougher than orbit decomposition. We also give sufficient and necessary conditions of the closure relations among the orbits.

0 Introduction

There have been many researches on the relations between orbit decompositions on flag manifolds and representation theory. For example for a real reductive algebraic group G, its minimal parabolic subgroup P_{G}(resp. a Borel subgroup B_{G} of the complexification G_{c} of G), and its algebraically defined closed subgroup H, Kobayashi-Oshima[3] proved the following theorem on the relationship between the H-orbits (resp. H_{c}-orbits) on the real flag variety G/P_{G} (resp. complex flag variety G_{c}/B_{G}) and the analysis on the homogeneous space G/H:

Theorem 0.1. *For an irreducible admissible representation π of G and a finite dimensional irreducible representation τ of H, set an integer $c_{\mathfrak{g},K}(\pi,\mathrm{Ind}_{H}^{G}\tau)$ to be the multiplicity of the underlying $(\mathfrak{g},K)$-module π_{K} in the space of section on the vector bundle on G/H associated with τ. Then the followings are equivalent;*

(1) H (resp. H_{c}) has an open orbit on G/P_{G} (resp. G_{c}/B_{G});

(2) for all irreducible admissible representations π of G and finite dimensional irreducible representations τ of H, $c_{\mathfrak{g},K}(\pi,\mathrm{Ind}_{H}^{G}\tau)<\infty$. (resp. bounded proportionally to $\dim\tau$).

A homogeneous space G/H satisfying the first condition in the theorem is called a real spherical variety (resp. spherical variety). Furthermore, Brion, Vinberg, Kimelfeld, Bien, Matsuki, and Kobayashi proved that G/H is real spherical (resp. spherical) if and only if the number of H-orbits (resp H_c-orbits) on the flag variety G/P_G (resp. G_c/B_G) is finite [1]. Remark that for a non-minimal parabolic subgroup P of G, it may occur that H has infinitely many orbits and some open orbits on G/P simultaneously.

In this note, we focus on the orbit decomposition of a 4-tuple flag manifold of $G = SL(3,\mathbb{C})$ under the diagonal action of G and observe behaviors of orbits, since this is the simplest case among the cases where H has some open orbits and uncountably infinitely many orbits simultaneously on a flag manifold G/P in some sense. More precisely, we will introduce a notion of new decomposition of multiple flag manifold which is slightly rougher than the orbit decomposition, and deal with the infinitely many orbits which lie in a piece with respect to the new decomposition together as a clump of infinitely many similar orbits.

1 Some basic properties

Let G_n be the special linear group $SL(n,\mathbb{C})$, and P_n be its parabolic (not minimal in general) subgroup defined as

$$P_n = \left\{ \begin{pmatrix} * & \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix} \right\},$$

then the flag variety G_n/P_n is isomorphic to $\mathbb{P}^{n-1}\mathbb{C}$ with the standard action of $G_n = SL(n,\mathbb{C})$ on $\mathbb{P}^{n-1}\mathbb{C} = (\mathbb{C}^n \setminus 0)/\mathbb{C}^\times$. Hence the m-tuple flag $\underbrace{G_n/P_n \times \cdots \times G_n/P_n}_{m} \simeq G_n^m/P_n^m$ is isomorphic to $(\mathbb{P}^n\mathbb{C})^m =: \tilde{X}_{n,m}$. From now on, we deal with $\tilde{X}_{n,m} = (\mathbb{P}^{n-1}\mathbb{C})^m$ under $SL(n,\mathbb{C}) = G_n \simeq \mathrm{diag}(G_n) \subset G_n^m = SL(n,\mathbb{C})^m$-action.

First of all, we see the relations among double coset spaces $\mathrm{diag}(G_n) \setminus G_n^m/P_n^m \simeq G_n \setminus (\mathbb{P}^{n-1}\mathbb{C})^m = G_n \setminus \tilde{X}_{n,m} =: X_{n,m}$ under the changing of (n,m). We can define the following inclusion map

$$\tilde{\iota}_{n,m} : \tilde{X}_{n,m} = (\mathbb{P}^{n-1}\mathbb{C})^m \to (\mathbb{P}^n\mathbb{C})^m = \tilde{X}_{n+1,m}$$

just by taking the m times direct product of the map defined as

$$\begin{array}{ccc} \mathbb{P}^{n-1}\mathbb{C} & \to & \mathbb{P}^n\mathbb{C} \\ \cup & & \cup \\ [x] & \mapsto & \left[\begin{pmatrix} x \\ 0 \end{pmatrix}\right] \end{array}$$

where $x \in \mathbb{C}^n \setminus \{0\}$. Clearly, this inclusion map intertwines the $G_n = SL(n,\mathbb{C})$-action and $G_{n+1} = SL(n+1,\mathbb{C})$-action under the identification of G_n with the subgroup $\left\{ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \mid g \in G_n \right\}$ of G_{n+1}, hence we can introduce a map

$$\iota_{n,m} : X_{n,m} = SL(n,\mathbb{C}) \setminus (\mathbb{P}^{n-1}\mathbb{C})^m \to SL(n+1,\mathbb{C}) \setminus (\mathbb{P}^n\mathbb{C})^m = X_{n+1,m}.$$

We can see by an easy computation that $\iota_{n,m}$ is still an inclusion map. Actually,we can arbitrarily choose a coordinate for attaching a new zero to define $\tilde{\iota}_{n,m}$ (in the definition above, we attach a new zero to the last coordinate of $\mathbb{C}^{n+1}$). However $\iota_{n,m}$ does not depend on this choice by thinking of permutation matrices in $G_{n+1} = SL(n+1, \mathbb{C})$ (if the signature of the permutation is -1, then we have just to switch the 1 in the first column to -1). Hence $\iota_{n,m}$ can be said to be the canonical inclusion map from $X_{n,m}$ to $X_{n+1,m}$.

Since we have seen that $\iota_{n,m}$ is an inclusion map, we now have to observe the image of $\iota_{n,m}$ to understand $\iota_{n,m}$ completely. We now introduce some subsets of $\tilde{X}_{n,m} = (\mathbb{P}^{n-1}\mathbb{C})^m$ defined by

$$\tilde{X}_{n,m}(r) = \left\{([x_j])_{j=1}^m \in \tilde{X}_{n,m} \mid x_j \in \mathbb{C}^n \setminus 0,\ \mathrm{dimSpan}_{\mathbb{C}}\{x_j \mid 1 \leq j \leq m\} = r\right\}.$$

Clearly we have a decomposition of $\tilde{X}_{n,m}$ as:

$$\tilde{X}_{n,m} = \coprod_{r=1}^{\min\{n,m\}} \tilde{X}_{n,m}(r),$$

by definition. It is also obvious that $\tilde{X}_{n,m}(r)$ are G_n-stable subsets, since the diagonal action of $G_n = GL(n, \mathbb{C})$ on $(\mathbb{C}^n)^m$ preserves the dimensions of subspaces spanned by some vectors of m-entries. Hence, we can define subsets $X_{n,m}(r) = G_n \setminus \tilde{X}_{n,m}(r)$ of $X_{n,m} = G_n \setminus \tilde{X}_{n,m}$ and obtain a stratification of $X_{n,m}$ as:

Proposition 1.1.

$$X_{n,m} = \coprod_{r=1}^{\min\{n,m\}} X_{n,m}(r).$$

Furthemore, we can see the following property of the image of $\iota_{n,m}$ and $X_{n,m}(r)$'s:

Lemma 1.2. *For $r \leq n$, $\iota_{n,m}(X_{n,m}(r)) = X_{n+1,m}(r)$.*

Remark that $X_{n,m}(r) = \emptyset$ if and only if $r \geq n+1$ or $r \geq m+1$.
This lemma immediatly implies the following stratification of $X_{n,m}(r)$:

Theorem 1.3.

$$X_{n,m} = \coprod_{r=1}^{\min\{n,m\}} \iota_{n-1,m} \circ \cdots \circ \iota_{r,m}(X_{r,m}(r)).$$

Combining these results, we get:

Corollary 1.4. *$\iota_{n,m}$ is a bijection if $n \geq m$.*

Hence all $\tilde{X}_{n,m}$ have the same orbit decomposition as that of $\tilde{X}_{m,m}$ if $n \geq m$.

Furthermore, $X_{n,m}(r)$ has a quite simple structure for some certain (n, m), and r. More precisely, the structures of $X_{m,m}(m)$, $X_{m-1,m}(m-1)$, and $X_{n,m}(1)$ are described as follows:

Proposition 1.5.

$$X_{m,m}(m) = \left\{G_m \cdot \left([e_j]\right)_{j=1}^{m}\right\}$$
$$X_{m-1,m}(m-1) = \coprod_{\substack{I \subset \{1,2,\cdots,m\} \\ \text{and } \#I \geq 2}} \left\{G_{m-1} \cdot \left([f_j^I]\right)_{j=1}^{m}\right\}$$
$$X_{n,m}(1) = \left\{G_n \cdot \left([e_1]\right)_{j=1}^{m}\right\}$$

where $\{e_j\}_{j=1}^{k}$ *denotes the standard basis of* $\mathbb{C}^k$ *for arbitarary* k *and* $\{f_j\}_{j=1}^{m} \subset \mathbb{C}^{m-1}$ *is defined as*

$$f_j^I = \begin{cases} e_j & 1 \leq j < \max I \\ \sum_{i \in I \setminus \{\max I\}} e_i & j = \max I \\ e_{j-1} & \max I < j \leq m \end{cases}.$$

Remark that for $\{f_j^I\}_{j=1}^{m} \subset \mathbb{C}^{m-1}$,

$$\dim \mathrm{Span}\{f_j^I \mid j \in J\} = \begin{cases} \#J & I \not\subset J \\ \#J - 1 & I \subset J. \end{cases}$$

Conversely, each orbit mentioned in the proposition above is explicitly expressed as follows:

Proposition 1.6.

$$G_m \cdot \left([e_j]\right)_{j=1}^{m} = \left\{\left([x_j]\right)_{j=1}^{m} \mid \dim \mathrm{Span}\{x_j \mid j \in J\} = \#J\right\}$$
$$G_{m-1} \cdot \left([f_j^I]\right)_{j=1}^{m} = \left\{\left([x_j]\right)_{j=1}^{m} \mid \dim \mathrm{Span}\{x_j \mid j \in J\} = \begin{cases} \#J & I \not\subset J \\ \#J - 1 & I \subset J \end{cases}\right\}$$
$$G_m \cdot \left([e_1]\right)_{j=1}^{m} = \left\{\left([x_j]\right)_{j=1}^{m} \mid \dim \mathrm{Span}\{x_j \mid j \in J\} = 1\right\}$$

This dimension condition plays a role in the following sections.
By these propositions and the stratification

$$X_{n,m} = \coprod_{r=1}^{\min\{n,m\}} \iota_{n-1,m} \circ \cdots \circ \iota_{r,m}\left(X_{r,m}(r)\right)$$

proved in Proposition 1.3, it can be said that non-elementary parts of the G_n-orbit decomposition of $\tilde{X}_{n,m}$ live in

$$\coprod_{r=2}^{\min\{n,m-2\}} \iota_{n-1,m} \circ \cdots \circ \iota_{r,m}\left(X_{r,m}(r)\right).$$

Since we focus on the case where $(n,m) = (3,4)$, the non-elementary part of the structure of $X_{3,4}$ lives only in

$$\iota_{3,4}(2) = \iota_{2,4}\left(X_{2,4}(2)\right) \simeq X_{2,4}(2).$$

2 Existence of open orbits and finiteness of orbits

It is well known that $\tilde{X}_{n,m}$ has only finitely many orbits if and only if $m \leq 3$, since this corresponds to the quivers of Dynkin type (classification of multiple flag varieties of general linear groups which is of finite type is given in [4]).

Also, we can easily see that

Theorem 2.1. *if $n \geq 2$, then $\tilde{X}_{n,m}$ has an open orbit if and only if $n \geq m-1$.*

Proof. If $n \geq m$, then $X_{n,m}$ inculudes $\iota_{n-1m,} \circ \cdots \circ \iota_{m,m}(X_{m,m}(m)) \simeq X_{m,m}(m)$. It is easily computed that

$$\iota_{n-1m,} \circ \cdots \circ \iota_{m,m}\left(G_m \cdot ([e_1], \cdots, [e_m])\right) = G_n \cdot ([e_1], \cdots, [e_m])$$

is an open orbit by computing the dimension. Similarly for $n = m-1$, $X_{m-1,m}(m)$ is included in $X_{m-1,n}$. Similar computation of dimension of

$$G_{m-1} \cdot \left(\left[f_1^{\{1,\cdots,m\}}\right], \cdots, \left[f_m^{\{1,\cdots,m\}}\right]\right) = G_{m-1} \cdot ([e_1], \cdots, [e_m], [e_1 + \cdots + e_m]) \quad .$$

implies that it is an open orbit. □

3 Main results

3.1 Explicit orbit decomposition

The first main result of this note is the description of orbit decomposition of $\tilde{X}_{3,4} = (\mathbb{P}^2\mathbb{C})^4$ under the $G_3 = SL((,\mathbb{C})3,\mathbb{C})$ action, which is equivalent to the problem giving explicit representatives of the double coset $\mathrm{diag}(SL(3,\mathbb{C})) \setminus SL(3,\mathbb{C})^4/P^4$.

Theorem 3.1. *G_3-orbit decomposition of $\tilde{X}_{3,4}$ is described as follows:*

$$\begin{aligned}
\tilde{X}_{3,4} =& \mathcal{O}(2) \\
& \amalg \mathcal{O}(4;1) \amalg \mathcal{O}(4;2) \amalg \mathcal{O}(4;3) \amalg \mathcal{O}(4;4) \\
& \amalg \mathcal{O}(4;1,2) \amalg \mathcal{O}(4;1,3) \amalg \mathcal{O}(4;1,4) \\
& \amalg \mathcal{O}(5;1,2) \amalg \mathcal{O}(5;1,3) \amalg \mathcal{O}(5;1,4) \amalg \mathcal{O}(5;2,3) \amalg \mathcal{O}(5;2,4) \amalg \mathcal{O}(5;3,4) \\
& \amalg \left(\coprod_{p \in (\mathbb{P}^1\mathbb{C})'} \mathcal{O}(5;p) \right) \\
& \amalg \mathcal{O}(6;1,2) \amalg \mathcal{O}(6;1,3) \amalg \mathcal{O}(6;1,4) \amalg \mathcal{O}(6;2,3) \amalg \mathcal{O}(6;2,4) \amalg \mathcal{O}(6;3,4) \\
& \amalg \mathcal{O}(7;1) \amalg \mathcal{O}(7;2) \amalg \mathcal{O}(7;3) \amalg \mathcal{O}(7;4) \\
& \amalg \mathcal{O}(8)
\end{aligned}$$

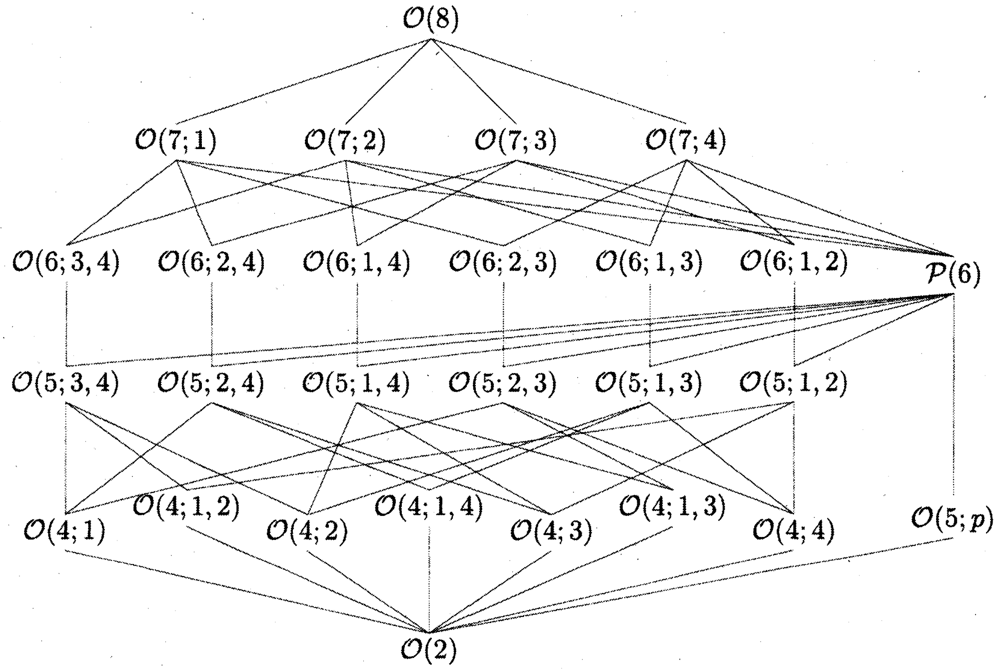
The definition of each orbit is given in the next section. For each orbit, the number before the semicolon means the dimension of the orbit, and following numbers are combinatorial indices. The parameter p in the 5th ray runs on an open dense subset $(\mathbb{P}^1\mathbb{C})'$ of $\mathbb{P}^1\mathbb{C}$.

The next second main result is the closure relations among orbits on $\tilde{X}_{3,4}$. We introduce a subset $\mathcal{P}(6)$ of $\tilde{X}_{3,4}$ as

$$\mathcal{P}(6) := \coprod_{p\in(\mathbb{P}^1\mathbb{C})'} \mathcal{O}(5;p)$$

; the whole union of orbits contained in a series which is parametrised by $(\mathbb{P}^1\mathbb{C})'$. Considering all orbits and $\mathcal{P}(6)$, we obtain the following result on closure relations:

Theorem 3.2. *The closure relations among all orbits and $\mathcal{P}(6)$ are as follows:*



4 Preliminaries

For the description of $SL(3,\mathbb{C})$-orbits on $\tilde{X}_{3,4}$, we introduce a new rough decomposition of $\tilde{X}_{3,4}$ into some finite G_3-stable subsets. A subset $\mathcal{P}(\varphi)$ of $\tilde{X}_{3,4} = (\mathbb{P}^2\mathbb{C})^4$ is defined as follows:

Definition 4.1. Let φ be a map from $2^{\{1,2,3,4\}}$ to $\mathbb{N}$, then we define a subset $\mathcal{P}(\varphi)$ of $\tilde{X}_{3,4}$ by

$$\mathcal{P}(\varphi) = \left\{ \begin{array}{l} ([x_1],[x_2],[x_3],[x_4]) \in \tilde{X}_{3,4} \mid \\ \quad x_1,x_2,x_3,x_4 \in \mathbb{C}^3, \\ \quad \dim\mathrm{Span}_{\mathbb{C}}\{x_i \mid i \in I\} = \varphi(I) \ ({}^\forall I \subset \{1,2,3,4\}) \end{array} \right\}.$$

In this definition, we regard $\mathbb{P}^2\mathbb{C}$ as an orbit space $(\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^\times$, and $[x]$ denotes the $\mathbb{C}^\times$-orbit through $x \in \mathbb{C}^3 \setminus \{0\}$.

The definition is well-defined because the dimensions of linear subspaces spanned by some vectors are NOT influenced by scalar multiplications: i.e. the action of $\mathbb{C}^\times$. We also see that $\mathcal{P}(\varphi)$ are $G_3 = SL(3,\mathbb{C})$-stable, since the projection $(\mathbb{C}^3 \setminus \{0\})^4 \to (\mathbb{P}^2\mathbb{C})^4 = \tilde{X}_{3,4}$ intertwines the linear diagonal action of $SL(3,\mathbb{C})$ on $(\mathbb{C}^3\setminus\{0\})^4$ and the $SL(3,\mathbb{C})$-diagonal action on $\tilde{X}_{3,4}$ which we are dealing with.

Hence, the first main result: giving the explicit orbit decomposition of X under the G-action: is reduced to the probem to classify the maps $\varphi : 2^{\{1,2,3,4\}} \to \mathbb{N}$ whether each φ satisfies

- $\mathcal{P}(\varphi)$ is an empty set;
- $\mathcal{P}(\varphi)$ is a single G_3-orbit;
- $\mathcal{P}(\varphi)$ is decomposed into plural G_3-orbits.

To classify these φ's, we now introduce some specific φ's for simplicity of notations.

Definition 4.2.

(1) In the case where $\dim \mathrm{Span}\{x_1,x_2,x_3,x_4\} = 1$, i.e. $([x_1],[x_2],[x_3],[x_4]) \in \tilde{X}_{3,4}(1)$, the map $\varphi(2) : 2^{\{1,2,3,4\}} \to \mathbb{N}$ is defined as follows:

$$\varphi(2) : \begin{cases} \emptyset & \mapsto 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto 1 \\ \{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\} & \mapsto 1 \\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\} & \mapsto 1 \\ \{1,2,3,4\} & \mapsto 1. \end{cases}$$

(2) In the case where $\dim \mathrm{Span}\{x_1,x_2,x_3,x_4\} = 2$, i.e. $([x_1],[x_2],[x_3],[x_4]) \in \tilde{X}_{3,4}(2)$,

ii) for if $\#\{[x_1],[x_2],[x_3],[x_4]\} = 2$;
the map $\varphi(4;1) : 2^{\{1,2,3,4\}} \to \mathbb{N}$ is defined as follows:

$$\varphi(4;1) : \begin{cases} \emptyset & \mapsto 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto 1 \\ \{2,3\},\{2,4\},\{3,4\} & \mapsto 1 \\ \{1,2\},\{1,3\},\{1,4\} & \mapsto 2 \\ \{2,3,4\} & \mapsto 1 \\ \{1,2,3\},\{1,2,4\},\{1,3,4\} & \mapsto 2 \\ \{1,2,3,4\} & \mapsto 2. \end{cases}$$

Respectively, we can define $\varphi(4;2)$, $\varphi(4;3)$, $\varphi(4;4)$ by permutating $\{1,2,3,4\}$;
the map $\varphi(4;1,2) : 2^{\{1,2,3,4\}} \to \mathbb{N}$ is defined as follows:

$$\varphi(4;1,2) : \begin{cases} \emptyset & \mapsto 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto 1 \\ \{1,2\},\{3,4\} & \mapsto 1 \\ \{1,3\},\{1,4\},\{2,3\},\{2,4\} & \mapsto 2 \\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\} & \mapsto 2 \\ \{1,2,3,4\} & \mapsto 2. \end{cases}$$

Respectively, we can define $\varphi(4;1,3)$, $\varphi(4;1,4)$.

iii) for if $\#\{[x_1], [x_2], [x_3], [x_4]\} = 3$,
the map $\varphi(5;1,2)$ is defined as

$$\varphi(5;1,2) : \begin{cases} \emptyset & \mapsto & 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto & 1 \\ \{1,2\} & \mapsto & 1 \\ \{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\} & \mapsto & 2 \\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\} & \mapsto & 2 \\ \{1,2,3,4\} & \mapsto & 2. \end{cases}$$

Respectively, we can define $\varphi(5;1,3)$, $\varphi(5;1,4)$, $\varphi(5;2,3)$, $\varphi(5;2,4)$, $\varphi(5;3,4)$.

iv) for if $\#\{[x_1], [x_2], [x_3], [x_4]\} = 4$,
$\varphi(6)$ is defined as

$$\varphi(6) : \begin{cases} \emptyset & \mapsto & 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto & 1 \\ \{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\} & \mapsto & 2 \\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\} & \mapsto & 2 \\ \{1,2,3,4\} & \mapsto & 2. \end{cases}$$

(3) In the case where $\dim \mathrm{Span}\{x_1, x_2, x_3, x_4\} = 3$, i.e. $([x_1], [x_2], [x_3], [x_4]) \in \tilde{X}_{3,4}(3)$,

iii) for if $\#\{[x_1], [x_2], [x_3], [x_4]\} = 3$,

$$\varphi(6;1,2) : \begin{cases} \emptyset & \mapsto & 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto & 1 \\ \{1,2\} & \mapsto & 1 \\ \{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\} & \mapsto & 2 \\ \{1,2,3\},\{1,2,4\} & \mapsto & 2 \\ \{1,3,4\},\{2,3,4\} & \mapsto & 3 \\ \{1,2,3,4\} & \mapsto & 3. \end{cases}$$

Respectively, we can define $\varphi(6;1,3)$, $\varphi(6;1,4)$, $\varphi(6;2,3)$, $\varphi(6;2,4)$, $\varphi(6;3,4)$.

iv) for if $\#\{[x_1], [x_2], [x_3], [x_4]\} = 4$,

$$\varphi(7;1) : \begin{cases} \emptyset & \mapsto & 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto & 1 \\ \{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\} & \mapsto & 2 \\ \{2,3,4\} & \mapsto & 2 \\ \{1,2,3\},\{1,2,4\},\{1,3,4\} & \mapsto & 3 \\ \{1,2,3,4\} & \mapsto & 3. \end{cases}$$

Respectively, we can define $\varphi(7;2)$, $\varphi(7;3)$, $\varphi(7;4)$.
Finally, we can define the map $\varphi(8)$ as:

$$\varphi(8) : \begin{cases} \emptyset & \mapsto & 0 \\ \{1\},\{2\},\{3\},\{4\} & \mapsto & 1 \\ \{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\} & \mapsto & 2 \\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\} & \mapsto & 3 \\ \{1,2,3,4\} & \mapsto & 3. \end{cases}$$

Combining with these definitions we introduce the following notations:

Definition 4.3.

(1) for each $\varphi(\cdot)$ listed in Definition4.2, $\mathcal{O}(\cdot)$ denotes $\mathcal{P}(\varphi(\cdot))$ unless $\varphi(\cdot)$ is $\varphi(6)$.

(2) $\mathcal{P}(6)$ denotes $\mathcal{P}(\varphi(6))$.

(3) i)

$$(\mathbb{P}^1\mathbb{C})' := \mathbb{P}^1\mathbb{C} \setminus \left\{ \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \right\}$$

ii)

$$\mathcal{O}(5;p) := \left\{ ([x_1], [x_2], [x_3], [x_4]) \in X \;\middle|\; \begin{array}{c} [x_1] \neq [x_2] \\ x_3 = x_1 + x_2 \\ x_4 = p^{(1)}x_1 + p^{(2)}x_2 \end{array} \right\}.$$

where $p = \left[\begin{pmatrix} p^{(1)} \\ p^{(2)} \end{pmatrix}\right] \in (\mathbb{P}^1\mathbb{C})'$

With these definitions, we have introduced all notations which take a role in the main results. The following lemma holds immedietly by definition:

Lemma 4.4.

$$\mathcal{P}(6) = \coprod_{p \in (\mathbb{P}^1\mathbb{C})'} \mathcal{O}(5;p).$$

Now we only have to prove the following lemma to prove the first main result:

Lemma 4.5.

(1) For each φ which is NOT listed in the Definition4.2, $\mathcal{P}(\varphi)$ is an empty set.

(2) Each $\mathcal{O}(\cdot)$ which is listed in the Definition4.3(1) is a single G-orbit.

(3) $\mathcal{P}(6)$ is decomposed into G-orbits as follows:

$$\mathcal{P}(6) = \coprod_{p \in (\mathbb{P}^1\mathbb{C})'} \mathcal{O}(5;p).$$

The proof of this lemma will be the key point of the proof of the main result of this section.

5 Proof of the description of the orbit decomposition

To begin the proof of the description of the orbit decomposition introduced in the previous subsection, we see the following statements at first.

Definition 5.1. An l-division of $\{1,2,\cdots,m\}$ is a family $\{J_i\}_{i=1}^l$ of subsets of $\{1,2,\cdots,m\}$ which satisfies

$$\{1,2,\cdots,m\} = \coprod_{i=1}^{l} J_i$$
$$J_i \neq \emptyset \ \ (1 \leq {}^\forall i \leq l).$$

Furthermore, let $\mathcal{J} = \{J_i\}_{i=1}^l$ and $\mathcal{J}' = \{J_i'\}_{i=1}^l$ are two l-divisions of $\{1,2,\cdots,m\}$, then we say $\mathcal{J} = \mathcal{J}'$ if there exists an element σ of the symmetric group $\mathfrak{S}_l$ such that

$$J_i = J'_{\sigma(i)} \ \ (1 \leq {}^\forall i \leq l).$$

However from now on, we fix an order of J_i's for each $\mathcal{J} = \{J_i\}_{i=1}^l$.

Lemma 5.2. *For an l-division $\mathcal{J} = \{J_i\}_{i=1}^l$ of $\{1,2,\cdots,m\}$;*

(1) there exists a canonical bijection

$$f_{\mathcal{J}} : \left\{ \left([x_i]\right)_{i=1}^{l} \in \tilde{X}_{n,l}(r) \mid [x_i] \neq [x_h] \text{ if } i \neq h \right\}$$
$$\to \left\{ \left([x_j]\right)_{j=1}^{m} \in \tilde{X}_{n,m}(r) \mid \begin{array}{l} [x_j] = [x_k] \ ({}^\forall j,k \in J_i,\ 1 \leq {}^\forall i \leq l) \\ [x_j] \neq [x_k] \text{ if } j \in J_i, k \in J_h \text{ and } i \neq h \end{array} \right\}$$

defined by

$$\left([x_i]\right)_{i=1}^{l} \mapsto \left([x_i] \ \text{ if } j \in J_i\right)_{j=1}^{m};$$

(2) and for a map $\psi : 2^{\{1,2,\cdots,l\}} \to \mathbb{N}$, let $\varphi_\psi^{\mathcal{J}}$ be a map from $2^{\{1,2,\cdots,m\}}$ to $\mathbb{N}$ defined by

$$J \mapsto \psi(I_J) \ \text{ where } I_J = \{1 \leq i \leq l \mid J \cap J_i \neq \emptyset\},$$

then the bijection $f_{\mathcal{J}}$ satisfies

$$f_{\mathcal{J}}(\mathcal{P}_{n,l}(\psi))) = \mathcal{P}_{n,m}(\varphi_\psi^{\mathcal{J}}).$$

Remark that $\mathcal{P}_{n,m}(\varphi)$ for $\varphi : 2^{\{1,\cdots,m\}} \to \mathbb{N}$ is a G_n-stable subset of $\tilde{X}_{n,m}$ defined analogously to $\mathcal{P}(\varphi)$ in $\tilde{X}_{3,4}$:

$$\mathcal{P}_{n,m}(\varphi) = \left\{ \left([x_j]\right)_{j=1}^{m} \in \tilde{X}_{n,m} \mid \begin{array}{l} x_j \in \mathbb{C}^n \setminus \{0\}, \\ \dim\mathrm{Span}_{\mathbb{C}}\{x_j \mid j \in I\} = \varphi(I) \ \left({}^\forall I \in 2^{\{1,\cdots,m\}}\right) \end{array} \right\}.$$

Clearly $\mathcal{P}(\varphi)$ we defined in the previous subsection is equal to $\mathcal{P}_{3,4}(\varphi)$.

By the map $f_{\mathcal{J}}$ we have introduced in the previous lemma, we get the following proposition. Remark that the two sets mentioned in the first statement of the previous lemma are G_n-stable. Hence, we can think of the G_n-orbit decompositions of them.

Proposition 5.3. *Under the notation in the previous lemma,*

(1) *if the decomposition of* $\left\{ ([x_i])_{i=1}^{l} \in \tilde{X}_{n,l}(r) \mid [x_i] \neq [x_h] \text{ if } i \neq h \right\}$ *into nonempty* $\mathcal{P}_{n,l}(\psi)$*'s is given as*

$$\coprod_{\nu=1}^{N} \mathcal{P}_{n,l}(\psi_\nu),$$

then the decomposition of

$$\left\{ ([x_j])_{j=1}^{m} \in \tilde{X}_{n,m}(r) \;\middle|\; \begin{array}{l} [x_j] = [x_k] \ (\forall j, k \in J_i,\ 1 \leq \forall i \leq l) \\ [x_j] \neq [x_k] \text{ if } j \in J_i, k \in J_h \text{ and } i \neq h \end{array} \right\}$$

into nonempty $\mathcal{P}_{n,m}(\varphi)$*'s is given as*

$$\coprod_{\nu=1}^{N} \mathcal{P}_{n,m}(\varphi_{\psi_\nu}^{\mathcal{J}}).$$

(2) *If the decomposition of* $\left\{ ([x_i])_{i=1}^{l} \in \tilde{X}_{n,l}(r) \mid [x_i] \neq [x_h] \text{ if } i \neq h \right\}$ *into nonempty* $\mathcal{P}_{n,l}(\psi)$*'s is given as*

$$\coprod_{\nu=1}^{N} \mathcal{P}_{n,l}(\psi_\nu),$$

then the decomposition of $\left\{ ([x_j])_{j=1}^{m} \in \tilde{X}_{n,m}(r) \mid \#\{[x_1], \cdots, [x_m]\} = l \right\}$ *into nonempty* $\mathcal{P}_{n,m}(\varphi)$*'s is given as*

$$\coprod_{\substack{\mathcal{J}:l-\text{division} \\ \text{of } \{1,\cdots,m\}}} \coprod_{\nu=1}^{N} \mathcal{P}_{n,m}(\varphi_{\psi_\nu}^{\mathcal{J}}).$$

We saw some properties of $\mathcal{P}(\cdot)$ in changing of m in the previous proposition. Next, we observe some properties of $\mathcal{P}(\cdot)$ in changing of n in the following proposition.

Proposition 5.4.

(1) *For* $\varphi : 2^{\{1,\cdots,m\}} \to \mathbb{N}$ *and* $n' \geq n \geq r$,

$$G_{n'} \cdot \left(\tilde{\iota}_{n'-1,m} \circ \cdots \circ \tilde{\iota}_{n,m} \left(\mathcal{P}_{n,m}(\varphi) \right) \right) = \mathcal{P}_{n',m}(\varphi).$$

(2) *If the decomposition of* $\tilde{X}_{r,m}(r)$ *into nonempty* $\mathcal{P}_{r,m}(\varphi)$*'s is given as*

$$\coprod_{\nu=1}^{N} \mathcal{P}_{r,m}(\varphi_\nu),$$

then for any $n \geq r$*, the decomposition of* $\tilde{X}_{n,m}(r)$ *into nonempty* $\mathcal{P}_{n,m}(\varphi)$*'s is given as*

$$\coprod_{\nu=1}^{N} \mathcal{P}_{n,m}(\varphi_\nu).$$

This proposition is proved by a similar argument to that of Proposition 1.1.

Finally, we think of some properties of the orbit decompositions of $\mathcal{P}(\cdot)$'s in changing of n and m:

Proposition 5.5.

(1) for an l-division of $\{1,\cdots,m\}$ and $\psi : 2^{\{1,\cdots,l\}} \to \mathbb{N}$, if $\mathcal{P}_{n,l}(\psi)$ is a single G_n-orbit, then $\mathcal{P}_{n,m}(\varphi_\psi^{\mathcal{J}})$ is;

(2) for $\varphi : 2^{\{1,\cdots,m\}} \to \mathbb{N}$ such that $\varphi(\{1,\cdots,m\}) = r$, if $\mathcal{P}_{r,m}(\varphi)$ is a single G_r-orbit, then $\mathcal{P}_{n,m}(\varphi)$ is a single G_n-orbit.

Proof. (1) By Lemma 5.2, $\mathcal{P}_{n,m}(\varphi_\psi^{\mathcal{J}}) = f_{\mathcal{J}}(\mathcal{P}_{n,l}(\psi))$ and $f_{\mathcal{J}}$ is G_n-intertwining. Hence the first statement holds.

(2) It is straightforward from the first statement of Proposition 5.4.

□

Preparing these propositions, we are ready to prove the first half of the main results: the description of the orbit decomposition of $\tilde{X}_{3,4}$ under the diagonal action of G_3 stated in the previous subsection. Recall that we only have to prove Lemma 4.5 to prove the description of the orbit decomposition of $\tilde{X}_{3,4}$.

Lemma 4.5.

(1) For each φ which is NOT listed in the Definition4.2, $\mathcal{P}(\varphi)$ is an empty set.

(2) Each $\mathcal{O}(\cdot)$ which is listed in the Definition4.3(1) is a single G-orbit.

(3) $\mathcal{P}(6)$ is decomposed into G-orbits as follows:

$$\mathcal{P}(6) = \coprod_{p \in (\mathbb{P}^1\mathbb{C})'} \mathcal{O}(5; p).$$

Proof.

(1) By Propositions 1.5 and 1.6, we already know that

$$\tilde{X}_{3,4}(1) = G_3 \cdot ([e_1], [e_1], [e_1], [e_1]) = \mathcal{P}(\varphi(2)).$$

(2) ii) At first, 2-divisions of $\{1,2,3,4\}$ are completely listed as follows: $\mathcal{J}(1) = \{\{1\},\{2,3,4\}\}$, $\mathcal{J}(2) = \{\{2\},\{1,3,4\}\}$, $\mathcal{J}(3) = \{\{3\},\{1,2,4\}\}$, $\mathcal{J}(4) = \{\{4\},\{1,2,3\}\}$, $\mathcal{J}(1,2) = \{\{1,2\},\{3,4\}\}$, $\mathcal{J}(1,3) = \{\{1,3\},\{2,4\}\}$, $\mathcal{J}(1,4) = \{\{1,4\},\{2,3\}\}$.

From Propositions 1.5 and 1.6,

$$\left\{([x_1],[x_2]) \in \tilde{X}_{2,2}(2) \mid [x_1] \neq [x_2]\right\} = \tilde{X}_{2,2}(2) = G_2 \cdot ([e_1],[e_2]) = \mathcal{P}_{2,2}(\psi)$$

where

$$\psi : 2^{\{1,2\}} \to \mathbb{N} : I \mapsto \#I.$$

Then by Proposition 5.4,

$$\left\{([x_1],[x_2]) \in \tilde{X}_{3,2}(2) \mid [x_1] \neq [x_2]\right\} = \mathcal{P}_{3,2}(\psi).$$

Hence by Proposition 5.3,

$$\begin{aligned}&\left\{([x_j])_{j=1}^4 \in \tilde{X}_{3,4}(2) \mid \#\{[x_1],\cdots,[x_4]\} = 2\right\}\\ &= \coprod_{\substack{\mathcal{J}:2\text{-division}\\ \text{of } \{1,2,3,4\}}} \mathcal{P}_{3,4}(\varphi_\psi^{\mathcal{J}}).\end{aligned}$$

Now, by easy computations we can see that $\varphi_\psi^{\mathcal{J}(i)} = \varphi(4;i)$, and $\varphi_\psi^{\mathcal{J}(i,j)} = \varphi(4;i,j)$. Also since $\mathcal{P}_{2,2}(\psi)$ is a single G_2-orbit, $\mathcal{P}_{3,2}(\psi)$ is a single G_3-orbit by Proposition 5.5. Furthermore for each 2-division $\mathcal{J}$, since $\mathcal{P}_{3,2}(\psi)$ is a single G_3-orbit, $\mathcal{P}_{3,4}(\varphi_\psi^{\mathcal{J}})$ is.

iii) At first, 3-divisions of $\{1,2,3,4\}$ are completely listed as follows: $\mathcal{J}(1,2) = \{\{1,2\},\{3\},\{4\}\}$, $\mathcal{J}(1,3) = \{\{1,3\},\{2\},\{4\}\}$, $\mathcal{J}(1,4) = \{\{1,4\},\{2\},\{3\}\}$, $\mathcal{J}(2,3) = \{\{2,3\},\{1\},\{4\}\}$, $\mathcal{J}(2,4) = \{\{2,4\},\{1\},\{3\}\}$, $\mathcal{J}(3,4) = \{\{3,4\},\{1\},\{2\}\}$.
From Propositions 1.5 and 1.6,

$$\begin{aligned}\tilde{X}_{2,3}(2) &= \coprod_{\substack{I\subset\{1,\cdots,m\},\\ \text{s.t. } \#I\geq 2}} G_2\cdot([f_1^I],[f_2^I],[f_3^I])\\ &= \coprod_{\substack{I\subset\{1,\cdots,m\},\\ \text{s.t. } \#I\geq 2}} \mathcal{P}_{2,3}(\psi_I)\end{aligned}$$

where

$$f_j^I = \begin{cases} e_j & 1\leq j < \max I\\ \sum_{i\in I\setminus\{\max I\}} e_i & j = \max I\\ e_{j-1} & \max I < j \leq 3\end{cases}$$

and

$$\psi_I : 2^{\{1,2,3\}} \to \mathbb{N} : J \mapsto \begin{cases} \#J & I \not\subset J\\ \#J - 1 & I\subset J\end{cases}$$

Hence,

$$\begin{aligned}&\left\{([x_1],[x_2],[x_3]) \in \tilde{X}_{2,3}(2) \mid [x_1]\neq[x_2]\neq[x_3]\neq[x_1]\right\}\\ &= G_2\cdot\left(\left[f_1^{\{1,2,3\}}\right],\left[f_2^{\{1,2,3\}}\right],\left[f_3^{\{1,2,3\}}\right]\right)\\ &= G_2\cdot([e_1],[e_2],[e_1+2_e]) = \mathcal{P}_{2,3}(\psi_{\{1,2,3\}}).\end{aligned}$$

Since $\left[f_j^{\{j,k\}}\right] = \left[f_k^{\{j,k\}}\right]$, only the summand which corresponds to $I = \{1,2,3\}$ is contained in the most left hand side. Hence the first equality holds.
Then by Proposition 5.4,

$$\left\{([x_1],[x_2],[x_3]) \in \tilde{X}_{3,3}(2) \mid [x_1]\neq[x_2]\neq[x_3]\neq[x_1]\right\} = \mathcal{P}_{3,3}(\psi_{\{1,2,3\}}).$$

Hence by Proposition 5.3,

$$\left\{([x_j])_{j=1}^4 \in \tilde{X}_{3,4}(2) \mid \#\{[x_1],\cdots,[x_4]\}=3\right\} = \coprod_{\substack{\mathcal{J}:3-\text{division}\\ \text{of } \{1,2,3,4\}}} \mathcal{P}_{3,4}(\varphi^{\mathcal{J}}_{\psi_{\{1,2,3\}}}).$$

Now, by easy computations we can see that $\varphi^{\mathcal{J}(i,j)}_{\psi_{\{1,2,3\}}} = \varphi(5;i,j)$.
Also since $\mathcal{P}_{2,3}(\psi_{\{1,2,3\}})$ is a single G_2-orbit, $\mathcal{P}_{3,3}(\psi_{\{1,2,3\}})$ is a single G_3-orbit by Proposition 5.5. Furthermore for each 3-division $\mathcal{J}$, since $\mathcal{P}_{3,2}(\psi_{\{1,2,3\}})$ is a single G_3-orbit, $\mathcal{P}_{3,4}(\varphi^{\mathcal{J}}_{\psi_{\{1,2,3\}}})$ is.

iv) Let $([x_j])_{j=1}^4$ be in $\left\{([x_j])_{j=1}^4 \in \tilde{X}_{3,4}(2) \mid \#\{[x_1],\cdots,[x_4]\}=4\right\}$, then for $J \subset \{1,2,3,4\}$;

$$\dim \operatorname{Span}\{x_j \mid j \in J\} = 2$$

if $\#J \geq 2$. Hence

$$\dim \operatorname{Span}\{x_j \mid j \in J\} = \begin{cases} \#J & 0 \leq \#J \leq 2 \\ 2 & 3 \leq \#I \leq 4 \end{cases},$$

and

$$\left\{([x_j])_{j=1}^4 \in \tilde{X}_{3,4}(2) \mid \#\{[x_1],\cdots,[x_4]\}=4\right\} = \mathcal{P}_{3,4}(\varphi(6))$$

(3) By Proposition 1.5, we have

$$\tilde{X}_{3,4}(3) = \coprod_{\substack{I\subset\{1,2,3,4\},\\ \text{and } \#I\geq 2}} G_3 \cdot \left([f_j^I]\right)_{j=1}^4$$

where f_j^I are defined as above.

Furthermore by Proposition 1.6, we have

$$G_3 \cdot \left([f_j^I]\right)_{j=1}^4 = \mathcal{P}_{3,4}(\varphi_I)$$

where

$$\varphi_I : 2^{\{1,\cdots,4\}} \to \mathbb{N} : J \mapsto \begin{cases} \#J & I \not\subset J \\ \#J-1 & I \subset J \end{cases}.$$

An easy computation immediatly leads to the fact that;

$$\varphi_{\{j,k\}} = \varphi(6;j,k) \text{ for } 1 \leq j < k \leq 4;$$
$$\varphi_{\{j,k,l\}} = \varphi(7;i) \text{ if } \{1,2,3,4\} = \{i,j,k,l\};$$
$$\varphi_{\{1,2,3,4\}} = \varphi(8).$$

From these arguments, we have proved that;

- $\mathcal{P}(\varphi)$ is nonempty if and only if φ is either one of the list in Definition 4.1;

- $\mathcal{P}(\varphi)$ is a single G_3-orbit for each φ listed in Definition 4.1 unless $\varphi = \varphi(6)$.

Hence, the remaining part of our proof is the description of the orbit decomposition of $\mathcal{P}(6)$.

Since we already know that

$$\mathcal{P}(6) = \coprod_{p \in (\mathbb{P}^1\mathbb{C})'} \mathcal{O}(5;p)$$

by Lemma 4.4, we only have to show that each $\mathcal{O}(5;p)$ is a G_3-orbit.

For $\left([x_1], [x_2], [x_1+x_2], [p^{(1)}x_1 + p^{(2)}x_2]\right) \in \mathcal{O}(5; \left[\begin{pmatrix} p^{(1)} \\ p^{(2)} \end{pmatrix}\right])$ and $g \in G_3$,

$$\begin{aligned} & g \cdot \left([x_1], [x_2], [x_1+x_2], \left[p^{(1)}x_1 + p^{(2)}x_2\right]\right) \\ &= \left([gx_1], [gx_2], [g(x_1+x_2)], \left[g(p^{(1)}x_1 + p^{(2)}x_2)\right]\right) \\ &= \left([gx_1], [gx_2], [gx_1+gx_2], \left[p^{(1)}(gx_1) + p^{(2)}(gx_2)\right]\right) \in \mathcal{O}(5; \left[\begin{pmatrix} p^{(1)} \\ p^{(2)} \end{pmatrix}\right]). \end{aligned}$$

Hence $\mathcal{O}(5;p)$ is G_3-stable.

Conversely for $\left([x_1], [x_2], [x_1+x_2], [p^{(1)}x_1 + p^{(2)}x_2]\right) \in \mathcal{O}(5; \left[\begin{pmatrix} p^{(1)} \\ p^{(2)} \end{pmatrix}\right])$, set an element $g \in G_3$ by putting x_1 and x_2 into the first two columns. Then clearly

$$\left([x_1], [x_2], [x_1+x_2], \left[p^{(1)}x_1 + p^{(2)}x_2\right]\right) = g \cdot \left([e_1], [e_2], [e_1+e_2], \left[p^{(1)}e_1 + p^{(2)}e_2\right]\right).$$

Hence $\mathcal{O}(5;p)$ is a single G_3-orbit. □

6 Proof of closure relations among orbits on $\tilde{X}_{3,4}$

The second main result in this section is to determine the closure relations among orbits and $\mathcal{P}_{3,4}(\varphi)'s$ stated in Theorem 3.2. To prove this theorem, we introduce the following criterion:

Proposition 6.1. *For maps $\varphi, \varphi' : 2^{\{1,\cdots,m\}} \to \mathbb{N}$ such that $\mathcal{P}_{n,m}(\varphi) \neq \emptyset$, the following conditions are equivalent;*

(1) $\mathcal{P}_{n,m}(\varphi') \subset \overline{\mathcal{P}_{n,m}(\varphi)}$;

(2) $\varphi'(I) \leq \varphi(I)$ for all $I \subset \{1, \cdots, m\}$.

Proof. Since

$$\mathcal{P}(\varphi) = \left\{ \left([x_j]\right)_{j=1}^m \mid \dim \operatorname{Span}\{x_j \mid j \in J\} = \varphi(J) \ \ (J \subset \{1, \cdots, m\}) \right\},$$

$\mathcal{P}(\varphi)$ is the set of m-tuples $\left([x_j]\right)_{j=1}^m$ satisfying $^\forall J \subset \{1, \cdots, m\}$,

- there exists an $\varphi(J)$-minor matrix of $(x_j)_{j \in J}$ whose determinant is non-zero;
- determinants of all $(\varphi(J)+1)$-minor matrices are zero.

Hence $\overline{\mathcal{P}(\varphi)}$ is the set of m-tuples $\left([x_j]\right)_{j=1}^m$ satisfying $^\forall J \subset \{1, \cdots, m\}$,

- determinants of all $(\varphi(J)+1)$-minor matrices are zero.

Now for an m-tuple $\left([x_j]\right)_{j=1}^m \in \mathcal{P}(\varphi')$, since determinants of all $(\varphi'(J)+1)$-minor matrices of $(x_j)_{j\in J}$ are zero and $\varphi'(J) \leq \varphi(J)$, determinants of all $(\varphi(J)+1)$-minor matrices of $(x_j)_{j\in J}$ are zero. Hence, $\mathcal{P}(\varphi') \subset \overline{\mathcal{P}(\varphi)}$.

Conversely for an m-tuple $\left([x_j]\right)_{j=1}^m$ satisfying that determinants of all $(\varphi(J)+1)$-minor matrices of $(x_j)_{j\in J}$ are zero,

$$\dim \operatorname{Span}\{x_j \mid j \in J\} \leq \varphi(J).$$

Hence $\left([x_j]\right)_{j=1}^m \in \mathcal{P}(\varphi')$ for a φ' which satisfies $\varphi' \leq \varphi$. □

Using this proposition, the Hasse diagram of all $\mathcal{P}(\varphi)$ is simply computed. Hence, we only have to compute the closure of $\mathcal{O}(5;p)$ to prove Theorem 3.2.

Proof. Since $\mathcal{O}(5;p) \subset \mathcal{P}(6)$, we have

$$\begin{aligned}\overline{\mathcal{O}(5;p)} &\subset \overline{\mathcal{P}(6)} \\ &= \mathcal{P}(6) \amalg \left(\coprod_{1\leq i<j\leq 4} \mathcal{P}(5;i,j) \right) \\ &\quad \amalg \left(\coprod_{\substack{\{i,j\}=\{1,2\} \\ \{1,3\},\{1,4\}}} \mathcal{P}(4;i,j) \right) \amalg \left(\coprod_{1\leq i\leq 4} \mathcal{P}(4;i) \right) \amalg \mathcal{P}(2) \\ &\subset \mathcal{P}(6) \amalg \left\{ \left([x_j]\right)_{j=1}^4 \mid \# \left\{[x_j]\right\}_{j=1}^4 \leq 3 \right\}.\end{aligned}$$

We can assume $[x_1] = [x_2]$ for $\left([x_j]\right)_{j=1}^4 \in \overline{\mathcal{O}(5;p)} \setminus \mathcal{O}(5;p)$ without loss of generality, hence $[x_1] = [x_2] = [x_3] = [x_4]$ from by the definition of $\mathcal{O}(5;p)$. Hence $\overline{\mathcal{O}(5;p)} = \mathcal{O}(5;p) \amalg \mathcal{P}(2)$. □

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