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Swirling flow of the axi-symmetric Navier-Stokes equations near a saddle point and no-slip boundary

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1 Introduction

This manuscript contains a summary of [5] and new related figures. As one of the violent flow, tornadoes occur in many place of the world. In order to reduce the loss of human lives and material damage caused by tornadoes, there are many research methods. One of the effective methods is numerical simulation. The swirling structure is significant both in mathematical analysis and the numerical simulations of tornadoes. In the work [5], we try to clarify the swirling structure. More precisely, we do numerical computations on axi-symmetric Navier-Stokes flows with no-slip flat boundary. We compare a hyperbolic flow with swirl and one without swirl and observe some phenomenons occur only in the swirl case. Our main purpose in this work is to combine the point of view from mathematical analysis (especially regularity results) and numerical approach to observe phenomenons which highly related to the structure of tornadoes.

More precisely, we consider a local behavior of the 3D-Navier-Stokes flow near a saddle point (with hyperbolic flow configuration) and no-slip flat boundary. The Navier-Stokes equations with no-slip flat boundary are expressed as

$$\begin{aligned} \partial_t v+(v\cdot\nabla)v-\nu\Delta v+\nabla p &= 0 \quad \text{in} \quad \mathbb{R}^3_+\times[0,T), \\ v_0=v|_{t=0},\quad v|_{\partial\mathbb{R}^3_+}=0,\quad \nabla\cdot v &= 0 \quad \text{in} \quad \mathbb{R}^3_+\times[0,T), \end{aligned} \tag{1.1}$$

where v is a vector field representing velocity of the fluid, and p is the pressure. The term "hyperbolic flow configuration" which used here and after means there is $\delta>0$ (depending on t) such that $v(t,x)\cdot e_z>0$, $v(t,x)\cdot e_r(x)<0$, or $v(t,x)\cdot e_z<0$, $v(t,x)\cdot e_r(x)>0$ for $0<|x_h|<\delta$ and $0<x_3<\delta$, where $e_z=(0,0,1)$, $e_r(x)=(x_1/|x_h|,x_2/|x_h|,0)$ and $|x_h|=\sqrt{x_1^2+x_2^2}$.

At first, let us look back the history of Navier-Stokes equations briefly. After the pioneering work of Leray (1934) and Hopf (1951), many different regularity criteria of solutions to (1.1) was established by many researchers working in the regularity theory of (1.1). For example, a regularity criterion along streamlines (characteristic curves) was constructed (see [2]). Besides these, other important works such as [4], in which type I blow up was excluded for solutions to (1.1) under a regularity condition on the vorticity direction in the half space to the case of the no-slip boundary conditon (see also [1], which is the pioneer work in this field), and [3, 8] for axisymmetric solutions to (1.1)

In other hand, from literatures (such as [7]), it is natural to consider that the hyperbolic flow with swirl and saddle point on the boundary might be the key structure of flow and probable place for unstability effects occur near the no-slip flat boundary. Although there are many fruitful results based on mathematical analysis as we recalled above, it is not easy to analyze locally such fluid mechanics to go a step further mathematically. Thus, it should be effective for us to attempt numerical approach.

In our numerical computation in [5], we use the following cylindrical domain

$$\Omega := \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : -\frac{1}{8} < x_3 < \frac{1}{2}, \sqrt{x_1^2 + x_2^2} < 1 \right\} \tag{1.2}$$

with no-slip boundary condition

$$v = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

We set the initial data in the following manner.

$$u_r = \mathrm{sign}(z)(\frac{1}{r^2+1})^2 \quad u_\theta = (\frac{1}{r^2+1})(\frac{1}{z^2+1}) \quad u_z = (\frac{1}{r^2+1})(\frac{1}{z^2+1})$$

As for the no-swirl case we only change $u_\theta = 0$. For the figures of initial data, we refer the readers to [5, figure 1].

The following are the main results in [5]. We showed the clear structure for the axisymmetric hyperbolic flow with swirl and observed the following phenomenons which are distinctly different from those without swirl: (1) The distance between the maximum point of the velocity and the z-axis is drastically changing around some time we called it turning point (Refer to [5, figure 2]). (2) The velocity increases and obtains its extreme value (maximum) near the axis of symmetry and the boundary when time is close to the turning points (Refer to [5, figure 2]). The comparison of these results with studies on tornadoes might help in understanding the behavior of the velocity of wind near the ground which is very significant in the research on tornadoes for reducing the damage cause by tornadoes or similar phenomena (Also refer to [5, figure 3] for no swirl case). (3) The downward flow near the z-axis in the swirl case was observed (Refer to [5, figure 6]). The downward wind inside the core of real tornado was also observed in a two-celled vortex structure in the studies of numerical simulations for time-averaged velocity, as shown in [6, figure 4(b)]. By comparing our observation with studies on tornadoes might enhance our understanding about the behavior inside the core of a tornado for a high swirl ratio.

For the details and more references we refer the readers to [5]. We present new related figures in the following section.

2 Numerical results

In this section we compare the axial velocity in the swirl case and the no swirl case. The work by [5] implies that the flow dissipates in a straightforward manner as t increases in the no swirl case and that an interesting flow structure is observed near the z-axis in the swirl case (Refer to [5, figure 6]). A downward flow arises near the z-axis at approximately $t = 0.3$, and the maximum value of $|v|$ is attained near the z-axis and the lower boundary at approximately the same time ($t = 0.3$). Those phenomena are observed only in the swirl case. Comparing to the swirl case, there are no such kind of downward flows in the no swirl case. We can observe from Figure 1 that the axial velocity near the z-axis is always in the same direction, which shows the phenomena near the z-axis is totally different for the swirl case and the no swirl case. This observation is highly related to the researches on the behavior inside the core of tornadoes for different swirl ratios. From the studies of tornadoes, the vortex structures are different for low swirl ratio (note that the swirl ratio of no swirl flow is zero) and high swirl ratio. The upward flow near the z-axis observed in [5] is also observed inside the core of tornadoes in the high swirl ratio flows.

For more references, refer to [4] for regularity results, refer to [9] for numerical studies of the Navier-Stokes and Euler equations, refer to [6] for studies on tornado-like vortices, and refer to [10, 11, 12, 13] for the stabilized Lagrange-Galerkin (finite element) scheme used in [5] and this summary manuscript.

References

[1] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations.* Indiana Univ. Math. J. 42, (1993) 775–789.

[2] C.-H. Chan and T. Yoneda, *On possible isolated blow-up phenomena and regularity criterion of the 3D Navier-Stokes equation along the streamlines.* Methods Appl. Anal., 19 (2012) 211–242.

[3] C.-C. Chen, R. M. Strain, T.-P. Tsai, and H.-T. Yau, *Lower bounds on the blow-up rate of the axisymmetric Navier-Stokes equations. II.* Comm. Partial Differential Equations, 34 (2009) 203–232.

[4] Y. Giga, P.-Y. Hsu and Y. Maekawa, *A Liouville theorem for the planer Navier-Stokes equations with the no-slip boundary condition and its application to a geometric regularity criterion.* Comm. Partial Differential Equations, 39 (2014) 1906-1935

[5] P.-Y. Hsu, H. Notsu and T. Yoneda, *A local analysis of the axi-symmetric Navier-Stokes flow near a saddle point and no-slip flat boundary.* To appear in J. Fluid Mech. Preprint available at http://arxiv.org/pdf/1505.03602

[6] T. Ishihara, S. Oh and Y. Tokuyama, *Numerical study on flow fields of tornado-like vortices using the LES turbulence model.* J. of Wind Engineering and Industrial Aerodynamics 99 (2011), 239–248.

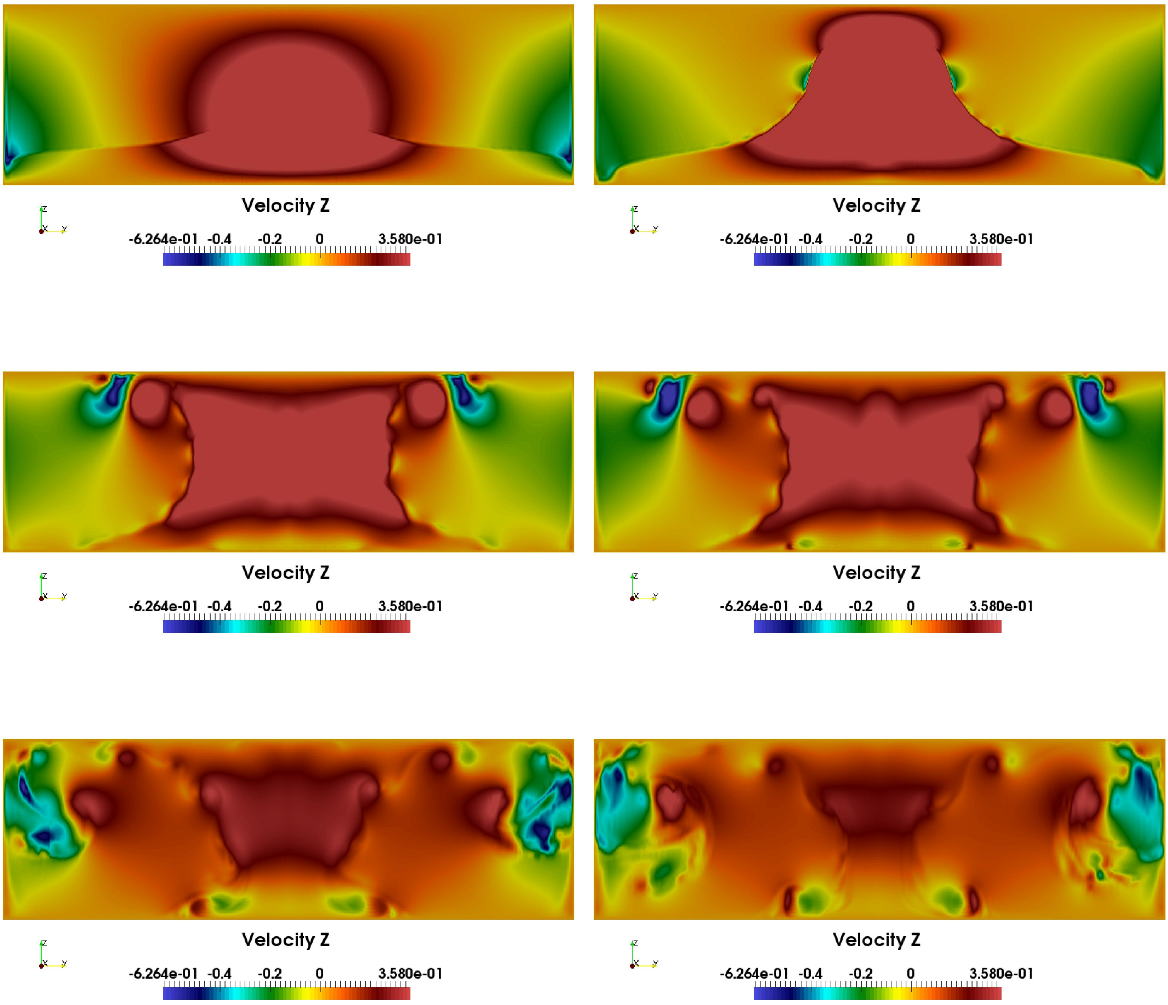

Figure 1: Time evolution of the axial velocity u_z on the plane $x_1 = 0$ in the no swirl case with $Re = 50,000$. $t = 0.1$ (top left), 0.3 (top right), 1.0 (middle left), 1.3 (middle right), 2.3 (bottom left) and 3.0 (bottom right). Note that the red and blue colors represent positive and negative values in this figure.

[7] K. Kang, Regularity of axially symmetric flows in a half-space in three dimensions, SIAM J. Math. Anal. 35, (2004) 1636–1643.

[8] G. Koch, N. Nadirashvili, G. Seregin and V. Sverak, *Liouville theorems for the Navier-Stokes equations and applications.* Acta Math. 203 (2009), no. 1, 83–105.

[9] G. Luo and T. Y. Hou, *Potentially singular solutions of the 3D incompressible Euler equations.* Proc. Natl Acad. Sci. USA 111 (2014), no. 36, 12968–12973.

[10] H. Notsu, *Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme.* Transactions of Japan Society for Computational Engineering and Science (2008), 20080032.

[11] H. Notsu and M. Tabata, *A combined finite element scheme with a pressure stabilization and a characteristic-curve method for the Navier-Stokes equations.* Transactions of the Japan Society for Industrial and Applied Mathematics (in Japanese) 18 (2008), 427–445.

[12] H. Notsu and M. Tabata, *Error estimates of a pressure-stabilized characteristics finite element scheme for the Oseen equations.*. Journal of Scientific Computing 65 (2015), 940–955.

[13] H. Notsu and M. Tabata, *Error estimates of a stabilized Lagrange-Galerkin scheme for the Navier-Stokes equations.* To appear in ESAIM: M2AN, doi:10.1051/m2an/2015047.