数理解析研究所講究録
第 1995 巻 2016 年 82-91

A norm-preserving self-adaptive moving mesh integrator for the short pulse equation

Shun Sato*, Takayasu Matsuo†, Bao-Feng Feng‡

Abstract

In this paper, we propose a self-adaptive moving mesh and norm-preserving scheme for the short pulse equation, which is a model equation of ultrashort optical pulses in nonlinear media. Our numerical method is based on the fact that the short pulse equation is associated with the coupled dispersionless equation via a hodograph transformation. Namely, instead of the numerical integration of the short pulse equation itself, we conduct the numerical integration of the coupled dispersionless equation under the corresponding initial and boundary conditions. An invariant-preserving numerical method for the coupled dispersionless equation leads to the discrete norm conservation law of the numerical solution for the short pulse equation.

1 Introduction

We consider the numerical integration of the short pulse (SP) equation

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}-\frac{1}{2}u^2\frac{\partial}{\partial x}\right)u=u, \tag{1}$$

which models the propagation of ultrashort optical pulses in nonlinear media [15]. Here, $t\in\mathbb{R}_+$ ($\mathbb{R}_+:=[0,+\infty)$) and $x\in\mathbb{R}$ denote temporal and spatial independent variables, respectively. We impose the periodic boundary condition $u(t,x+L)=u(t,x)$ for any $x\in\mathbb{R}$ and $t\in\mathbb{R}_+$. The dependent variable $u=u(t,x)$ represents the magnitude of the electric field. Brunelli [2] showed that the short pulse equation (1) has the first integral

$$\mathcal{H}(u):=\frac{1}{2}\int_0^L u^2\,\mathrm{d}x, \tag{2}$$

which we call "norm" hereafter.

Sakovich–Sakovich [13] showed that the SP equation is associated with the sine-Gordon equation

$$\theta_{\tau s}=\sin\theta$$

in light-cone coordinates via a hodograph transformation, and thus it is integrable. Here, subscripts τ and s denote partial differentiations. Feng–Inoguchi–Kajiwara–Maruno–Ohta [3]

*Graduate School of Information Science and Technology, The University of Tokyo, E-mail: shun_sato@mist.i.u-tokyo.ac.jp

†Graduate School of Information Science and Technology, The University of Tokyo, E-mail: matsuo@mist.i.u-tokyo.ac.jp

‡School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, E-mail: baofeng.feng@utrgv.edu

derived an integrable discretization of the SP equation from an integrable discretization of the sine-Gordon equation. The numerical integrator obtained via a discrete hodograph transformation has an advantage that it automatically works as a moving grid method. (Feng et al. [3] also considered the Wadati–Konno–Ichikawa (WKI) elastic beam equation [16] and the complex Dym equation). Then the team including the present authors [11] constructed a more general numerical scheme for the SP equation by applying the discrete variational derivative method [5, 7] to the sine-Gordon equation. In this method, the boundary condition for the SP equation is ensured by the discrete conservation law of the sine-Gordon equation. In this case, since the structure-preservation in solving the sine-Gordon equation is solely used to satisfy the boundary condition of the SP equation, it is hard to preserve another first integral of the SP equation.

On the other hand, Feng–Maruno–Ohta [4] showed that the SP equation can also be associated with the coupled dispersionless (CD) equation [9], and derived an integrable discretization of the SP equation from an integrable discretization of the CD equation.

In this study, our aim is to follow this line and to construct a general self-adaptive moving mesh and structure-preserving scheme for the SP equation again by using the discrete variational derivative method, but this time for the CD equation. Since the formulation of the CD equation provides the almost automatic translation of the boundary condition, it enables us to use structure-preservation for an additional feature. Thus, in this study, our aim is to construct a numerical integrator for the SP equation, which possesses self-adaptive moving mesh and norm-preserving properties.

The rest of the paper is organized as follows: Section 2 is devoted to preliminaries such as the hodograph transformation between SP equation and CD equation. Then, a self-adaptive moving mesh and norm-preserving integrator is constructed in Section 3. Concluding remarks are given in Section 4.

2 Preliminaries

2.1 Translation from the SP equation to the CD equation

The short pulse (SP) equation (1) can be transformed into the coupled dispersionless (CD) equation

$$\begin{cases} \rho_\tau = \left(-\dfrac{1}{2}u^2\right)_s, \\ u_{\tau s} = \rho u \end{cases} \tag{3}$$

via a hodograph transformation [4]

$$\mathrm{d}x = \rho \mathrm{d}s - \frac{1}{2}u^2 \mathrm{d}\tau, \tag{4}$$

$$\mathrm{d}t = \mathrm{d}\tau, \tag{5}$$

where $\tau \in \mathbb{R}^+$ and $s \in \mathbb{R}$ denote the new temporal and spatial independent variables, and $S \in \mathbb{R}_+$ is arc-length. Because of the relation $\rho = x_s$, the constants L and S are related to each other:

$$\int_0^S \rho\, \mathrm{d}s = L, \qquad \int_0^L \sqrt{1+|y_x|^2}\mathrm{d}x = S.$$

Here, the first equation of the CD equation (3) actually ensures that the transformation (4) is well-defined: since the relation (4) implies $x_s = \rho$ and $x_\tau = -u^2/2$, the first equation of (3) ensures $x_{s\tau} = x_{\tau s}$. On the other hand, since the transformation (4) and (5) yields

$$\frac{\partial}{\partial s} = \rho\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} - \frac{u^2}{2}\frac{\partial}{\partial x},$$

the second equation of the CD equation (3) can be transformed into the SP equation (1). Note that, the boundary condition for the CD equation corresponding to the periodic boundary condition $u(t, x + L) = u(t, x)$ for the SP equation is, in fact, periodic:

$$u(\tau, s + S) = u(\tau, s), \quad \rho(\tau, s + S) = \rho(\tau, s) \qquad (\forall s \in \mathbb{R}).$$

2.2 Conservation laws

Since the norm $\mathcal{H}$ defined by (2) can be transformed into

$$\frac{1}{2}\int_0^L (u(t,x))^2\,\mathrm{d}x = \frac{1}{2}\int_0^S (u(\tau,s))^2\,\rho(\tau,s)\mathrm{d}s$$

by using the hodograph transformation (4) and (5), the counterpart

$$\mathcal{H}'(u,\rho) := \frac{1}{2}\int_0^S u^2\rho\,\mathrm{d}s$$

of the norm $\mathcal{H}$ is a first integral of the CD equation (3). Moreover, Kakuhata–Konno [8] showed that the CD equation can be written in the variational form

$$\begin{cases} \rho_\tau = -\dfrac{\partial}{\partial s}\dfrac{\delta\mathcal{H}'(u,\rho)}{\delta\rho} = -\dfrac{\partial}{\partial s}\left(\dfrac{1}{2}u^2\right), \\ u_{\tau s} = \dfrac{\delta\mathcal{H}'(u,\rho)}{\delta u} = \rho u, \end{cases} \tag{6}$$

where the variational derivatives $\delta\mathcal{H}'/\delta u$ and $\delta\mathcal{H}'/\delta\rho$ are defined as

$$\frac{\delta\mathcal{H}'(u,\rho)}{\delta\rho} := \frac{1}{2}u^2, \qquad \frac{\delta\mathcal{H}'(u,\rho)}{\delta u} := \rho u.$$

From the variational structure (6), the conservation law of $\mathcal{H}'$ can be derived.

Proposition 1. *For the solution (u, ρ) of the coupled dispersionless equations (6) under the periodic boundary condition, it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{H}'(u,\rho) = 0.$$

Proof. It can be shown by the simple calculation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{H}' &= \int_0^S \left(\frac{\delta\mathcal{H}'}{\delta\rho}\rho_\tau + \frac{\delta\mathcal{H}'}{\delta u}u_\tau\right)\mathrm{d}s = \int_0^S \left\{-\frac{\delta\mathcal{H}'}{\delta\rho}\left(\frac{\partial}{\partial s}\frac{\delta\mathcal{H}'}{\delta\rho}\right) + (u_{\tau s})\,u_\tau\right\}\mathrm{d}s \\ &= \frac{1}{2}\left[-\left(\frac{\delta\mathcal{H}'}{\delta\rho}\right)^2 + (u_\tau)^2\right]_0^S = \frac{1}{2}\left[-\left(\frac{1}{2}u^2\right)^2 + (u_\tau)^2\right]_0^S = 0. \end{aligned}$$

□

3 Proposed method

In this section, we construct the self-adaptive moving mesh and norm-preserving scheme for the short pulse equation. Let us introduce the symbols $u_k^{(m)}, \rho_k^{(m)} \in \mathbb{R}$, which are used for the approximation of $u(m\Delta\tau, k\Delta s), \rho(m\Delta\tau, k\Delta s)$ $(k = 1, \dots, K; m = 1, \dots, M)$, where $\Delta\tau$ and Δs are the temporal and spatial mesh sizes. Moreover, the notation $u^{(m)}$ and $\rho^{(m)}$ denote $u^{(m)} := (u_1^{(m)}, \dots, u_K^{(m)})^\top$ and $\rho^{(m)} := (\rho_1^{(m)}, \dots, \rho_K^{(m)})^\top$, respectively. Let us also introduce the spatial forward, backward, and central difference operators δ_s^+, δ_s^-, and $\delta_s^{\langle 1 \rangle}$, and the spatial forward, backward, and central average operators μ_s^+, μ_s^-, and $\mu_s^{\langle 1 \rangle}$ as follows:

$$
\begin{aligned}
\delta_s^+ u_k^{(m)} &:= \frac{u_{k+1}^{(m)} - u_k^{(m)}}{\Delta s}, & \mu_s^+ u_k^{(m)} &:= \frac{u_{k+1}^{(m)} + u_k^{(m)}}{2}, \\
\delta_s^- u_k^{(m)} &:= \frac{u_k^{(m)} - u_{k-1}^{(m)}}{\Delta s}, & \mu_s^- u_k^{(m)} &:= \frac{u_k^{(m)} + u_{k-1}^{(m)}}{2}, \\
\delta_s^{\langle 1 \rangle} u_k^{(m)} &:= \frac{u_{k+1}^{(m)} - u_{k-1}^{(m)}}{2\Delta s}, & \mu_s^{\langle 1 \rangle} u_k^{(m)} &:= \frac{u_{k+1}^{(m)} + u_{k-1}^{(m)}}{2}.
\end{aligned}
$$

The temporal counterparts $\delta_\tau^+, \delta_\tau^-, \delta_\tau^{\langle 1 \rangle}, \mu_\tau^+, \mu_\tau^-, \mu_\tau^{\langle 1 \rangle}$ are defined similarly.

3.1 Outline

The outline of the proposed method for the short pulse equation with the initial condition $u(0, x) = u_0(x)$ and the periodic boundary condition $u(t, x + L) = u(t, x)$ is as follows:

Step 0. Prepare the initial value $\left(u_k^{(0)}, \rho_k^{(0)}\right)$ $(k = 1, \dots, K)$ of the CD equation, which corresponds to the initial condition u_0 of the SP equation.

Step 1. Obtain the numerical solution $\left(u_k^{(m)}, \rho_k^{(m)}\right)$ $(k = 1, \dots, K; m = 1, \dots, M)$ by using a $\mathcal{H}'$-preserving scheme for the CD equation with the periodic boundary condition $u_{k+K}^{(m)} = u_k^{(m)}$, $\rho_{k+K}^{(m)} = \rho_k^{(m)}$.

Step 2. Compute $x_k^{(m)} \approx x(m\Delta\tau, k\Delta s)$ $(k = 1, \dots, K;\ m = 1, \dots, M)$ by using the discrete counterpart of the hodograph transformation.

We construct a $\mathcal{H}'$-preserving scheme for the CD equation in Section 3.2. Then, Section 3.3 is devoted to describe a discrete counterpart of the hodograph transformation in Step 2. After that we summarize the properties of the numerical solutions in Section 3.4. Finally some numerical experiments are conducted in Section 3.5.

3.2 $\mathcal{H}'$-preserving numerical scheme

In this section, we describe the $\mathcal{H}'$-preserving scheme. In order to construct it, we can use the concept of the "discrete variational derivative method," which utilizes the variational structure and the conservation law of a conservative system (see the monograph [6] for more details). We define a discrete counterpart $\mathcal{H}'_{\mathrm{d}}$ of the conserved quantity $\mathcal{H}'$ as

$$
\mathcal{H}'_{\mathrm{d}}\left(u^{(m)}, \rho^{(m)}\right) := \frac{1}{2} \sum_{k=1}^{K} \left(u_k^{(m)}\right)^2 \rho_k^{(m)} \Delta s.
$$

Then, the corresponding discrete variational derivatives can be derived as

$$\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_k} = \frac{1}{2}\mu_\tau^+\left(u_k^{(m)}\right)^2,$$

$$\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(u^{(m+1)},u^{(m)}\right)_k} = \left(\mu_\tau^+\rho_k^{(m)}\right)\left(\mu_\tau^+u_k^{(m)}\right).$$

Here, we obtain a $\mathcal{H}'$-preserving numerical scheme

$$\begin{cases} \delta_\tau^+\rho_k^{(m)} = -\delta_s^{\langle 1\rangle}\dfrac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_k} & (k=1,\dots,K), \\ \delta_s^{\langle 1\rangle}\delta_\tau^+u_k^{(m)} = \dfrac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(u^{(m+1)},u^{(m)}\right)_k} & (k=1,\dots,K). \end{cases} \tag{7}$$

Thanks to the skew-symmetry of the central difference operator, we can follow the line of the discussion in the proof of Proposition 1 to prove Proposition 2 below.

Proposition 2. *Under the periodic boundary condition, for the numerical solution $u_k^{(m)}, \rho_k^{(m)}$ of the scheme* (7)*, it holds that*

$$\delta_\tau^+\mathcal{H}'_{\mathrm{d}}\left(u^{(m)},\rho^{(m)}\right) = 0.$$

Proof. It can be shown by the property of discrete variational derivative (first equality) and the skew-symmetric property of the central difference operator (third equality):

$$\begin{aligned}
&\delta_\tau^+\mathcal{H}'_{\mathrm{d}}\left(u^{(m)},\rho^{(m)}\right) \\
&= \sum_{k=1}^{K}\left\{\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_k}\delta_\tau^+\rho_k^{(m)} + \frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(u^{(m+1)},u^{(m)}\right)_k}\delta_\tau^+u_k^{(m)}\right\}\Delta s \\
&= \sum_{k=1}^{K}\left\{-\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_k}\left(\delta_s^{\langle 1\rangle}\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_k}\right) + \left(\delta_s^{\langle 1\rangle}\delta_\tau^+u_k^{(m)}\right)\delta_\tau^+u_k^{(m)}\right\}\Delta s \\
&= -\frac{1}{2}\left\{\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_K}\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_{K+1}} - \frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_1}\frac{\delta \mathcal{H}'_{\mathrm{d}}(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_0}\right\} \\
&\quad + \frac{1}{2}\left\{\left(\delta_\tau^+u_K^{(m)}\right)\left(\delta_\tau^+u_{K+1}^{(m)}\right) - \left(\delta_\tau^+u_1^{(m)}\right)\left(\delta_\tau^+u_0^{(m)}\right)\right\} \\
&= 0.
\end{aligned}$$

□

3.3 Discrete hodograph transformation

In this section, we consider a discrete counterpart of the hodograph transformation (4) and (5).

First of all, we review the following (trivial) fact:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(x(\tau,s+S)-x(\tau,s)\right) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\int_s^{s+S}x_s(\tau,s)\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_s^{s+S}\rho\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^{S}\rho\,\mathrm{d}s \\
&= \int_0^S\rho_\tau\mathrm{d}s = \int_0^S\left(-\frac{1}{2}u^2\right)_s\mathrm{d}s = \left[-\frac{1}{2}u^2\right]_0^S = 0,
\end{aligned}$$

which means that the solution $(x(\tau,s),u(\tau,s))$ obtained from $(u(\tau,s),\rho(\tau,s))$ automatically satisfies the boundary condition "$x(\tau,s+S)-x(\tau,s)=L$ and $u(\tau,s+S)=u(\tau,s)$ hold for any $s\in\mathbb{R},\tau\in\mathbb{R}_+$," which corresponds to the periodic boundary condition $u(t,x+L)=u(t,x)$ $(x\in\mathbb{R})$.

In order to rigorously ensure the discrete periodic boundary condition $x_{k+K}^{(m)}=x_k^{(m)}+L$, a discrete counterpart of the transformation should allow the numerical solution to replicate the line of discussion above. If we define

$$\delta_s^- x_k^{(m)}=\rho_k^{(m)}, \tag{8}$$

we see that

$$\begin{aligned}\delta_\tau^+\left(x_K^{(m)}-x_0^{(m)}\right)&=\delta_\tau^+\sum_{k=1}^K\rho_k^{(m)}\Delta s=\sum_{k=1}^K\delta_\tau^+\rho_k^{(m)}\Delta s=\sum_{k=1}^K-\delta_s^{\langle 1\rangle}\frac{\delta\mathcal{H}_{\mathrm{d}}'(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_k}\Delta s\\&=-\mu_s^+\left\{\frac{\delta\mathcal{H}_{\mathrm{d}}'(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_K}-\frac{\delta\mathcal{H}_{\mathrm{d}}'(u,\rho)}{\delta\left(\rho^{(m+1)},\rho^{(m)}\right)_0}\right\},\end{aligned}$$

and it vanishes due to the periodic boundary condition $u_{k+K}^{(m)}=u_k^{(m)}$.

The discrete transformation (8) corresponds to $x_s=\rho$. In order to complete the definition of the discrete transformation, we should define a discrete counterpart of $x_\tau=-u^2/2$, which is indispensable to the computation of $x_k^{(m)}$'s; i.e., we need a way to elevate $x_k^{(m)}$ to $x_k^{(m+1)}$ for at least one $k\in\{1,\dots,K\}$ so that (8) can be used to determine $x_k^{(m+1)}$'s for other k's. This can be done, for example, by a simple finite difference update $x_0^{(m+1)}=x_0^{(m)}-\Delta\tau\left(u_0^{(m)}\right)^2/2$. This surely works, but a drawback in this simple formulation is that the discrete transformation becomes different if we choose a different point, say $k=1$, as the elevating point. This ambiguity never happens in the continuous context. In view of this, let us here consider a slightly better formulation; we consider the following condition

$$\delta_\tau^+\delta_s^- x_k^{(m)}=\delta_s^-\delta_\tau^+ x_k^{(m)}, \tag{9}$$

which corresponds to the identity $x_{s\tau}=x_{\tau s}$. By using the discrete transformation (8) and the first equality of the discrete CD equation (7), we see

$$\begin{aligned}\delta_\tau^+\delta_s^- x_k^{(m)}&=\delta_\tau^+\rho_k^{(m)}=-\delta_s^{\langle 1\rangle}\left(\frac{1}{2}\mu_\tau^+\left(u_k^{(m)}\right)^2\right)=-\delta_s^-\mu_s^+\left(\frac{1}{2}\mu_\tau^+\left(u_k^{(m)}\right)^2\right)\\&=\delta_s^-\left(-\frac{1}{2}\mu_s^+\mu_\tau^+\left(u_k^{(m)}\right)^2\right).\end{aligned}$$

From the equality above and the condition (9), we should define the discrete counterpart of $x_\tau=-u^2/2$ as

$$\delta_\tau^+ x_k^{(m)}=-\frac{1}{2}\mu_s^+\mu_\tau^+\left(u_k^{(m)}\right)^2. \tag{10}$$

The condition (9) ensures that the value $x_k^{(m+1)}$ obtained from $x_{k-1}^{(m)}$ does not depend on the order of the computation: Let us consider how to find $x_k^{(m+1)}$ starting from $x_{k-1}^{(m)}$. We can do this in the following two different ways; moving first right and then upwards $x_{k-1}^{(m)}\mapsto x_k^{(m)}\mapsto x_k^{(m+1)}$, or in the reversed order $x_{k-1}^{(m)}\mapsto x_{k-1}^{(m+1)}\mapsto x_k^{(m+1)}$. In the discretization above, these two are rigorously equivalent, and the ambiguity in the simple formulation mentioned above does not occur.

For any k and m, the value $x_k^{(m)}$ can be computed from $x_0^{(0)}$ by using $u_l^{(n)}, \rho_l^{(n)}$ ($l = 0, 1, \dots, K; n = 0, 1, \dots, M$) via the discrete transformation (8) and (10). Thanks to the condition (9), the value $x_k^{(m)}$ does not depend on the order of computation.

3.4 Properties of the numerical solutions

Here, we summarize the properties of the numerical solutions. Since the numerical solution $(x_k^{(m)}, u_k^{(m)})$ is an approximation of $(x(m\Delta\tau, k\Delta s), u(m\Delta\tau, k\Delta s))$, the sample point is distributed uniformly with respect to the arc-length. Moreover, Proposition 3 ensures the periodic boundary condition and the norm-preservation property, which are already proved in the previous sections.

Proposition 3. *Let $(x_k^{(m)}, u_k^{(m)})$ be the numerical solution obtained from the numerical solution $(u_k^{(m)}, \rho_k^{(m)})$ of (7) via the discrete hodograph transformation (8) and (10). Then, the following properties hold for any $m = 0, 1, \dots, M-1$:*

- $x_K^{(m+1)} - x_0^{(m+1)} = x_K^{(m)} - x_0^{(m)}$ *holds.*
- $\delta_\tau^+ \mathcal{H}_{\mathrm{d}}(x^{(m)}, u^{(m)}) = 0$ *holds, where*

$$\mathcal{H}_{\mathrm{d}}(x^{(m)}, u^{(m)}) := \frac{1}{2}\sum_{k=1}^{K} \left(u_k^{(m)}\right)^2 \left(x_k^{(m)} - x_{k-1}^{(m)}\right).$$

3.5 Numerical experiments

3.5.1 Smooth pulse solution

The SP equation has the breather soliton solution [14]:

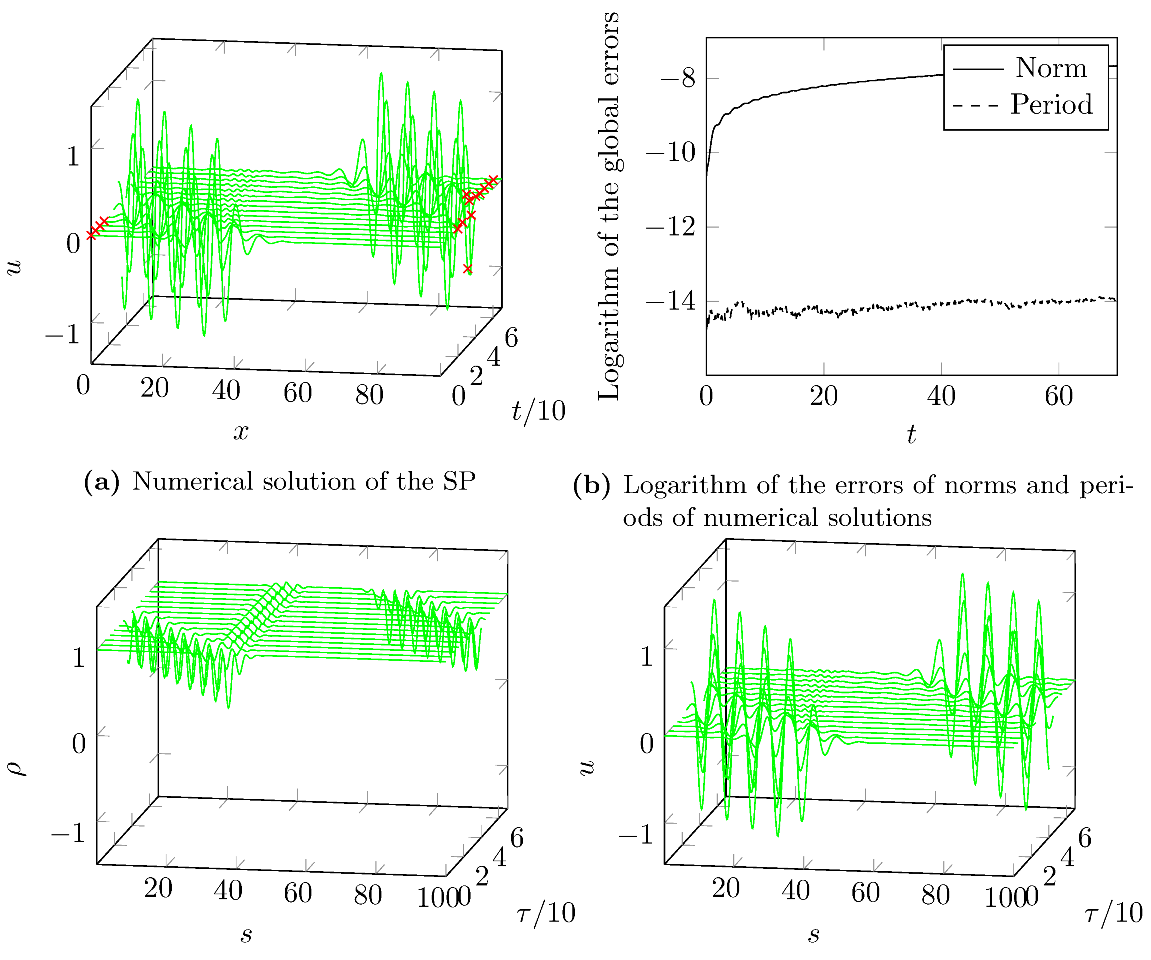
$$\begin{cases} u(\tau, s) = 4\xi\zeta \dfrac{\xi \sin\psi \sinh\phi + \zeta \cos\psi \cosh\phi}{\xi^2 \sin^2\psi + \zeta^2 \cosh^2\phi}, \\ x(\tau, s) = s + 2\xi\zeta \dfrac{\xi \sin 2\psi - \zeta \sinh 2\phi}{\xi^2 \sin^2\psi + \zeta^2 \cosh^2\phi}, \end{cases} \tag{11}$$

where the parameter $\xi \in (0, 1)$ and

$$\phi = \xi(s + t), \quad \psi = \zeta(s - t), \quad \zeta = \sqrt{1 - \xi^2}.$$

The exact solution with $\xi < \xi_{\mathrm{cr}} = \sin(\pi/8) \approx 0.383$ is nonsingular, whereas the singularities $u_x \to \pm\infty$ appear in the solution when $\xi > \xi_{\mathrm{cr}}$. The breather soliton is also called the smooth pulse, and the name "short pulse" comes from this solution. The numerical solutions obtained by the proposed method are shown in Fig. 1 ($\xi = 0.3$, $S = 100$, $K = 513$, $\Delta t = 0.1$). The nonlinear equation is solved by the function "fsolve" of MATLAB R2013a. Note that, strictly speaking, the solution (11) is not an exact solution on the periodic domain.

As shown in Fig. 1, the numerical solutions of the proposed method seem fine (see, (a),(c), and (d)). Moreover, the norm $\mathcal{H}$ and period $x(\tau, S) - x(\tau, 0)$ are preserved (see, (b)).



(a) Numerical solution of the SP

(b) Logarithm of the errors of norms and periods of numerical solutions

(c) Numerical solution ρ of the CD

(d) Numerical solution u of the CD

Figure 1: Numerical solution of the proposed method: smooth pulse solution (11) with $\xi = 0.3$ ($S = 100$, $K = 513$, $\Delta t = 0.1$). Solid lines represents the numerical solution, and the red symbol $\times$ represents the point $(x_0^{(m)}, u_0^{(m)})$.

3.5.2 Exact periodic solutions

In order to show some numerical examples, we review several exact solutions under the periodic boundary condition devised by Parkes [12] (see also Matsuno [10] for various periodic solutions). All numerical experiments are done under the condition $\Delta t = 0.1$ and $K = 129$. The nonlinear equation is again solved by "fsolve" of MATLAB R2013a. Since the norm $\mathcal{H}$ and period $x(\tau, S) - x(\tau, 0)$ are preserved and their behaviors are quite similar to the previous case, we will show only the profiles of the numerical solutions.

Periodic hump solution The SP equation has a periodic hump solution

$$\begin{cases} x(\tau, s) = v\tau + x_0 - s + \dfrac{1}{\alpha^2}\tau + \dfrac{2}{\alpha} E\left(\alpha s - \dfrac{\tau}{\alpha} \,\middle|\, \xi\right), \\ u(\tau, s) = \pm\dfrac{2\sqrt{\xi}}{\alpha}\,\mathrm{cn}\left(\alpha s - \dfrac{\tau}{\alpha} \,\middle|\, \xi\right), \end{cases} \tag{12}$$

where $v > 0$ represents the velocity of the traveling-wave, $\xi \in (0, 1/2)$ denotes the parameter related to the shape of the wave, and $\alpha := \sqrt{(1 - 2\xi)/v}$. Here, $\mathrm{cn}(w|\xi)$ is a Jacobbian elliptic function, and $E(w|\xi)$ is the elliptic integral of the second kind (the notation is the same as those used in [1]). We also use another two Jacobbian elliptic functions $\mathrm{sn}(w|\xi)$ and $\mathrm{dn}(w|\xi)$. In view of the periodicity, we set $S = 8K(\xi)/\alpha$, where $K(\xi)$ is the complete elliptic integral of the first kind.

The corresponding initial condition

$$\begin{cases} \rho(0,s) = -1 + 2\,\mathrm{dn}^2\,(\alpha s \mid \xi)\,, \\ u(0,s) = \pm\dfrac{2\sqrt{\xi}}{\alpha}\,\mathrm{cn}\,(\alpha s \mid \xi)\,, \end{cases}$$

for the CD equation can be obtained from (12). The numerical solution of the proposed method finely reproduces the periodic humps as shown in Fig. 2 (a).

Periodic upright loop solution The SP equation has a periodic upright loop solution

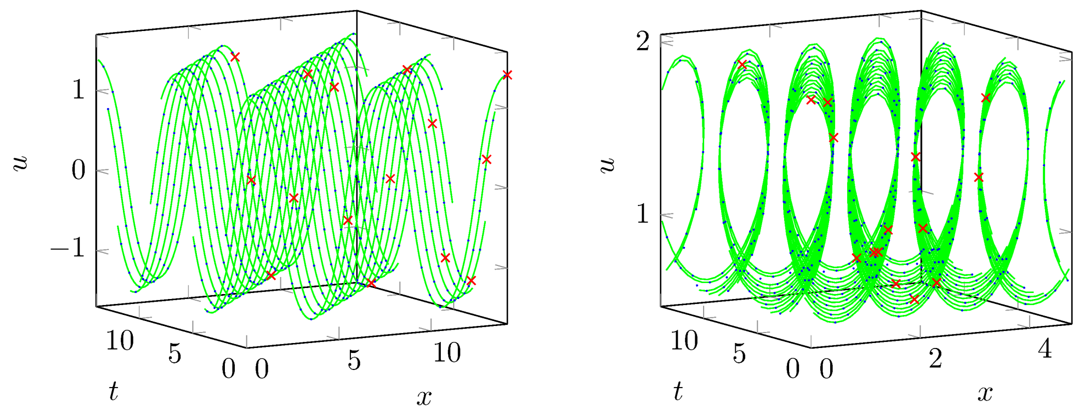
$$\begin{cases} x(\tau,s) = -v\tau + x_0 - \xi\alpha^2 s + \dfrac{1}{\alpha}\tau + \dfrac{2}{\xi\alpha}E\left(\alpha s - \dfrac{\tau}{\xi\alpha^2}\,\middle|\,\xi\right), \\ u(\tau,s) = \pm\dfrac{2}{\xi\alpha}\mathrm{dn}\left(\alpha s - \dfrac{\tau}{\xi\alpha^2}\,\middle|\,\xi\right), \end{cases} \tag{13}$$

where $v > 0$ represents the velocity of the traveling-wave, $\xi \in (0,1)$ denotes the parameter related to the shape of the wave, and $\alpha := \sqrt{(2-\xi)/(\xi^2 v)}$. In view of the periodicity, we set $S = 8K(\xi)/\alpha$, where $K(\xi)$ is the complete elliptic integral of the first kind.

The corresponding initial condition

$$\begin{cases} \rho(0,s) = 1 - 2\,\mathrm{cn}^2\,(\alpha s \mid \xi)), \\ u(0,s) = \pm\dfrac{2}{\xi\alpha}\,\mathrm{dn}\,(\alpha s \mid \xi)\,, \end{cases}$$

for the CD equation can be obtained from (13). The numerical solution of the proposed method again beautifully reproduces the periodic loops as shown in Fig. 2 (b).



(a) Periodic humps ($\xi = 0.25$) (b) Periodic loops ($\xi = 0.9$)

Figure 2: Numerical solutions of the proposed method: exact periodic solutions ($K = 129, \Delta t = 0.1$). Solid lines represents the numerical solution, blue dots are for every four points, and the red symbol $\times$ represents the point $(x_0^{(m)}, u_0^{(m)})$.

4 Concluding remarks

In this paper, we proposed a numerical scheme for the short pulse equation under the periodic boundary condition, which preserves the discrete norm $\mathcal{H}_\mathrm{d}$ and possesses self-adaptive moving mesh property. Some numerical experiments confirmed the validity and theory of the proposed integrator. However, in some unshown preliminary numerical tests, we observed that the numerical solutions can be sometimes unstable in the long run due to the central difference operator. We are now working on its stabilization, which will be reported elsewhere.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.

[2] J. C. Brunelli. The short pulse hierarchy. *J. Math. Phys.*, 46(12):123507, 9, 2005.

[3] B.-F. Feng, J. Inoguchi, K. Kajiwara, K. Maruno, and Y. Ohta. Discrete integrable systems and hodograph transformations arising from motions of discrete plane curves. *J. Phys. A*, 44:395201 (19pp), 2011.

[4] B.-F. Feng, K. Maruno, and Y. Ohta. Self-adaptive moving mesh schemes for short pulse type equations and their Lax pairs. *Pacific Journal of Mathematics for Industry*, 6(1):1–14, 2014.

[5] D. Furihata. Finite difference schemes for $\frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial x}\right)^\alpha \frac{\delta G}{\delta u}$ that inherit energy conservation or dissipation property. *J. Comput. Phys.*, 156:181–205, 1999.

[6] D. Furihata and T. Matsuo. *Discrete variational derivative method–A structure-preserving numerical method for partial differential equations*. CRC Press, Boca Raton, 2011.

[7] D. Furihata and M. Mori. General derivation of finite difference schemes by means of a discrete variation (in Japanese). *Trans. Japan Soc. Indust. Appl.*, 8(3):317–340, 1998.

[8] H. Kakuhata and K. Konno. Lagrangian, Hamiltonian and conserved quantities for coupled integrable, dispersionless equations. *J. Phys. Soc. Japan*, 65(1):1–2, 1996.

[9] K. Konno and H. Oono. New coupled integrable dispersionless equations. *J. Phys. Soc. Jpn.*, 63(2):377–378, 1994.

[10] Y. Matsuno. Periodic solutions of the short pulse model equation. *J. Math. Phys.*, 49(7):073508, 18, 2008.

[11] K. Oguma, S. Sato, T. Matsuo, and B.-F. Feng. A self-adaptive moving mesh method for the short-pulse equation via its sine-Gordon form. In preparation.

[12] E. J. Parkes. Some periodic and solitary travelling-wave solutions of the short-pulse equation. *Chaos Solitons Fractals*, 38:154–159, 2008.

[13] A. Sakovich and S. Sakovich. The short pulse equation is integrable. *J. Phys. soc. Japan*, 74:239–241, 2005.

[14] A. Sakovich and S. Sakovich. Solitary wave solutions of the short pulse equation. *J. Phys. A*, 39(22):L361–L367, 2006.

[15] T. Schäfer and C. E. Wayne. Propagation of ultra-short optical pulses in cubic nonlinear media. *Phys. D*, 196(1-2):90–105, 2004.

[16] M. Wadati, K. Konnno, and Y. Ichikawa. New integrable nonlinear evolution equations. *J. Phys. soc. Japan*, 47(5):1698–1700, 1979.