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FINITE SIMPLE C^*-ALGEBRAS OF LABELED SPACES

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ABSTRACT. The C^*-algebras of directed graphs are introduced in the 1990s and its study is extended to larger classes of C^*-algebras in many ways, among which is the class of labeled graph C^*-algebras started by Bates and Pask. In this paper we survey some of our recent results on finite labeled graph C^*-algebras.

1. INTRODUCTION

A class of C^*-algebras $C^*(E)$ associated to directed graphs E was introduced in [14, 15]. Cuntz-Krieger algebras are now regarded as graph C^*-algebras of finite graphs (graphs with finitely many vertices and edges). The graph C^*-algebra $C^*(E)$ is the C^*-algebra generated by a universal Cuntz-Krieger E-family consisting of projections $\{p_v\}_{v\in E^0}$ and partial isometries $\{s_e\}_{e\in E^1}$, indexed by the vertex set E^0 and the edge set E^1 of E, which are subject to the relations determined by the graph E. If a graph E has condition (K), a condition on the loop structure of E, it is known [14] that the ideal structure of the C^*-algebra $C^*(E)$ can be fully understood from the graph E itself. Also, if $C^*(E)$ is simple, it must be either AF or purely infinite. Cuntz algebras and simple Cuntz Krieger algebras are standard examples of those simple purely infinite graph C^*-algebras.

By a labeled graph, we mean a graph E with a labeling map $\mathcal{L} : E^1 \to \mathcal{A}$ of E^1 onto the alphabet $\mathcal{A}$. If a set $\mathcal{B} \subset 2^{E^0}$ of vertex subsets satisfies certain conditions (see Chapter 2), we call it an accommodating set and the triple $(E, \mathcal{L}, \mathcal{B})$ a labeled space. With the alphabet $\mathcal{A} = E^1$ and the trivial labeling map $\mathcal{L}_{id} := id : E^1 \to \mathcal{A}$, we have a trivial labeled space $(E, \mathcal{L}_{id}, \mathcal{B})$ associated to a graph E, where $\mathcal{B}$ is the accommodating set of all vertex sets that are either finite or cofinite. To each labeled space $(E, \mathcal{L}, \mathcal{B})$ with some mild conditions, one can associate a C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ generated by a universal family of projections $p_A (A \in \mathcal{B})$ and partial isometries $s_a (a \in \mathcal{A})$ that obey some relations given by the labeled space $(E, \mathcal{L}, \mathcal{B})$. This is a similar but more complicated way to the construction of graph C^*-algebras associated with graphs, and by construction every graph C^*-algebra is the C^*-algebra of the trivial labeled space $(E, \mathcal{L}_{id}, \mathcal{B})$.

As for the simplicity of labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$, it is often enough to check the structure of labeled paths in the labeled space $(E, \mathcal{L}, \mathcal{B})$ as in the case of graph C^*-algebras which was well known back in the 1990s ([2, 7]).

While AF graph C^*-algebras are exactly the C^*-algebras $C^*(E)$ of graphs E with no loops, it is not so clear when a labeled graph C^*-algebra $C^*(E, \mathcal{L}, \mathcal{B})$ is AF. We review the discussion on this problem given in [8] in Section 3 after setting up some notation in Section 2. Then in Section 4 we present the construction (given in [9]) of finite simple labeled graph C^*-algebras that are not AF, which shows that the class of simple labeled graph C^*-algebras is strictly larger than the simple graph C^*-algebras. For this construction we use generalized Morse sequences ω to label the uderlying graph $E_{\mathbb{Z}}$ with the vertices $E^0_{\mathbb{Z}} := \mathbb{Z}$ and the edges $E^1_{\mathbb{Z}} := \{e_n \mid s(e_n) = n,\ r(e_n) = n+1,\ n \in \mathbb{Z}\}$, and show that the C^*-algebras of these labeled graphs $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ are simple and non-AF (with non-zero K_1), but finite admitting unique tracial states.

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2. Preliminaries

2.1. **Labeled spaces.** For notational conventions we refer to [14], [2] and [3]. A (directed) *graph* $E = (E^0, E^1, r, s)$ consists of a countable set of vertices E^0, a countable set of edges E^1, and the range, source maps $r,\ s : E^1 \to E^0$. E^n denotes the set of all finite paths $\lambda = \lambda_1 \cdots \lambda_n$ of *length* n ($|\lambda| = n$). We write $E^{\leq n}$ and $E^{\geq n}$ for the sets $\cup_{i=1}^{n} E^i$ and $\cup_{i=n}^{\infty} E^i$, respectively. The maps r and s naturally extend to $E^{\geq 0}$, where $r(v) = s(v) = v$ for $v \in E^0$. One can consider an infinite path $\lambda_1\lambda_2\lambda_3\cdots$ with the source $s(\lambda_1\lambda_2\lambda_3\cdots) := s(\lambda_1)$ if $r(\lambda_i) = s(\lambda_{i+1})$ for all i, and by E^∞ we denote the set of all infinite paths. For a vertex subset $A \subset E^0$, $A_{\rm sink}$ denotes the sinks $A \cap E^0_{\rm sink}$ in A, and for $\mathcal{B} \subset 2^{E_0}$, we simply denote the set $\{A_{\rm sink} : A \in \mathcal{B}\}$ by $\mathcal{B}_{\rm sink}$. For $\mathcal{B} \subset 2^{E_0}$ and $A \subset E_0$, with abuse of notation, we write

$$\mathcal{B} \cap A := \{B \in \mathcal{B} :\ B \subset A\}.$$

A *labeled graph* $(E, \mathcal{L})$ over a countable alphabet $\mathcal{A}$ consists of a graph E and a *labeling map* $\mathcal{L} : E^1 \to \mathcal{A}$. For $\lambda = \lambda_1 \cdots \lambda_n \in E^{\geq 1}$, we call $\mathcal{L}(\lambda) := \mathcal{L}(\lambda_1)\cdots\mathcal{L}(\lambda_n)$ a (*labeled*) *path*, and will use notation $\mathcal{L}^*(E) := \mathcal{L}(E^{\geq 1})$. Similarly we can define an infinite labeled path $\mathcal{L}(\lambda)$ for $\lambda \in E^\infty$. If a path α is of the form $\alpha = \beta \cdots \beta$ for some $\beta \in \mathcal{L}^*(E)$, we call α a *repetition* of β. A labeled graph $(E, \mathcal{L})$ is said to have a *repeatable path* β if $\beta^n := \beta\cdots\beta$(repeated n-times) $\in \mathcal{L}^*(E)$ for all $n \geq 1$. The *range* $r(\alpha)$ and *source* $s(\alpha)$ of $\alpha \in \mathcal{L}^*(E)$ are subsets of E^0 defined by

$$r(\alpha) = \{r(\lambda) : \lambda \in E^{\geq 1},\ \mathcal{L}(\lambda) = \alpha\},$$
$$s(\alpha) = \{s(\lambda) : \lambda \in E^{\geq 1},\ \mathcal{L}(\lambda) = \alpha\}.$$

The *relative range of* $\alpha \in \mathcal{L}^*(E)$ *with respect to* $A \subset 2^{E^0}$ is defined to be

$$r(A, \alpha) = \{r(\lambda) : \lambda \in E^{\geq 1},\ \mathcal{L}(\lambda) = \alpha,\ s(\lambda) \in A\}.$$

We denote the subpath $\alpha_i \cdots \alpha_j$ of $\alpha = \alpha_1\alpha_2\cdots\alpha_{|\alpha|} \in \mathcal{L}^*(E)$ by $\alpha_{[i,j]}$ for $1 \leq i \leq j \leq |\alpha|$. A subpath of the form $\alpha_{[1,j]}$ is called an *initial path* of α. The symbol ϵ is regarded as an initial path of every path.

Let $\mathcal{B} \subset 2^{E^0}$ be a collection of subsets of E^0. If $r(A, \alpha) \in \mathcal{B}$ for all $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$, $\mathcal{B}$ is said to be *closed under relative ranges* for $(E, \mathcal{L})$. We call $\mathcal{B}$ an *accommodating set* for $(E, \mathcal{L})$ if it is closed under relative ranges, finite intersections and unions and contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$. The triple $(E, \mathcal{L}, \mathcal{B})$ is called a *labeled space* when $\mathcal{B}$ is accommodating for $(E, \mathcal{L})$.

For $A, B \in 2^{E^0}$ and $n \geq 1$, let

$$AE^n = \{\lambda \in E^n : s(\lambda) \in A\},\quad E^nB = \{\lambda \in E^n : r(\lambda) \in B\}.$$

We write E^nv for $E^n\{v\}$ and vE^n for $\{v\}E^n$, and will use notations like $AE^{\geq k}$ and vE^∞ which should have their obvious meaning. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is *set-finite* (*receiver set-finite*, respectively) if for every $A \in \mathcal{B}$ and $l \geq 1$ the set $\mathcal{L}(AE^l)$ ($\mathcal{L}(E^lA)$, respectively) is finite. A labeled space $(E, \mathcal{L}, \mathcal{B})$ is *finite* if there are only finitely many labels.

We call $(E, \mathcal{L}, \mathcal{B})$ *weakly left-resolving* (*left-resolving*, respectively) if

$$r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$$

for all $A, B \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$ ($\mathcal{L} : r^{-1}(v) \to \mathcal{A}$ is injective for each $v \in E^0$, respectively). Every left-resolving labeled space is weakly left-resolving.

Assumptions. We assume that a labeled space $(E, \mathcal{L}, \mathcal{B})$ considered in this paper always satisfies the following:

(i) $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving.
(ii) $(E, \mathcal{L}, \mathcal{B})$ is set-finite and receiver set-finite.

For $v, w \in E^0$, we write $v \sim_l w$ if $\mathcal{L}(E^{\leq l}v) = \mathcal{L}(E^{\leq l}w)$ as in [2]. Then $\sim_l$ defines an equivalence relation on E^0, and the equivalence class $[v]_l$ of v is called a *generalized vertex*. If $k > l$, $[v]_k \subset [v]_l$ is obvious and $[v]_l = \cup_{i=1}^m [v_i]_{l+1}$ for some vertices $v_1, \ldots, v_m \in [v]_l$ ([2, Proposition 2.4]).

Notation 2.1. Let $(E, \mathcal{L})$ be a labeled graph.

(i) For a labeled space $(E, \mathcal{L}, \mathcal{B})$, we denote by $\overline{\mathcal{B}}$ the smallest accommodating set that contains $\mathcal{B} \cup \mathcal{B}_{\text{sink}}$ and is *normal* (closed under relative complements). The existence of $\overline{\mathcal{B}}$ follows clearly from considering the intersection of all those accommodating sets. $\overline{\mathcal{E}}$ will denote the smallest accommodating set that is closed under relative complements and contains the sets in $\{r(\alpha) : \alpha \in \mathcal{L}^*(E)\}$.

(ii) $\mathcal{L}^{\#}(E)$ will denote the union of all labeled paths $\mathcal{L}^*(E)$ and empty word ϵ, where ϵ is a symbol such that $r(\epsilon) = E^0$, $r(A, \epsilon) = A$ for all $A \subset E^0$.

Proposition 2.2. ([2, Remark 2.1 and Proposition 2.4.(ii)] and [8, Proposition 2.3]) *Let $(E, \mathcal{L})$ be a labeled graph. Then $A \in \overline{\mathcal{E}}$ is of the form*

$$A = \big(\cup_{i=1}^{n_1} [v_i]_l\big) \cup \big(\cup_{j=1}^{n_2} ([u_j]_l)_{\text{sink}}\big) \cup \big(\cup_{k=1}^{n_3} [w_k]_l \setminus ([w_k]_l)_{\text{sink}}\big)$$

for some $v_i, u_j, w_k \in \Omega_0(E) := E^0 \setminus \{$source vertices$\}$ and $l \geq 1$, $n_1, n_2, n_3 \geq 0$. If $(E, \mathcal{L})$ has no sinks and sources, $\overline{\mathcal{E}}$ contains all generalized vertices; moreover every $A \in \overline{\mathcal{E}}$ is a finite union of generalized vertices, that is $A = \cup_{i=1}^n [v_i]_l$ for some $v_i \in E^0$, $l \geq 1$, and $n \geq 1$.

2.2. **Labeled graph C^*-algebras.**

Definition 2.3. ([1, Definition 4.1], [2, Remark 3.2], [3, Definition 2.1], and [8, Definition 2.4]) Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space such that $\overline{\mathcal{E}} \subset \mathcal{B}$. A *representation* of $(E, \mathcal{L}, \mathcal{B})$ consists of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ such that for $A, B \in \mathcal{B}$ and $a, b \in \mathcal{A}$,

(i) $p_\emptyset = 0$, $p_A p_B = p_{A\cap B}$, and $p_{A\cup B} = p_A + p_B - p_{A\cap B}$,
(ii) $p_A s_a = s_a p_{r(A,a)}$,
(iii) $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$,
(iv) for each $A \in \mathcal{B}$,

$$p_A = \sum_{a\in\mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^* + p_{A_{\text{sink}}}.$$

By $C^*(p_A, s_a)$ we denote the C^*-algebra generated by $\{s_a, p_A : a \in \mathcal{A},\ A \in \mathcal{B}\}$.

Remark 2.4. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space such that $\overline{\mathcal{E}} \subset \mathcal{B}$.

(i) There exists a C^*-algebra generated by a universal representation $\{s_a, p_A\}$ of $(E, \mathcal{L}, \mathcal{B})$ (see the proof of [1, Theorem 4.5] and [7, Remark 2.5]). If $\{s_a, p_A\}$ is a universal representation of $(E, \mathcal{L}, \mathcal{B})$, we call $C^*(s_a, p_A)$, denoted $C^*(E, \mathcal{L}, \mathcal{B})$, the *labeled graph C^*-algebra* of $(E, \mathcal{L}, \mathcal{B})$. Note that $s_a \neq 0$ and $p_A \neq 0$ for $a \in \mathcal{A}$ and $A \in \mathcal{B}$, $A \neq \emptyset$, and that $s_\alpha p_A s_\beta^* \neq 0$ if and only if $A \cap r(\alpha) \cap r(\beta) \neq \emptyset$. By definition of representation and [1, Lemma 4.4],

$$C^*(E, \mathcal{L}, \mathcal{B}) = \overline{span}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^{\#}(E),\ A \in \mathcal{B}\}, \tag{1}$$

where s_ϵ is regarded as the unit of the multiplier algebra of $C^*(E, \mathcal{L}, \mathcal{B})$.

(ii) Universal property of $C^*(E,\mathcal{L},\mathcal{B}) = C^*(s_a,p_A)$ defines a strongly continuous action $\gamma:\mathbb{T}\to Aut(C^*(E,\mathcal{L},\mathcal{B}))$ such that
$$\gamma_z(s_a)=zs_a \ \text{ and } \ \gamma_z(p_A)=p_A$$
for $a\in\mathcal{L}(E^1)$ and $A\in\mathcal{B}$, which we call the *gauge action.*

(iii) The fixed point algebra of the gauge action γ is equal to
$$C^*(E,\mathcal{L},\mathcal{B})^\gamma=\overline{\operatorname{span}}\{s_\alpha p_A s_\beta^* : |\alpha|=|\beta|,\ A\in\mathcal{B}\}, \tag{2}$$
and it is an AF algebra. Moreover, since $\mathbb{T}$ is a compact group, there exists a faithful conditional expectation
$$\Psi: C^*(E,\mathcal{L},\mathcal{B})\to C^*(E,\mathcal{L},\mathcal{B})^\gamma.$$

(iv) From Definition 2.3(iv), we have for each $n\geq 1$,
$$p_A=\sum_{\alpha\in\mathcal{L}(AE^n)} s_\alpha p_{r(A,\alpha)}s_\alpha^* + \sum_{\gamma\in\mathcal{L}(AE^{\leq n-1})} s_\gamma p_{r(A,\gamma)_{\rm sink}} s_\gamma^*,$$
where $\sum_{\gamma\in\mathcal{L}(AE^0)} s_\gamma p_{r(A,\gamma)_{\rm sink}} s_\gamma^* := p_{A_{\rm sink}}$.

Recall [2, 7] that for a labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$, a path $\alpha\in\mathcal{L}([v]_l E^{\geq 1})$ is *agreeable* for a generalized vertex $[v]_l$ if $\alpha=\beta^k\beta'$ for some $\beta\in\mathcal{L}([v]_l E^{\leq l})$ and its initial path β', and $k\geq 1$. A labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$ is said to be *disagreeable* if every $[v]_l$, $l\geq 1$, $v\in E^0$, is disagreeable in the sense that there is an $N\geq 1$ such that for all $n\geq N$ there is a path $\alpha\in\mathcal{L}([v]_l E^{\geq n})$ which is not *agreeable.*

Remark 2.5. If $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable, every representation $\{s_a,p_A\}$ such that $p_A\neq 0$ for all non-empty set $A\in\overline{\mathcal{E}}$ gives rise to a C^*-algebra $C^*(s_a,p_A)$ isomorphic to $C^*(E,\mathcal{L},\overline{\mathcal{E}})$ ([2, Theorem 5.5] and [?, Corollary 2.5]). A labeled space $(E,\mathcal{L},\overline{\mathcal{E}})$ is disagreeable if there is no repeatable paths in $(E,\mathcal{L})$ ([8, Proposition 4.12]).

2.3. **K-theory of labeled graph C^*-algebras.** K-theory of labeled graph C^*-algebras was obtained in [3]. Let E have no sinks and $(E,\mathcal{L},\mathcal{B})$ be a normal labeled space. Then the set $\mathcal{B}_J$ given in (2) of [3] coincides with $\mathcal{B}$, and by [3, Theorem 4.4] the linear map $(1-\Phi):\operatorname{span}_{\mathbb{Z}}\{\chi_A : A\in\mathcal{B}\}\to\operatorname{span}_{\mathbb{Z}}\{\chi_A : A\in\mathcal{B}\}$ given by
$$(1-\Phi)(\chi_A)=\chi_A-\sum_{a\in\mathcal{L}(AE^1)}\chi_{r(A,a)},\ \ A\in\mathcal{B} \tag{3}$$
determines the K-groups of $C^*(E,\mathcal{L},\mathcal{B})$ as follows:
$$K_0(C^*(E,\mathcal{L},\mathcal{B}))\cong\operatorname{span}_{\mathbb{Z}}\{\chi_A : A\in\mathcal{B}\}/\operatorname{Im}(1-\Phi) \tag{4}$$
$$K_1(C^*(E,\mathcal{L},\mathcal{B}))\cong\ker(1-\Phi). \tag{5}$$
In (4), the isomorphism is given by $[p_A]_0\mapsto\chi_A+\operatorname{Im}(1-\Phi)$ for $A\in\mathcal{B}$.

2.4. **Generalized Morse sequences.** We review from [10] definitions and basic properties of (generalized) Morse sequences. Let
$$\Omega:=\{\omega=\cdots\omega_{-2}\omega_{-1}\omega_0\omega_1\cdots : \omega_i\in\{0,1\},\ i\in\mathbb{Z}\}$$
be the space of all two-sided sequences of zeros and ones, and let
$$\Omega_+:=\{x=x_0x_1\cdots : x_i\in\{0,1\},\ i\geq 0\}$$
the space of one-sided sequences. $\mathfrak{B}$ denotes the set of all finite blocks (finite sequences) of zeros and ones. For $b=b_0\cdots b_n\in\mathfrak{B}$, its *length* is $|b|:=n+1$. For $\omega\in\Omega$ ($x\in\Omega_+$, respectively), the

set of all finite blocks appearing in ω (x, respectively) will be denoted by $\mathfrak{B}_\omega$ ($\mathfrak{B}_x$, respectively). For $x \in \Omega_+$, the set of all two-sided sequences ω such that $\mathfrak{B}_\omega \subset \mathfrak{B}_x$ is denoted by

$$\mathcal{O}_x := \{\omega \in \Omega : \mathfrak{B}_\omega \subset \mathfrak{B}_x\}.$$

For $\omega \in \Omega$, we write $\omega_{[t_1,t_2]} := \omega_{t_1} \cdots \omega_{t_2} \in \mathfrak{B}_\omega$ which is a block at the position t_1 ($t_1 \leq t_2$) of ω. Similarly, $\omega_{[t_1,\infty)}$ and $\omega_{(-\infty,t_2]}$ mean the infinite sequences $\omega_{t_1}\omega_{t_1+1}\cdots$ and $\cdots\omega_{t_2-1}\omega_{t_2}$, respectively.

The space Ω (and similarly Ω_+) endowed with the product topology becomes a totally disconnected compact Hausdorff space such that the clopen *cylinder sets*

$${}_t[b] := \{\omega \in \Omega : \omega_{[t,t+n]} = b\},$$

$t \in \mathbb{Z}$, $b \in \mathfrak{B}$, $|b| = n+1 \geq 1$, form a base for the topology. For convenience, we use the following notation:

$$[.b] := {}_0[b], \quad [b.] := {}_{-|b|}[b], \quad [b.c] := {}_{-|b|}[bc]$$

for $b, c \in \mathfrak{B}$. Note that on the right side of the dot is the zeroth position.

The *shift map*

$$T : \Omega \to \Omega \ \text{ given by } \ (T\omega)_i = \omega_{i+1},$$

$\omega \in \Omega$, $i \in \mathbb{Z}$, is easily seen to be a homeomorphism. For $\omega \in \Omega$, the closure of the orbit of ω will be denoted by $\mathcal{O}_\omega := \overline{\{T^i(\omega) : i \in \mathbb{Z}\}} \subset \Omega$.

Each block $b \in \mathfrak{B}$ defines a block $\tilde{b}$, the *mirror image* of b, such that $\tilde{b}_i = b_i + 1 \pmod 2$. For $c = c_0 \cdots c_n \in \mathfrak{B}$, the product $b \times c$ of b and c denotes the block (of length $|b| \times |c|$) obtained by putting $n+1$ copies of either b or $\tilde{b}$ next to each other according to the rule of choosing the ith copy as b if $c_i = 0$ and $\tilde{b}$ if $c_i = 1$.

Let $\{b^i := b^i_0 \cdots b^i_{|b^i|-1}\}_{i\geq 1} \subset \mathfrak{B}$ be a sequence of blocks with length $|b^i| \geq 2$ such that $b^i_0 = 0$ for all $i \geq 0$. Then one can consider a (one-sided) *recurrent* sequence of the form

$$x = b^0 \times b^1 \times b^2 \times \cdots \in \Omega_+$$

(see [10, Definition 7]). We call such an $x = b^0 \times b^1 \times b^2 \times \cdots \in \Omega_+$ a *(generalized) one-sided Morse sequence* if it is non-periodic and $\sum_{i=0}^{\infty} \min(r_0(b^i), r_1(b^i)) = \infty$, where $r_a(b)$ is the *relative frequency of occurrence* of a ($a = 0$ or 1) in $b \in \mathfrak{B}$ (see [10, p.338]).

Recall that $\mathcal{O}_\omega$ is *uniquely ergodic* if $\mathcal{O}_\omega$ admits exactly one T-invariant probability measure m_ω. Such a unique measure is automatically ergodic.

Theorem 2.6. ([10, Lemma 2, Lemma 4, Theorem 3]) *Let $x \in \Omega_+$ be a non-periodic recurrent sequence. Then we have the following:*

(i) *x is almost periodic; for any cylinder set $[.b]$, $b \in \mathfrak{B}_x$, there exists $d \geq 1$ such that for any $n \geq 0$, $T^{n+j}x \in [.b]$ for some $0 \leq j \leq d$.*
(ii) *There exists $\omega \in \mathcal{O}_x$ with $x = \omega_{[0,\infty)}$. Moreover, x is a one-sided Morse sequence if and only if $\mathcal{O}_\omega$ is minimal and uniquely ergodic, and if this is the case, then $\mathcal{O}_\omega = \mathcal{O}_x$.*

Definition 2.7. By a *generalized Morse sequence*, we mean a two-sided sequence $\omega \in \Omega$ such that $x := \omega_{[0,\infty)}$ is a one-sided Morse sequence and $\mathfrak{B}_\omega = \mathfrak{B}_x$.

Remark 2.8. For a generalized Morse sequence ω, the unital commutative AF algebra $C(\mathcal{O}_\omega)$ of all continuous functions on $\mathcal{O}_\omega$ admits a (tracial) state

$$f \mapsto \int_{\mathcal{O}_\omega} f \mathrm{d}m_\omega : C(\mathcal{O}_\omega) \to \mathbb{C}$$

which we also write m_ω. Since m_ω is T-invariant, it easily follows that $m_\omega(\chi_{t[b]}) = m_\omega(\chi_{t[b]}\circ T) = m_\omega(\chi_{t+1[b]})$, and hence

$$m_\omega(\chi_{t[b]}) = m_\omega(\chi_{[.b]}) \tag{6}$$

holds for all $t \in \mathbb{Z}$ and $b \in \mathfrak{B}_\omega$.

Example 2.9. **(Thue-Morse sequence)** Let $b^i := 01 \in \mathfrak{B}$ for all $i \geq 0$. Then the recurrent sequence

$$x := b^0 \times b^1 \times b^2 \times \cdots = 01 \times b^1 \times \cdots = 0110 \times b^2 \times \cdots = 01101001 \times b^3 \times \cdots$$

is a one-sided Morse sequence called the Thue-Morse sequence and

$$\omega := x^{-1}.x = \cdots 10010110.011010011001 \cdots \in \mathcal{O}_x$$

is a generalized Morse sequence, where $x^{-1} := \cdots x_2x_1x_0$ is the sequence obtained by writing $x = x_0x_1\cdots$ in reverse order. In fact, ω is the sequence constructed from the proof of Theorem 2.6(ii) (see [10, Lemma 4]), and it is well known [6] that ω has no blocks of the form bbb_0 for any $b = b_0\cdots b_{|b|-1} \in \mathfrak{B}_\omega$.

3. AF LABELED GRAPH C^*-ALGEBRAS

Recall that a path $x \in E^{\geq 1}$ in a directed graph E is a *loop* if $s(x) = r(x)$. It is well known [14, Theorem 2.4] that for a graph C^*-algebra $C^*(E)$ to be AF it is a sufficient and necessary condition that E has no loops. To find conditions of a labeled space which arises an AF C^*-algebras, we define following generalized notion of loop.

Definition 3.1. Let $(E, \mathcal{L}, \mathcal{B})$ be a labeled space and $\alpha \in \mathcal{L}^*(E)$ a labeled path.

(a) α is a *generalized loop* at $A \in \mathcal{B}$ if $\alpha \in \mathcal{L}(AE^{\geq 1}A)$.
(b) α is a *loop* at $A \in \mathcal{B}$ if it is a generalized loop such that $A \subset r(A, \alpha)$.
(c) A loop α at $A \in \mathcal{B}$ has an *exit* if one of the following holds:
    (i) $\{\alpha_{[1,k]} : 1 \leq k \leq |\alpha|\} \subsetneq \mathcal{L}(AE^{\leq|\alpha|})$,
    (ii) $r(A, \alpha_{[1,i]})_{\text{sink}} \neq \emptyset$ for some $i = 1, \ldots, |\alpha|$,
    (iii) $A \subsetneq r(A, \alpha)$.

Remark 3.2. Let (s_a, p_A) be a representation of $(E, \mathcal{L}, \mathcal{B})$.

(i) A generalized loop α at a minimal set $A \in \mathcal{B}$ is necessarily a loop. A labeled graph $(E, \mathcal{L})$ might have a (generalized) loop α even when the underlying graph E has no loops at all.
(ii) If α is a loop at $A \in \mathcal{B}$ then $p_A \leq p_{r(A,\alpha)}$.

Proposition 3.3. *Let $(E, \mathcal{L})$ be a labeled graph and α be a loop at $A \in \overline{\mathcal{E}}$ with an exit. Then $p_{r(A,\alpha)}$ is an infinite projection in $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$.*

Theorem 3.4. *If $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is an AF algebra, the labeled space $(E, \mathcal{L}, \overline{\mathcal{E}})$ has no loops.*

Since the accommodating set $\overline{\mathcal{E}}$ of a labled graph $(E, \mathcal{L}_{id})$ with the trivial labeling $\mathcal{L}_{id}$ contains all the single vertex sets $\{v\}$, $v \in E^0$, the following are equivalent for a path $x = x_1\cdots x_m \in E^{\geq 1}(= \mathcal{L}^*_{id}(E))$:

(i) x is a loop in E,
(ii) $\{r(x)\} = r(\{r(x)\}, x)$,

(iii) x is *repeatable*, that is, $x^n \in E^{\geq 1}$ for all $n \geq 1$,
(iv) $(A_1x_1A_2x_2\cdots A_mx_m)^n(A_1x_1A_2x_2\cdots A_ix_i) \in \mathcal{L}^*_{id}(E)$ for all $n \geq 1$ and $1 \leq i \leq m$, where $A_i = \{s(x_i)\} \in \overline{\mathcal{E}}$.

From this we can obtain several equivalent conditions for a graph C^*-algebra $C^*(E)$ to be AF as follows.

Proposition 3.5. *Let $(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$ be a labeled space with the trivial labeling $\mathcal{L}_{id}$ so that $C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}}) \cong C^*(E)$. Then the following are equivalent:*
(i) *$C^*(E, \mathcal{L}_{id}, \overline{\mathcal{E}})$ is AF,*
(ii) *E has no loops,*
(iii) *$A \not\subset r(A, x)$ for all $A \in \overline{\mathcal{E}}$ and $x \in \mathcal{L}^*_{id}(E)$,*
(iv) *there are no repeatable paths in $\mathcal{L}^*_{id}(E)$,*
(v) *if $\{A_1, \ldots, A_m\}$ is a finite collection of sets from $\overline{\mathcal{E}^0}$ and $K \geq 1$, there is an $m_0 \geq 1$ such that $A_{i_1}E^{\leq K}A_{i_2}\cdots E^{\leq K}A_{i_{n+1}} = \emptyset$ for all $n > m_0$.*

Let $A_1E^{\geq 1}A_2\cdots E^{\geq 1}A_{n+1}$ denote the following set

$$\{x = x_1x_2\cdots x_n \in E^{\geq 1} : x_k \in A_kE^{\geq 1}A_{k+1},\ 1 \leq k \leq n\}.$$

Theorem 3.6. *Let $(E, \mathcal{L})$ be a labeled graph. Assume that if $A_1, A_2, \ldots$ is a sequence of sets in $\overline{\mathcal{E}}$ such that*

$$A_1E^{\geq 1}A_2E^{\geq 1}A_3\cdots E^{\geq 1}A_n \neq \emptyset$$

for all $n \geq 1$, the set $\{A_1, A_2, \ldots\}$ is infinite. Then $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is AF.

For a labeled graph C^*-algebra $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ and a set $A \in \overline{\mathcal{E}}$, we denote by I_A the ideal of $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ generated by the projection p_A as before.

Lemma 3.7. *Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ be the C^*-algebra of a labeled graph $(E, \mathcal{L})$ with no sinks or sources. For $A, B \in \overline{\mathcal{E}}$, we have $p_A \in I_B$ if and only if there exist an $N \geq 1$ and finitely many paths $\{\mu_i\}_{i=1}^n$ in $\mathcal{L}(BE^{\geq 0})$ such that*

$$\cup_{|\beta|=N}\, r(A, \beta) \subset \cup_{i=1}^n r(B, \mu_i).$$

Lemma 3.8. *Let $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(s_a, p_A)$ be the C^*-algebra of a labeled graph $(E, \mathcal{L})$ and let $\alpha \in \mathcal{L}^*(E)$ satisfy $\alpha^n \in \mathcal{L}^*(E)$ for all $n \geq 1$. If $p_{r(\alpha^m)}$ does not belong to the ideal generated by a projection $p_{r(\alpha^m)\setminus r(\alpha^{m+1})}$ for some $m \geq 1$, then $C^*(E, \mathcal{L}, \overline{\mathcal{E}})$ is not AF.*

Recall that the set $[v]_{l_{\text{sink}}}$ of all sinks of $[v]_l$ is a member of $\overline{\mathcal{E}}$ and that $\overline{\mathcal{E}} \cap [v]_{l_{\text{sink}}}$ denotes the set $\{A \in \overline{\mathcal{E}} : A \subset [v]_{l_{\text{sink}}}\}$. The ideal $I_{[v]_{l_{\text{sink}}}}$ of $C^*(E, \mathcal{L}, \overline{\mathcal{E}}) = C^*(p_A, s_a)$ generated by the projection $p_{[v]_{l_{\text{sink}}}}$ is equal to

$$\overline{\text{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^\#(E) \text{ and } A \in \overline{\mathcal{E}} \cap [v]_{l_{\text{sink}}}\}.$$

Lemma 3.9. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a labeled space and $v \in \Omega_0(E)$. If $[v]_{l_{\text{sink}}}$ is the disjoint union of finitely many minimal sets $A_i \in \overline{\mathcal{E}}$, $i = 1, \ldots, N$,*

$$I_{[v]_{l_{\text{sink}}}} = \oplus_{i=1}^N \overline{\text{span}}\{s_\alpha p_{A_i} s_\beta^* : \ \alpha, \beta \in \mathcal{L}^\#(E)\} \cong \oplus_{i=1}^N \mathcal{K}(\ell^2(\mathcal{L}(E^{\geq 0}A_i))),$$

where $\mathcal{L}(E^0A_i) := \{\epsilon\}$.

Proposition 3.10. *Let $(E, \mathcal{L}, \overline{\mathcal{E}})$ be a finite labeled space such that there exists an $l \geq 1$ for which $(E, \mathcal{L}, \overline{\mathcal{E}})$ has no generalized loops at $[v]_l$ for all $[v]_l \in \Omega_l(E)$. Then*

$$C^*(E, \mathcal{L}, \overline{\mathcal{E}}) \cong \oplus_{[v]_l \in \Omega_l(E)} I_{[v]_{l_{\rm sink}}}.$$

Moreover, the ideal $I_{[v]_{l_{\rm sink}}}$ is finite dimensional whenever $\overline{\mathcal{E}} \cap [v]_{l_{\rm sink}}$ is a finite set.

4. Non-AF finite simple labeled graph C^*-algebras

Recall C^*-algebra is said to be *infinite* if it has an infinite projection. A unital C^*-algebra $A(\neq \mathbb{C})$ is *purely infinite* if for each nonzero positive element $a \in A$ there is a $b \in A$ satisfying $b^*ab = 1$. A purely infinite C^*-algebra A is always simple since the ideal generated by any positive nonzero element contains the unit of A. (For nonsimple purely infinite C^*-algebras see [11, 12].) It is an easy observation that a simple unital C^*-algebra A is purely infinite if and only if every nonzero hereditary C^*-subalgebra $\overline{aAa}$ of A has a projection $a^{1/2}b(a^{1/2}b)^*$ equivalent to the unit $1 = (a^{1/2}b)^*(a^{1/2}b)$. Thus if A is purely infinite, every nonzero projection is always infinite. A simple C^*-algebra without unit is called purely infinite if every nonzero hereditary C^*-subalgebra contains an infinite projection.

We call a C^*-algebra A *finite* when A has no infinite projections. A simple unital C^*-algebra A with a tracial state τ (τ is automatically faithful since A is simple) is always finite because the faithfulness of τ implies that if a projection $p \in A$ is equivalent to its subprojection $q \leq p$ in A, with $p = vv^*$ and $q = v^*v$ for $v \in A$, then $\tau(p-q) = \tau(vv^* - v^*v) = 0$ and so $p - q = 0$ by faithfulness of τ.

Besides commutative C^*-algebras, all finite dimensional C^*-algebras are obviously finite, and moreover all AF algebras are also finite. On the other hand, the Cuntz-algebras $\mathcal{O}_n$ ($n = 2, 3, \ldots, \infty$) [4] or more generally simple Cuntz-Krieger algebras are well known to be purely infinite.

In [2, Proposition 7.2], Bates and Pask provide an example of a simple unital purely infinite labeled graph C^*-algebra which is not isomorphic to any unital graph C^*-algebra. We also know from [16] that there exist simple higher rank graph C^*-algebras which are neither AF nor purely infinite; there exist such simple C^*-algebras which are stably isomorphic to irrational rotation algebras or Bunce-Deddens algebras. This fact leads us to ask if there exists a simple unital labeled graph C^*-algebra which is neither AF nor purely infinite. To this question we answer in Theorem 4.4 that there really exists a simple unital finite, but non-AF labeled graph C^*-algebra $C^*(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$. This is a C^*-algebra associated to a labeled space $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ which is labeled by a generalized Morse sequence ω.

Throughout this section, $E_\mathbb{Z}$ will denote the following graph:

$$\cdots \underset{v_{-4}}{\bullet} \xrightarrow{-4} \underset{v_{-3}}{\bullet} \xrightarrow{-3} \underset{v_{-2}}{\bullet} \xrightarrow{-2} \underset{v_{-1}}{\bullet} \xrightarrow{-1} \underset{v_{0}}{\bullet} \xrightarrow{0} \underset{v_{1}}{\bullet} \xrightarrow{1} \underset{v_{2}}{\bullet} \xrightarrow{2} \underset{v_{3}}{\bullet} \xrightarrow{3} \underset{v_{4}}{\bullet} \cdots .$$

Given a two-sided sequence $\omega = \cdots \omega_{-1}\omega_0\omega_1 \cdots \in \Omega$ of zeros and ones, we obtain a labeled graph $(E_\mathbb{Z}, \mathcal{L}_\omega)$ shown below

$$(E_\mathbb{Z}, \mathcal{L}_\omega) \quad \cdots \underset{v_{-4}}{\bullet} \xrightarrow{\omega_{-4}} \underset{v_{-3}}{\bullet} \xrightarrow{\omega_{-3}} \underset{v_{-2}}{\bullet} \xrightarrow{\omega_{-2}} \underset{v_{-1}}{\bullet} \xrightarrow{\omega_{-1}} \underset{v_{0}}{\bullet} \xrightarrow{\omega_{0}} \underset{v_{1}}{\bullet} \xrightarrow{\omega_{1}} \underset{v_{2}}{\bullet} \xrightarrow{\omega_{2}} \underset{v_{3}}{\bullet} \xrightarrow{\omega_{3}} \underset{v_{4}}{\bullet} \cdots ,$$

where the labeling map $\mathcal{L}_\omega : E^1_\mathbb{Z} \to \{0, 1\}$ is given by $\mathcal{L}_\omega(n) = \omega_n$ for $n \in E^1_\mathbb{Z}$. Then we also have a labeled space $(E_\mathbb{Z}, \mathcal{L}_\omega, \overline{\mathcal{E}}_\mathbb{Z})$ with the smallest accommodating set $\overline{\mathcal{E}}_\mathbb{Z}$ which is closed under relative complements.

Let $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}}) = C^*(s_a,p_A)$ be the labeled graph C^*-algebra associated with the labeled space $(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ of a generalized Morse sequence ω. Then by (2) the fixed point algebra of the gauge action γ is generated by elements of the form $s_\alpha p_A s_\beta^*$ ($|\alpha|=|\beta|$ and $A\subset r(\alpha)\cap r(\beta)$) which is nonzero only when $\alpha=\beta$, and hence

$$C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma = \overline{\operatorname{span}}\{s_\alpha p_A s_\alpha^* : A\in\overline{\mathcal{E}}_{\mathbb{Z}},\ A\subset r(\alpha)\}.$$

Moreover $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ is easily seen to be a commutative C^*-algebra. For each $k\geq 1$, let

$$F_k := \operatorname{span}\{s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* : \alpha,\alpha'\in\mathcal{L}_\omega(E_{\mathbb{Z}}^k)\}.$$

The (finitely many) elements $s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*$ in F_k are linearly independent and actually orthogonal to each other so that F_k is a finite dimensional subalgebra of $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$. Moreover F_k is a subalgebra of F_{k+1} because

$$s_\alpha p_{r(\alpha'\alpha)} s_\alpha^* = \sum_{b\in\{0,1\}} s_{\alpha b} p_{r(\alpha'\alpha b)} s_{\alpha b}^* = \sum_{a,b\in\{0,1\}} s_{\alpha b} p_{r(a\alpha'\alpha b)} s_{\alpha b}^*.$$

This gives rise to an inductive sequence $F_1 \xrightarrow{\iota_1} F_2 \xrightarrow{\iota_2} \cdots$ of finite dimensional C^*-algebras, where the connecting maps $\iota_k : F_k \to F_{k+1}$ are inclusions for $k\geq 1$, from which we obtain an AF algebra $\varinjlim F_k$. Then

$$C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma = \varinjlim F_k,$$

and thus the fixed point algebra is an AF algebra.

Proposition 4.1. *Let $(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ be the labeled space of a generalized Morse sequence ω. Then there is a surjective isomorphism*

$$\rho : C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \to C(\mathcal{O}_\omega) \tag{7}$$

such that $\rho(s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*) = \chi_{[\alpha'.\alpha]}$ for $s_\alpha p_{r(\alpha'\alpha)} s_\alpha^*\in F_k$, $k\geq 1$.

Lemma 4.2. *Let $(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ be the labeled space of a generalized Morse sequence ω and let $\rho : C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \to C(\mathcal{O}_\omega)$ be the isomorphism in (7). Then the unique T-invariant ergodic measure $m_\omega : C(\mathcal{O}_\omega)\to\mathbb{C}$ defines a tracial state*

$$\tau_0 := m_\omega\circ\rho : C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma \to \mathbb{C}$$

on the fixed point algebra $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})^\gamma$ such that for $\alpha,\beta\in\mathcal{L}_\omega^*(E_{\mathbb{Z}})$,

$$\tau_0(s_\alpha p_{r(\beta\alpha)} s_\alpha^*) = \tau_0(p_{r(\beta\alpha)}).$$

The following lemma can be proved by straightforward computation.

Lemma 4.3. *Let $(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ be the labeled space of a generalized Morse sequence ω. Then*

$$\tau_0\circ\Psi : C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}}) \to \mathbb{C}$$

is a tracial state.

Theorem 4.4. *Let ω be a generalized Morse sequence of zeros and ones. Then the C^*-algebra $C^*(E_{\mathbb{Z}},\mathcal{L}_\omega,\overline{\mathcal{E}}_{\mathbb{Z}})$ is*

(i) *simple unital,*
(ii) *non AF,*
(iii) *finite with a unique tracial state τ which satisfies*

$$\tau(s_\alpha p_{r(\sigma\alpha)} s_\beta^*) = \tau(\Psi(s_\alpha p_{r(\sigma\alpha)} s_\beta^*)) = \delta_{\alpha,\beta}\tau(p_{r(\sigma\alpha)})$$

for $\alpha,\beta,\sigma\in\mathcal{L}_\omega^*(E_{\mathbb{Z}})$.

In particular, $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ is not stably isomorphic to a graph C^*-algebra.

Let $\omega \in \Omega$ be a generalized Morse sequence. Then the shift map $T : \mathcal{O}_\omega \to \mathcal{O}_\omega$ induces an automorphism $\sigma_T : C(\mathcal{O}_\omega) \to C(\mathcal{O}_\omega)$, $\sigma_T(f) = f \circ T^{-1}$. In particular, for each $A \in \overline{\mathcal{E}}_{\mathbb{Z}}$ we have

$$\sigma_T(\chi_A) = \chi_A \circ T^{-1} = \chi_{T(A)}.$$

The following can be shown by universal property of the labeled graph C^*-algebra $C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ since one can find a representation of $(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}})$ in the crossed product $C(\mathcal{O}_\omega) \rtimes_{\sigma_T} \mathbb{Z}$. The proof will be contained in the revised version of [9]. Note that $(\mathcal{O}_\omega, T)$ is a Cantor system, so that we can apply the results known in [5] to identify the isomorphism classes of the crossed products.

Theorem 4.5. *Let $\omega \in \Omega$ be a generalized Morse sequence and $T : \mathcal{O}_\omega \to \mathcal{O}_\omega$ be the shift map. There exists an isomorphism*

$$\pi : C^*(E_{\mathbb{Z}}, \mathcal{L}_\omega, \overline{\mathcal{E}}_{\mathbb{Z}}) \to C(\mathcal{O}_\omega) \rtimes_{\sigma_T} \mathbb{Z}.$$

References

[1] T. Bates and D. Pask, *C^*-algebras of labeled graphs*, J. Operator Theory, **57**(2007), 101–120.

[2] T. Bates and D. Pask, *C^*-algebras of labeled graphs II - simplicity results*, Math. Scand. **104**(2009), no. 2, 249–274.

[3] T. Bates, T. M. Carlsen, and D. Pask, *C^*-algebras of labeled graphs III - K-theory computations*, arXiv:1203.3072v2 [math.OA].

[4] J. Cuntz, *K-theory for certain C^*-algebras*, Ann. Math. **113**(1981), no. 1, 181–197.

[5] T. Giordano, I. F. Putnam, and C. F. Skau, *Topological orbit equivalence and C∗-crossed products*, J. reine. angew. Math., **469** (1995), 51-111.

[6] W. H. Gottschalk and G. A. Hedlund, *A characterization of the Morse minimal set*, Proc. Amer. Math. Soc., **15** (1964), 70-74.

[7] J. A Jeong and S. H. Kim, *On simple labeled graph C^*-algebras*, J. Math. Anal. Appl. **386**(2012), 631–640.

[8] J. A Jeong, E. J. Kang and S. H. Kim, *AF labeled graph C^*-algebras*, J. Funct. Anal. **266**(2014), 2153–2173.

[9] J. A Jeong, E. J. Kang, S. H. Kim, and G. H. Park, *Finite simple unital non-AF labeled graph C^*-algebras of generalized Morse sequences*, arXiv:1504.03455.

[10] M. Keane, *Generalized Morse sequences*, Z. Wahrscheinlichkeitstheorie verw. Geb. **10**(1968), 335-353.

[11] E. Kirchberg, M. Rørdam, *Non-simple purely infinite C^*-algebras*, Amer. J. Math. **122**(2000), 637–666.

[12] E. Kirchberg, M. Rørdam, *Infinite Non-simple C^*-algebras: Absorbing the Cuntz Algebra $\mathcal{O}_\infty$*, Adv. in Math. **167**(2002), 195-264.

[13] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6**(2000), 1–20.

[14] A. Kumjian, D. Pask and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184**(1998), 161–174.

[15] A. Kumjian, D. Pask, I. Raeburn and J. Renault *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144**(1997), 505–541.

[16] D. Pask, I. Raeburn, M. Rørdam, and A. Sims, *Rank-two graphs whose C^*-algebras are direct limits of circle algebras*, J. Funct. Anal. **239**(2006), 137–178.

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