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Some examples of operator monotone functions

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1 Introduction and preliminaries

Let f be a real valued continuous function on $(0,\infty)$. We call f n-matrix monotone on $(0,\infty)$ if it holds $f(A)\leq f(B)$, for $n\times n$ self-adjoint matrices A, B with $0\leq A\leq B$, where $A\leq B$ means

$$(A\xi,\xi)\leq(B\xi,\xi) \qquad \forall\xi\in\mathbb{C}^{n}.$$

When f is n-matrix monotone on $(0,\infty)$ for any positive integer $n\in\mathbb{N}$, f is called operator monotone on $(0,\infty)$. By Löwner's theorem, it is known that f is operator monotone on $(0,\infty)$ if and only if f is a Pick function on $(0,\infty)$, which means the function $f:(0,\infty)\longrightarrow\mathbb{R}$ has the analytic continuation $f(z)$ on the upper half plane $\mathbb{H}_{+}=\{z\in\mathbb{C}:\Im z>0\}$ and satisfies the condition $f(\mathbb{H}_{+})$ is contained in the closure of $\mathbb{H}_{+}$ ([1], [3], [7]). For any positive integer $n\in\mathbb{N}$, a real number $\gamma\in\mathbb{R}$, and positive numbers α_{i}, β_{i} $(1\leq i\leq n)$ with $\alpha_{i}\neq\beta_{j}$ $(1\leq i,j\leq n)$, we define the function $f(t)$ on $(0,\infty)$ as follows:

$$f(t)=t^{\gamma}\prod_{i=1}^{n}\frac{\beta_{i}}{\alpha_{i}}\frac{t^{\alpha_{i}}-1}{t^{\beta_{i}}-1} \qquad (t\neq 1)$$

and $f(1)=1$. In [9], the author gave the method to investigate the operator monotonicity of functions $f(t)$. Using this result, we consider the operator monotonicity of the function $f(t)$ with some special form.

In section 2, we treat the following functions related to the power difference mean:

$$h(t)=\frac{b}{a}\frac{t^{a}-1}{t^{b}-1}, \qquad t\in(0,\infty),$$

for any real number a and b. In section 3, we treat the following functions (extended Petz-Hasegawa's functions) :

$$h(t)=\frac{ab(t-1)^{2}}{(t^{a}-1)(t^{b}-1)}, \qquad t\in(0,\infty),$$

for any real number a and b, where we use the notation

$$\frac{t^0-1}{0} = \log t \ (= \lim_{a\to 0} \frac{t^a-1}{a}).$$

We remark that the point-wise limit function $f(t)$ of $\{f_m(t)\}_{m=1}^{\infty}$ is n-matrix monotone if $f_m(t)$ is n-matrix monotone for all m.

Let, for $\gamma \in \mathbb{R}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ with $\alpha_i, \beta_i > 0$,

$$f(t) = t^\gamma \prod_{i=1}^{n} \frac{\beta_i}{\alpha_i} \frac{t^{\alpha_i}-1}{t^{\beta_i}-1} \qquad (t \neq 1)$$

and $f(1) = 1$. We introduce two quantities $F_0(\alpha, \beta)$ and $F(\alpha, \beta)$ for $f(t)$. The following two lemmas related to these quantities are used to determine the operator monotonicity of functions in section 3.

When $0 < \alpha_i, \beta_i \leq 2$, we define

$$\arg \frac{z^{\alpha_i}-1}{z^{\beta_i}-1} = 0 \qquad \text{for } z \in (0, \infty)$$

and continuously define the argument of $\frac{z^{\alpha_i}-1}{z^{\beta_i}-1}$ on $z \in \mathbb{H}_+$. So we can define, for $\alpha_i, \beta_i \leq 2$ $(i = 1, \ldots, n)$,

$$\arg f(z) = \gamma \arg z + \sum_{i=1}^{n} \arg \frac{z^{\alpha_i}-1}{z^{\beta_i}-1}, \qquad \text{for } z \in \mathbb{H}_+,$$

and $\arg f(t) = 0$ for $t \in (0, \infty)$.

If $f(t)$ is non-constant operator monotone, then its analytic continuation $f(z)$ has no zeros and no singular points on $\mathbb{H}_+$ since f is Pick function. It is known (see, [9]:Proposition 3.1) that $f(z)$ has no zeros and no singular points on $\mathbb{H}_+$ if and only if $|\gamma| \leq 2$ and $0 < \alpha_i, \beta_i \leq 2$ $(1 \leq i \leq n)$. When $|\gamma| > 2$ or $\max\{\alpha_i, \beta_i : 1 \leq i \leq n\} > 2$, $f(t)$ is not operator monotone.

Lemma 1.1 ([9]:Theorem 1.1, Lemma 2.3 and Proposition 3.2). *Let* $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i \leq 2$ $(1 \leq i \leq n)$.

(1) $f(t)$ *is operator monotone on* $(0, \infty)$ *if and only if*

$$\gamma + G_0(\alpha, \beta) \geq 0 \text{ and } \gamma + F_0(\alpha, \beta) \leq 1,$$

where we set

$$g(t) = \prod_{i=1}^{n} \frac{\beta_i}{\alpha_i} \frac{t^{\alpha_i}-1}{t^{\beta_i}-1}$$

and define $G_0(\alpha, \beta) = \inf\{\arg g(re^{\pi i}) : r \in (0, \infty)\}/\pi$ *and* $F_0(\alpha, \beta) = \sup\{\arg g(re^{\pi i}) : r \in (0, \infty)\}/\pi$.

(2) $G_0(\alpha,\beta) = -F_0(\beta,\alpha)$ *and* $F_0(\alpha,\beta) \geq 0$.

(3) $F_0(\alpha,\beta) + G_0(\alpha,\beta) = \sum_{i=1}^{n}(\alpha_i - \beta_i)$.

(4) *When* $0 < b < a \leq 2$,

$$0 < b \leq 1 \Leftrightarrow G_0(a,b) \geq 0 \Leftrightarrow F_0(a,b) \leq a-b,$$

where we use the notation a *(resp.* b*) instead of* $\alpha = (a)$ *(resp.* $\beta = (b)$*).*

For $0 \leq a, b \leq 2$, we define

$$F(a,b) = \begin{cases} a-b & \text{if } a \geq b, 0 \leq b \leq 1 \\ a-1 & \text{if } 1 < a, b \leq 2 \\ 0 & \text{if } a < b, 0 \leq a \leq 1 \end{cases}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ $(0 < \alpha_i, \beta_i \leq 2)$ and σ and τ permutations on $\{1, \ldots, n\}$ satisfying with $\alpha_{\sigma(1)} \leq \alpha_{\sigma(2)} \leq \cdots \leq \alpha_{\sigma(n)}$ and $\beta_{\tau(1)} \leq \beta_{\tau(2)} \leq \cdots \leq \beta_{\tau(n)}$. Then we define

$$F(\alpha,\beta) = \sum_{i=1}^{n} F(\alpha_{\sigma(i)}, \beta_{\tau(i)}).$$

Lemma 1.2 ([9]:Theorem 1.2). *For* $|\gamma| \leq 2$, $0 < \alpha_i, \beta_i \leq 2$, *the function*

$$f(t) = t^{\gamma} \prod_{i=1}^{n} \frac{\beta_i}{\alpha_i} \frac{t^{\alpha_i} - 1}{t^{\beta_i} - 1}$$

becomes operator monotone on $(0,\infty)$ *if*

$$\gamma - F(\beta,\alpha) \geq 0 \text{ and } \gamma + F(\alpha,\beta) \leq 1.$$

2 Functions related the power difference mean

The following characterization is well-known:

Lemma 2.1 ([6]:Theorem 2.4.3). *The function* $h : (0,\infty) \longrightarrow \mathbb{R}$ *is* 2*-matrix monotone if and only if* h *is in* $C^1(0,\infty)$ *and*

$$\begin{pmatrix} h^{[1]}(\lambda_1,\lambda_1) & h^{[1]}(\lambda_1,\lambda_2) \\ h^{[1]}(\lambda_2,\lambda_1) & h^{[1]}(\lambda_2,\lambda_2) \end{pmatrix} \geq 0$$

for any $\lambda_1, \lambda_2 \in (0,\infty)$, *where*

$$h^{[1]}(\lambda,\mu) = \begin{cases} \dfrac{h(\lambda) - h(\mu)}{\lambda - \mu} & \lambda \neq \mu \\ h'(\lambda) & \lambda = \mu \end{cases}.$$

Theorem 2.2. *Let a, b be real numbers and*

$$h(t) = \frac{b}{a}\frac{t^a - 1}{t^b - 1}, \qquad t \in (0, \infty).$$

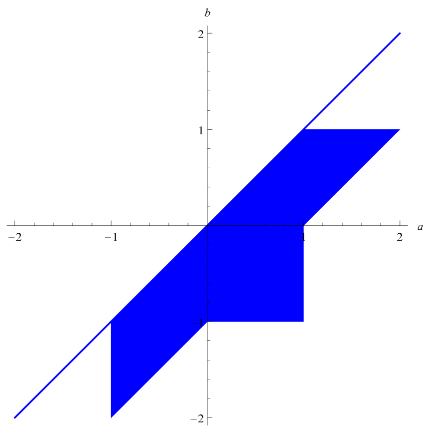
Then we have

(1) *h is increasing on $(0, \infty)$ if $a > b$ and decreasing if $a < b$.*

(2) *h becomes operator monotone on $(0, \infty)$ if and only if the point (a, b) belongs to the set*

$$\Omega = \{a = b\} \cup \{(a, b) : 0 \leq a - b \leq 1, a \geq -1, b \leq 1\} \cup ([0, 1] \times [-1, 0])$$

in the (a, b)-plane:



(3) *h is 2-matrix monotone on $(0, \infty)$ if and only if h is operator monotone on $(0, \infty)$.*

Proof. (1) We set

$$\frac{dh(t)}{dt} = \frac{t^{b-1}}{(t^b - 1)^2} k(t),$$

where $k(t) = \frac{b}{a}((a - b)t^a - at^{a-b} + b)$. Since

$$k(1) = 0, \qquad \frac{dk(t)}{dt} = b(a - b)t^{a-1}(1 - t^{-b}),$$

we have $k(t) \geq 0$ for $t \in (0, \infty)$ if $a > b$. This means $h(t)$ is positive and increasing on $(0, \infty)$ if $a > b$.

Remarking the fact

$$h(t) = \frac{b}{a}\frac{t^a - 1}{t^b - 1} = \frac{1}{\frac{a}{b}\frac{t^b - 1}{t^a - 1}},$$

$h(t)$ is decreasing on $(0,\infty)$ if $a<b$.

(2) This has been proved in [9]:Example 3.4(1).

(3) It suffices to show that $(a,b)\in\Omega$ if $h(t)$ is 2-matrix monotone on $(0,\infty)$.

We assume that h is 2-matrix monotone and not constant. By (1) we have $a>b$ and

$$h'(t)=\frac{b}{a}\frac{k_1(t)}{(t^b-1)^2}\geq 0 \quad \text{for all } t\in(0,\infty),$$

where $k_1(t)=(a-b)t^{a+b-1}-at^{a-1}+bt^{b-1}$. So we have, for any $s,t>0$,

$$\begin{pmatrix} h'(s) & h^{[1]}(s,t) \\ h^{[1]}(s,t) & h'(t) \end{pmatrix}\geq 0,$$

equivalently, $h'(s)h'(t)-(h^{[1]}(s,t))^2\geq 0$. Then we set

$$\begin{aligned} D(s,t)&=h'(s)h'(t)-(h^{[1]}(s,t))^2\\ &=\frac{b^2}{a^2}\frac{1}{(s^b-1)^2(t^b-1)^2(s-t)^2}\times(k_1(s)k_1(t)(s-t)^2-k_2(s,t)), \end{aligned}$$

where

$$\begin{aligned} k_2(s,t)&=((s^a-1)(t^b-1)-(s^b-1)(t^a-1))^2\\ &=((s^a-1)t^b-(s^b-1)t^a-s^a+s^b)^2, \end{aligned}$$

and we remark $k_1(s)k_1(t)(s-t)^2$, $k_2(s,t)\geq 0$ for all $s,t\in(0,\infty)$.

When $b>0$ and $a+b+1<2a$, we have

$$\lim_{t\to\infty}D(s,t)<0 \quad \text{for some } s$$

because

$$\begin{aligned} &\lim_{t\to\infty}t^{-2a}(k_1(s)k_1(t)(s-t)^2-k_2(s,t))\\ &=\lim_{t\to\infty}t^{-2a+(a+b+1)}(((a-b)-at^{-b}+bt^{-a})k_1(s)(st^{-1}-1)^2)\\ &\qquad-((s^a-1)t^{b-a}-(s^b-1)-(s^a-s^b)t^{-a})^2\\ &=-(s^b-1)^2<0. \end{aligned}$$

This contradicts to the assumption of h. The highest degree d_1 (resp. d_2) of t in $k_1(s)k_1(t)(s-t)^2$ (resp. $k_2(t)$) is

$$(d_1,d_2)=\begin{cases}(a+b+1,2a) & (0<b<a)\\ (a+1,2a) & (b<0<a)\\ (a+1,0) & (b<a<0)\end{cases},$$

and the lowest degree d_1' (resp. d_2') of t in $k_1(s)k_1(t)(s-t)^2$ (resp. $k_2(t)$) is

$$(d_1', d_2') = \begin{cases} (b-1, 0) & (0 < b < a) \\ (b-1, 2b) & (b < 0 < a) \\ (a+b-1, 2b) & (b < a < 0) \end{cases}.$$

By using the similar argument as above, when $d_1 < d_2$ or $d_1' > d_2'$, we have

$$D(s,t) < 0 \text{ for a sufficiently large } t(>0) \text{ and some fixed } s,$$

or

$$D(s,t) < 0 \text{ for a sufficiently small } t(>0) \text{ and some fixed } s.$$

So 2-matrix monotonicity of h implies

$$d_1 \geq d_2 \text{ and } d_1' \leq d_2'.$$

This means that $(a,b) \in \Omega$ if h is 2-matrix monotone. □

3 Extended Petz-Hasegawa's functions

We consider the operator monotonicity of the function

$$h(t) = \frac{ab(t-1)^2}{(t^a-1)(t^b-1)}, \qquad t \in (0,\infty),$$

for any real number a and b. When $b = 1-a$ and $-1 \leq a \leq 2$, this function is called Petz-Hasegawa's function and becomes operator monotone on $(0,\infty)$ (see [4], [5], [7]).

Theorem 3.1. *Let a, b be real numbers and*

$$h(t) = \frac{ab(t-1)^2}{(t^a-1)(t^b-1)}.$$

Then h becomes operator monotone on $(0,\infty)$ if and only if the point (a,b) belongs to the following set:

$$\Omega = \{(a,b) : \ a \in [-1,2], \ g_1(a) \leq b \leq g_2(a)\},$$

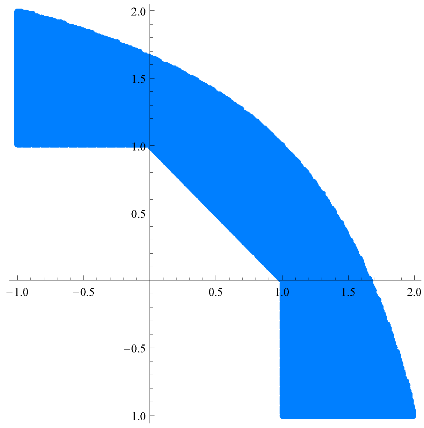
where

$$g_1(a) = \begin{cases} 1, & a \in [-1,0] \\ 1-a, & a \in [0,1) \\ -1, & a \in [1,2] \end{cases}$$

and $g_2(a)$ is satisfying $1-a \le g_2(a) \le 2-a$ and the following equation:

$$\left(\frac{a-g_2(a)}{a}\frac{\sin a\pi}{\sin(a+g_2(a))\pi}\right)^a = \left(\frac{g_2(a)-a}{g_2(a)}\frac{\sin g_2(a)\pi}{\sin(g_2(a)+a)\pi}\right)^{g_2(a)}.$$

This set Ω in the (a,b)-plane is as follows:



,

where the boundary curve $g_2(a)$ is given by computations of approximate values.

Proof. The function h is symmetric for a and b. So we may assume that $a \ge b$. We can rewrite $h(t)$ as follows:

$$\begin{aligned} h(t) &= ab \cdot \frac{(t-1)^2}{(t^a-1)(t^b-1)} && a \ge b \ge 0 \\ &= a(-b)t^{-b} \cdot \frac{(t-1)^2}{(t^a-1)(t^{-b}-1)} && b < 0 \le a \\ &= (-a)(-b)t^{-a-b} \cdot \frac{(t-1)^2}{(t^{-a}-1)(t^{-b}-1)} && b \le a < 0 \end{aligned}$$

By the remark before Lemma 1.1, we have $|a|, |b| \le 2$ if h is operator monotone.

Case (1) $0 < b \le a \le 2$: We can consider $\gamma = 0$, $\alpha = (1,1)$, and $\beta = (a,b)$. Since

$$\lim_{r\to\infty} \arg h(re^{\pi i}) = \lim_{r\to\infty} \arg abr^{2-a-b} \cdot \frac{(e^{\pi i}-1/r)^2}{(e^{a\pi i}-1/r^a)(e^{b\pi i}-1/r^b)} = (2-a-b)\pi$$

and $G_0(\alpha,\beta) \le 2-a-b \le F_0(\alpha,\beta)$, it follows that, by Lemma 1.1,

$$1 \le a+b \le 2$$

if h is operator monotone.

When $0 < b \le a \le 1$, we have

$$F(\alpha,\beta) = F(1,a) + F(1,b) = (1-a) + (1-b) = 2 - a - b,$$
$$-F(\beta,\alpha) = -(F(a,1) + F(b,1)) = -(0+0) = 0.$$

By Lemma 1.2, $a + b \ge 1$ and $0 < b \le a \le 1$ implies that h is operator monotone.

Case (2) $-2 \le b \le 0 < a \le 2$: We can consider $\gamma = -b$, $\alpha = (1,1)$, and $\beta = (a,-b)$. Since

$$\begin{aligned}\lim_{r\to\infty} \arg h(re^{\pi i}) &= \lim_{r\to\infty} \arg a(-b)r^{2-a}e^{-b\pi i}\cdot\frac{(e^{\pi i} - 1/r)^2}{(e^{a\pi i} - 1/r^a)(e^{-b\pi i} - 1/r^{-b})}\\ &= (-b + 2 - a - (-b))\pi = (2-a)\pi\end{aligned}$$

and

$$\begin{aligned}\lim_{r\to 0+} \arg h(re^{\pi i}) &= \lim_{r\to 0+} \arg a(-b)r^{-b}e^{-b\pi i}\cdot\frac{(re^{\pi i} - 1)^2}{(r^a e^{a\pi i} - 1)(r^{-b}e^{-b\pi i} - 1)}\\ &= -b\pi,\end{aligned}$$

we have $1 \le a \le 2$ and $-1 \le b \le 0$ if h is operator monotone.

When $1 \le a \le 2$ and $-1 \le b \le 0$, we have

$$-b + F(\alpha,\beta) = -b + F(1,a) + F(1,-b) = -b + 0 + (1 - (-b)) = 1,$$
$$-b + G(\alpha,\beta) = -b - F(a,1) - F(-b,1) = -b - (a-1) - 0 = 1 - (a+b).$$

By Lemma 1.2, $a + b \ge 1$, $1 \le a \le 2$, and $-1 \le b \le 0$ implies that h is operator monotone.

Case (3) $-2 \le a, b < 0$: We can consider $\gamma = -a - b$, $\alpha = (1,1)$, and $\beta = (-a,-b)$. Since

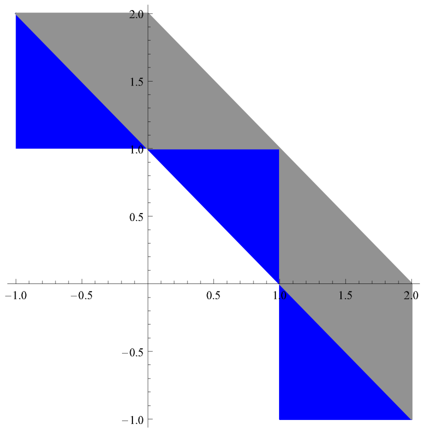
$$\lim_{r\to\infty} \arg h(re^{\pi i}) = \lim_{r\to\infty} \arg abr^2 e^{(-a-b)\pi i}.\frac{(e^{\pi i} - 1/r)^2}{(e^{-a\pi i} - 1/r^{-a})(e^{-b\pi i} - 1/r^{-b})} = 2\pi,$$

h is not operator monotone.

So we have that Ω is contained in

$$\{(a,b) : -1 \le a \le 0,\ g_1(a) \le b \le 2\} \cup \{(a,b) : 0 \le a \le 2,\ g_1(a) \le b \le 2 - a\}$$

and h is operator monotone if the point (a,b) is contained in the following three triangles:



By the numerical computation of $F_0(\alpha,\beta)$ and $G_0(\alpha,\beta)$, we can replace the above figure to the figure in the statement of Theorem 3.1.

We consider the function $g_2(a)$. By the symmetry of a and b, we only consider the case $1 \leq a \leq 2$ and $1-a \leq b \leq 2-a$. We remark that $f(t)$ is operator monotone on $(0,\infty)$ if and only if $\Im f(re^{\pi i}) \geq 0$ for all $r \geq 0$ by Lemma 1.1(1). Since

$$\begin{aligned}
&\Im f(re^{\pi i})\\
=&\frac{(r+1)^2}{|(r^a e^{a\pi i}-1)(r^b r^{b\pi i}-1)|^2}\Im ab(r^a e^{-a\pi i}-1)(r^{-b\pi i}-1)\\
=&\frac{(r+1)^2 r^b}{|(r^a e^{a\pi i}-1)(r^b r^{b\pi i}-1)|^2}ab(-r^a\sin(a+b)\pi+\sin b\pi+r^{a-b}\sin a\pi),
\end{aligned}$$

the signature of $\Im f(re^{\pi i})$ is equal to that of

$$k(r) = ab(-r^a\sin(a+b)\pi+\sin b\pi+r^{a-b}\sin a\pi)$$

for all $r > 0$. We can see that the solution r_0 of $k'(r) = 0$ is

$$\left(\frac{(a-b)\sin a\pi}{a\sin(a+b)\pi}\right)^{1/b},$$

and $k(r)$ is decreasing on $(0, r_0)$ and increasing on (r_0,∞). So we have $f(t)$ is operator monotone on $(0,\infty)$ if and only if $k(r_0) \geq 0$. As a relation of a and b satisfying with $k(r_0) = 0$, we can get the following:

$$\left(\frac{a-b}{a}\frac{\sin a\pi}{\sin(a+b)\pi}\right)^a = \left(\frac{b-a}{b}\frac{\sin b\pi}{\sin(b+a)\pi}\right)^b.$$

So we can get the desired relation. □

The following is a program drawing a part of this figure in $[1,2]\times[-1,1]$ by Mathematica.

```
pick1={};

f10[a_,b_,z_]:= -Arg[z^a-1]-Arg[z^b-1];

minmaxf10[a_,b_]:=Module[{zval,zmin,zmax},
  zval=Table[f10[a,b,-0.001*i],{i,1,1000}];
  zmin=Min[zval];
  zmax=Max[zval];
  {Min[{zmin,(2-a-b)*Pi-zmax}],Max[{zmax,(2-a-b)*Pi-zmin}]}]

Do[   {  c = minmaxf10[a,b];
  If[  ( c[[1]]>=0 ) && ( c[[2]] <= Pi) ,
    pick1 = Append[ pick1, {a,b} ] ; ] } ,
  {a,1.01, 2.0, 0.01}, {b, 0.01, 1.0, 0.01}]

pick2={};

f00[a_,b_,z_]:= -Arg[z^a-1]-Arg[z^(-b)-1];

minmaxf00[a_,b_]:=Module[{zval,zmin,zmax},
  zval=Table[f00[a,b,-0.001*i],{i,1,1000}];
  zmin=Min[zval];
  zmax=Max[zval];
  {Min[{zmin,(2-a+b)*Pi-zmax}] + (-b)*Pi,
   Max[{zmax,(2-a+b)*Pi-zmin}]+ (-b)*Pi}]

Do[   {  c = minmaxf00[a,b];
  If[  ( c[[1]]>=0 ) && ( c[[2]] <= Pi) ,
    pick2 = Append[ pick2, {a,b} ]; ]} ,
  {a,1.01, 2.0, 0.01}, {b, -0.01, -1.0, -0.01}]

ListPlot[ {pick1, pick2},  AspectRatio->Automatic,
  AxesOrigin->{0,0},PlotStyle->PointSize[0.01]]
```

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