数理解析研究所講究録
第 1996 巻 2016 年 45-53



# RELATIVE OPERATOR ENTROPY AND KARCHER MEAN

JUN ICHI FUJII[1] AND YUKI SEO[2]

## 1. Introduction

Lawson and Lim [10] showed that the Karcher equation for positive invertible operators $A_j$ $(j=1,2,\ldots,n)$, $X$ on a Hilbert space and a weight $\{\omega_j\}$ $(\omega_j\geq 0, \sum_j\omega_j=1)$:

**(KE)**
$$0=\sum_{j=1}^{n}\omega_j\log\left(X^{-\frac{1}{2}}A_jX^{-\frac{1}{2}}\right)$$

has a unique positive invertible solution

$$X=G_K(\omega_j;A_j)=G_K(\omega;\mathbb{A})$$

for $\omega=(\omega_1,\ldots,\omega_n)$ and $\mathbb{A}=(A_1,\ldots,A_n)$. It is called the (weighted $n$-variable) Karcher mean. This definition depends on the invertibility of operators: Their theory depends on the Thompson metric $d(A,B)=\|\log(A^{-1/2}BA^{-1/2}\|$ and the power operator mean corresponding to the power function $f_{\mathrm{m}_{r,t}}(x)=(1-t+tx^r)^{\frac{1}{r}}$: The Karcher mean of positive invertible operators coincides with the strong-operator limit of the power means $P_t(\omega;\mathbb{A})$ as $t\to 0$:

$$G_K(\omega;\mathbb{A})=\operatorname*{s-lim}_{t\to 0}P_t(\omega;\mathbb{A}).$$

Thus it needs substantially the invertibility of positive operators.

In this note, we extend it to a mean for (non-invertible) positive operators by virtue of the relative operator entropy. So, we observe the properties for the relative operator entropy with the existence conditions which are closely related the kernels and ranges for operators. Defining the Karcher mean for non-invertible operators as the strong-operator limit, we discuss their properties and kernels. Then we see that the original Kubo-Ando geometric mean satisfies the Karcher equation with the kernel condition. Also we verify that the Karcher mean for operators with the closed ranges is a unique solution of this equation.

## 2. Relative operator entoropy

First we review the *relative operator entropy* $S(A|B)$ for positive (bounded linear) operators $A$, $B$ on a Hilbert space, see [5, 6, 7, 2, 3, 4].

Nakamura and Umegaki [11] extended the notion of the entropy formulated by von Neumann and gave the entorpy by $-A\log A$ for a positive operator $A$ on a Hilbert space. Also, Umegaki [13] introduced the relative entorpy as a noncommutative version of the Kullback-Leibler entorpy, which is given by the trace of $A\log A-A\log B$ for positive operators $A,B$ affiliated with a semifinite von Neumann algebra. In [5], Fujii and Kamei

2000 *Mathematics Subject Classification.* Primary 47A64; Secondary 47A63, 47A58, 94A17.
*Key words and phrases.* Operator mean, relative operator entropy, Karcher equation, Karcher mean.

introduced the relative operator entorpy which is a relative version of the operator entorpy defined by Nakamura-Umegaki: Let $A$ and $B$ be positive operators on a Hilbert space $H$. If $B$ is invertible, then the relative operator entorpy is defined by

$$S(A|B) = B^{\frac{1}{2}}\eta\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)B^{\frac{1}{2}}$$

where $\eta$ is the entropy function:

$$\eta(x) = -x\log x \quad \text{if } x > 0, \qquad \eta(0) = 0.$$

In addition, if $A$ is invertible, then the relative operator entropy is rewrited as follows:

$$S(A|B) = A^{\frac{1}{2}}\log\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

In the case, since $\log t$ is operator monotone, $S(A|B)$ has the right monotonicity:

$$B \leq C \quad \text{implies} \quad S(A|B) \leq S(A|C).$$

Moreover, by using the fact that $\lim_{t\to 0}\frac{x^t-1}{t} = \log t$, it is constructed by Uhlmann's way [12]:

$$S(A|B) = \operatorname*{s-lim}_{t\to 0}\frac{A\#_tB - A}{t},$$

where the geometric operator mean $A\#_tB$ is defined by

$$A\#_tB = A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2}.$$

Next, let $A$ and $B$ be non-invertible. For each $\varepsilon > 0$, since $B+\varepsilon$ is invertible, the relative operator entropy $S(A|B+\varepsilon)$ exists and

$$S(A|B+\varepsilon) = \operatorname*{s-lim}_{\delta\downarrow 0} S(A+\delta|B+\varepsilon)$$

in the strong operator topology. Since for each $\delta > 0$, $S(A+\delta|B+\varepsilon)$ has the right-term monotone decreasing property as $\varepsilon\downarrow 0$, we have

$$S(A+\delta|B+\varepsilon) \geq S(A+\delta|B+\varepsilon') \quad \text{for } \varepsilon > \varepsilon' > 0$$

and so $S(A|B+\varepsilon) \geq S(A|B+\varepsilon')$ as $\delta\downarrow 0$. Therefore, in the case of non-invertible $A$ and $B$, we can define

$$S(A|B) = \operatorname*{s-lim}_{\varepsilon\downarrow 0} S(A|B+\varepsilon)$$

if the strong-operator limit exists as a bounded operator. In fact, by the right-term monotonicity, the lower boundedness for $\{S(A|B+\varepsilon)\,|\,\varepsilon > 0\}$ guarantees the strong-operator limit. But, in general, $S(A|B)$ does not always exist. For example, $S(A|B)$ does not exists for invertible $A$ and non-invertible $B$.

Here, we state a existence condition of the relative operator entropy:

**Lemma 2.1.** *Let $A, B$ be positive operators. The relative operator entropy $S(A|B)$ exists if and only if $E(\alpha) = (-1-\log\alpha)A + \alpha B$ is bounded below for all $\alpha > 0$.*

Next, we consider another equivalent definition of Uhlmann's type. For this, based on the fact that $\frac{x^t-1}{t} \searrow \log t$ as $t\downarrow 0$, it follows that $\frac{A\#_tB-A}{t}$ is monotone-decreasing as $t\downarrow 0$, so that another equivalent definition of Uhlmann's type is the derivative one for the path of geometric means $A\#_tB$ for $t\in(0,1]$:

$$S(A|B) = \operatorname*{s-lim}_{t\downarrow 0}\frac{A\#_tB - A}{t}$$

if the strong-operator limit exists. In fact, put $X = \text{s-lim}_{t\downarrow 0}\frac{A\#_t B - A}{t}$ and since

$$\frac{A\#_t B - A}{t} \le \frac{(A+\varepsilon)\#_t(B+\varepsilon)-(A+\varepsilon)}{t} \quad \text{for each } \varepsilon > 0,$$

we have $X \le \text{s-lim}_{t\downarrow 0}\frac{(A+\varepsilon)\#_t(B+\varepsilon)-(A+\varepsilon)}{t} = S(A+\varepsilon|B+\varepsilon)$ as $t \downarrow 0$ and so $X \le S(A|B)$ as $\varepsilon \downarrow 0$. On the other hand, since

$$\frac{(A+\varepsilon)\#_t(B+\varepsilon)-(A+\varepsilon)}{t} \ge S(A+\varepsilon|B+\varepsilon) \qquad \text{for each } \varepsilon > 0 \text{ and } t \in (0,1],$$

we have $\frac{A\#_t B - A}{t} \ge S(A|B)$ as $\varepsilon \downarrow 0$ for $t > 0$ and so $X \ge S(A|B)$. Therefore, $X = S(A|B)$ and we have another equivalent definition of Uhlmann's type for non-invertible case.

If $A$ and $B$ are commuting and $S(A|B)$ is defined, then

$$S(A|B) = A\log B - A\log A,$$

in particular, $S(0|B) = 0$ for all $B \ge 0$. In fact, since $S(A|B)$ exists, by Lemma 2.1, there exists a scalar $c \in \mathbb{R}$ such that for all $\alpha > 0$

$$\begin{aligned} c &\le (-1-\log\alpha)A + \alpha B \\ &\le (-1-\log\alpha)A + \alpha(B+\varepsilon) \quad \text{for each } \varepsilon > 0 \end{aligned}$$

and so

$$\begin{aligned} c &\le (B+\varepsilon)(-(B+\varepsilon)^{-1}A\log(B+\varepsilon)^{-1}A) \\ &= -A\log A + A\log(B+\varepsilon). \end{aligned}$$

Therefore, since $A\log B - A\log A$ exists, we have $A\log B - A\log A = S(A|B)$.

Under the existence, we have the following properties of $S(A|B)$ for positive operators $A$ and $B$ by those for operator means:

**Lemma 2.2.** *Under the existence, the following properties like operator means hold:*

(1) **right monotonicity:** *If* $B \le B'$, *then* $S(A|B) \le S(A|B')$.
(2) **transformer inequality:** $T^*S(A|B)T \le S(T^*AT|T^*BT)$ *for all* $T$ (*the equality holds for invertible* $T$).
(2′) **informational monotonicity:** $\Phi(S(A|B)) \le S(\Phi(A)|\Phi(B))$ *for all normal positive linear maps* $\Phi$.
(3) **sub-additivity:** $S(A_1|B_1) + (A_2|B_2) \le S(A_1 + A_2|B_1 + B_2)$.
(3′) **joint concavity:** $(1-t)S(A_1|B_1) + tS(A_2|B_2) \le S((1-t)A_1 + tA_2|(1-t)B_1 + tB_2)$ *for all* $t \in [0,1]$.
(4) **upper bound:** $S(A|B) \le B - A$.
(5) **kernel inclusion:** $\ker S(A|B) \supset \ker A$.
(6) **orthogonality:** $S\left(\bigoplus_k A_k \mid \bigoplus_k B_k\right) = \bigoplus_k S(A_k|B_k)$.
(7) **affine parametrization:** $S(A|A\#_t B) = t\, S(A|B)$ *for all* $t \in [0,1]$.

Here we recall the equality condition for the transformer inequality (2) of Lemma 2.2:

**Lemma 2.3.** *Let $A$ and $B$ be positive operators. Under the existence of $S(A|B)$, the transformer equality*

$$T^*S(A|B)T = S(T^*AT|T^*BT)$$

*holds for an operator $T$ with* $\ker T^* \subset \ker A \cap \ker B$.

We have the following relations around the existence condition for the relative operator entropy:

**Theorem 2.4.** *The implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold in the following conditions for a pair of* $A$, $B \geq 0$ *and each converse does not always hold.*

(1) **majorization** *or* **range inclusion:** $\exists \alpha > 0$; $A \leq \alpha B$, *i.e.*, $\operatorname{ran} A^{\frac{1}{2}} \subset \operatorname{ran} B^{\frac{1}{2}}$.
(2) **existence condition**: $S(A|B)$ *exists as a bounded operator, i.e.*,

$$\inf_{\alpha>0} \left[\frac{1}{\alpha} B - A + (\log \alpha) A\right] > -\infty.$$

(3) $B$-**absolute continuity:** $A = [B]A \left(= A^{\frac{1}{2}} P_M A^{\frac{1}{2}} = \lim_{t \downarrow 0} A \#_t B\right)$.
(4) **kernel inclusion:** $\ker A \supset \ker B$.

**Remark 2.1.** If both ranges of $A$ and $B$ are closed, in particular, for the case of matrices, the above conditions in Theorem 2.4 are all equivalent since the relation $\operatorname{ran} A^{\frac{1}{2}} = \overline{\operatorname{ran} A} = (\ker A)^{\perp}$ holds for all positive operators $A$.

For invertible positive operators $A$ and $B$, it is easy to see that the positivity (resp. negativity) of $S(A|B)$ is equivalent to $B \geq A$ (resp. $A \geq B$) and hence $S(A|B) = 0$ if and only if $A = B$. Second we discuss non-invertble case:

**Theorem 2.5.** *Let $A$ and $B$ be positive operators. Suppose $S(A|B)$ exists. $S(A|B) \geq 0$ if and only if $B \geq A$. If* $\ker A$ *is trivial, then $S(A|B) \leq 0$ if and only if $A \geq B$. Consequently, for $A$ with the trivial kernel, $S(A|B) = 0$ if and only if $A = B$.*

## 3. Karcher mean for non-invertible positive operators

For non-invertible positive operators $A_j$ ($j = 1, 2, \ldots, n$), for each $\varepsilon > 0$ the Karcher mean $X_\varepsilon = G_K(\omega_j; A_j + \varepsilon) \geq 0$ exists and the monotonicity of $G_K$ guarantees the strong-operator limit:

$$X_0 = \operatorname*{s-lim}_{\varepsilon \to 0} X_\varepsilon = \operatorname*{s-lim}_{\varepsilon \to 0} G_K(\omega_j; A_j + \varepsilon).$$

Naturally we write $X_0 = G_K(\omega_j; A_j)$ for non-invertible $A_j$ and call it *the Karcher mean* again.

Here we extend the extremal means with a weight $\{\omega_j\}$ synchronously to $G_K$: The arithmetic mean $A$ and the harmonic one $H$ for non-invertible $A_j$ are defined by

$$A(\omega_j; A_j) = \sum_j \omega_j A_j$$

$$H(\omega_j; A_j) = \operatorname*{s-lim}_{\varepsilon \to 0} H(\omega_j; A_j + \varepsilon) = \operatorname*{s-lim}_{\varepsilon \to 0} (\sum_j \omega_j (A_j + \varepsilon)^{-1})^{-1}.$$

As for this construction of corresponding mean, we say '$H$ is the adjoint of $A$' as in the Kubo-Ando theory.

**Theorem 3.1.** *For a weight $\{\omega_j\}$ and positive operators $A_j$ ($j = 1, 2, \ldots, n$), the following properties hold:*

(1) **monotonicity**: *If $A_j \leq B_j$ for all $j$, then $G_K(\omega_j; A_j) \leq G_K(\omega_j; B_j)$.*
(2) **transformer inequality**: $T^* G_K(\omega_j; A_j) T \leq G_K(\omega_j; T^* A_j T)$ *for all $T$*
*(the equality holds for invertible $T$).*

(2′) **informational monotonicity**: $\Phi(G_K(\omega_j; A_j)) \le G_K(\omega_j; \Phi(A_j))$ *for all normal positive linear maps* $\Phi$.

(3) **sub-additivity**: $G_K(\omega_j; A_j) + G_K(\omega_j; B_j) \le G_K(\omega_j; A_j + B_j)$.

(3′) **joint concavity**:

$$(1-t)G_K(\omega_j; A_j) + tG_K(\omega_j; B_j) \le G_K(\omega_j; (1-t)A_j + tB_j) \quad \textit{for all } t \in [0,1].$$

(4) **consistency with scalars**:

*If all* $A_j$ *are commuting, then* $G_K(\omega_j; A_j) = \prod_{j=1}^{n} A_j^{\omega_j}$ *with convention* $A^0 = I$.

(5) **self-adjointness**: $G_K(\omega_j; A_j) = \text{s-lim}_{\varepsilon \downarrow 0} G_K(\omega_j; (A_j + \varepsilon)^{-1})^{-1}$.

(6) **joint homogeneity**: $G_K(\omega_j; c_j A_j) = \prod_{j=1}^{n} c_j^{\omega_j} G_K(\omega_j; A_j)$ *for non-negative numbers* $c_j \ge 0$ $(j = 1, 2, \ldots, n)$.

(7) **AGH weighted mean inequality**:

$$H(\omega_j; A_j) \le G_K(\omega_j; A_j) \le A(\omega_j; A_j).$$

(8) **orthogonality**: $G_K\left(\omega_j; \bigoplus_m A_{j,m}\right) = \bigoplus_m G_K(\omega_j; A_{j,m})$.

Note that if $\omega_k = 0$ for some $k$, then $n$-mean $G_K(\omega_j; A_j)$ is nothing but $(n-1)$-mean without $\omega_k$, $A_k$. So we call $G_K(\omega_j; A_j)$ the *proper* Karcher mean if $\omega_j > 0$ for all $j$. Then we also call the weight $\{\omega_j\}$ *proper*.

**Lemma 3.2.** *For a proper weight* $\{\omega_j\}$ *and positive operators* $A_j$ $(j = 1, 2, \ldots, n)$,

$$\operatorname{ran} A(\omega_j; A_j)^{\frac{1}{2}} = \bigvee_j \operatorname{ran} A_j^{\frac{1}{2}} \quad \textit{and} \quad \operatorname{ran} H(\omega_j; A_j)^{\frac{1}{2}} = \bigcap_j \operatorname{ran} A_j^{\frac{1}{2}}.$$

By Lemma 3.2 and the transformer inequaity (2) of Theorem 3.1, we have the following kernel condition for the Karcher mean.

**Theorem 3.3.** *Let* $A_j$ *be positive operators for* $j = 1, 2, \ldots, n$. *For a proper weight* $\{\omega_j\}$,

$$\ker G_K(\omega_j; A_j) = \bigvee_j \ker A_j.$$

For any projections $P$ and $Q$, it is known in [9] that $P \#_t B = P \bigwedge Q$ for $t \in [0,1]$. The following result is an extension of 2-variable case:

**Corollary 3.4.** *For a proper weight* $\{\omega_j\}$,

$$G_K(\omega_j; P_j) = \bigwedge_j P_j \qquad \textit{for projections } P_j \ (j = 1, 2, \ldots, n).$$

In this invertible case, note that (KE) is equivalent to a simple equation by the relative operator entoropy

$$(**) \qquad 0 = \sum_{j=1}^{n} \omega_j S(X|A_j),$$

which also makes sense for non-invertible $A_j$. But this equation always has a trivial solution $X = 0$ since $S(0|A_j) = 0$. For the case of Corollary 3.4, the entropy is $S(P|P_j) =$

$P \log P_j - P \log P = 0$, and hence $P$ is indeed a solution of the equation $(**)$. But this consideration shows that each projection $Q$ with $0 \le Q \le P$ is a solution of $(**)$.

Thereby, a reasonable extension of (KE) is the following EKE(*Extended Karcher equation*) with the kernel condition under the existence of each $S(X|A_j)$:

$$\textbf{(EKE)} \qquad 0 = \sum_{j=1}^{n} \omega_j S(X|A_j) \quad \text{with} \quad \ker X = \bigvee_{\omega_j > 0} \ker A_j.$$

**Remark 3.1.** If operators $A_j$ are commuting for all $j$, then $X_0 = G_K(\omega_j; A_j) = \prod_j A_j^{\omega_j}$ and $X_0$ is a solution of (EKE). But, if the kernel condition is removed, the following example gives another solution $X$ even if $X$ commutes with all $A_j$.

**Example 1.** For diagonal matrices $A = \operatorname{diag}(a, b, c, 0)$ and $B = \operatorname{diag}(\frac{1}{a}, b, 0, d)$, take $X_1 = \operatorname{diag}(0, b, 0, 0)$. Then, all matrices are commuting and hence

$$\begin{aligned} S(X_1|A) + S(X_1|B) &= -2X_1 \log X_1 + X_1 \log A + X_1 \log B \\ &= \operatorname{diag}(0, -2b\log b, 0, 0) + \operatorname{diag}(0, b\log b, 0, 0) + \operatorname{diag}(0, b\log b, 0, 0) = 0. \end{aligned}$$

So $X_1$ is a solution while $X_1 \ne X_0 = \operatorname{diag}(1, b, 0, 0)$.

**Remark 3.2.** For the case of projections $A_j = P_j$, the above $P$ is a unique solution of (EKE). In fact, suppose $Y$ is another solution. Then the kernel condition $P \le P_j$ shows $PYP = Y$ and hence we have $YP_j = P_jY = Y$ and $Y \log P_j = YP_j \log P_j = 0$. Therefore

$$\begin{aligned} 0 &= \sum_j \omega_j S(Y|P_j) = \sum_j \omega_j (Y \log P_j - Y \log Y) \\ &= \sum_j \omega_j (-Y \log Y) = -Y \log Y, \end{aligned}$$

which implies that $Y$ must be a projection and consequently $Y = P$ by the kernel condition.

Moreover note that $S(A|B)$ does not always exist as a bounded self-adjoint operator as in the preceding section. But $S(X_0|A_j)$ indeed exists for the Karcher mean $X_0 = G_K(\omega_j; A_j)$:

**Lemma 3.5.** *Let $A_j$ be positive operators and $\{\omega_j\}$ a weight. For the Karcher mean $X_0 = G_K(\omega_j; A_j)$, each entropy $S(X_0|A_j)$ exists for $\omega_j > 0$. For $\omega_k > 0$, bounds are expressed by*

$$-\frac{M_k}{\omega_k} \le S(X_0|A_k) \le M_k$$

*for $M_k = \max_{j \ne k} \| A_j \| + 1$.*

Thus far, we have not showed that our Karcher mean $X_0 = \text{s-lim}_{\varepsilon \to 0} X_\varepsilon$ satisfies (EKE) for general positive operators. Here we can obtain only the inequality:

**Lemma 3.6.** *Let $A_j$ be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. Then*

$$0 \le \sum_{j=1}^{n} \omega_j S(X_0|A_j) \qquad \textit{with} \qquad \ker X_0 = \bigvee_{\omega_j > 0} \ker A_j.$$

However, in the case of $n = 2$, we have the following result:

**Theorem 3.7.** *For positive operators $A$ and $B$ and $t \in (0,1)$, the original geometric mean $A\#_t B$ satisfies* (EKE).

Recall that a non-invertible positive operators $A$ has the closed range if and only if 0 is an isolated point in $\sigma(A)$. Any positive semi-definite matrix has the closed range. Finally in this section, we show that the Karcher mean for such operators is a unique solution of (EKE). To see this, we verify the following fact:

**Lemma 3.8.** *If $A_j$ $(j = 1, 2, \ldots, n)$ are positive opereators whose ranges are closed, then so is $X_0 = G_K(\omega_j; A_j)$.*

So we have a unique solution of (EKE):

**Theorem 3.9.** *If $A_j$ are positive opereators whose ranges are closed for $j = 1, 2, \ldots, n$, the Karcher mean $X_0 = G_K(\omega_j; A_j)$ is a unique solution of* (EKE).

Therefore, we have the following result in the matrix case:

**Corollary 3.10.** *For positive semi-definite matrices $A_j$ for $j = 1, 2, \ldots, n$ and a weight $\{\omega_j\}$, the Karcher mean $X = G_K(\omega_j; A_j)$ is the unique solution of* (EKE):

$$0 = \sum_{j=1}^{n} \omega_j S(X|A_j) \quad \textit{with} \quad \ker X = \bigvee_{\omega_j > 0} \ker A_j.$$

## 4. Power means for non-invertible case

Let $A_j$ be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. For each $\varepsilon > 0$, similarly to the Karcher mean $X_\varepsilon = G_K(\omega_j; A_j + \varepsilon)$, the power means $P_t(\omega_i; A + \varepsilon)$ for $t \in (0, 1]$ is the unique positive invertible solution of the power mean equation

$$X = \sum_{j=1}^{n} \omega_j (X \#_t (A_j + \varepsilon)).$$

Then the Karcher mean for invertible case is the strong-operator limit of the power means:

$$\text{s-}\lim_{t \downarrow 0} P_t(\omega_j; A_j + \varepsilon) = X_\varepsilon.$$

The power means $P_t(\omega_j; A_j + \varepsilon)$ are monotone decreasing for $\varepsilon \downarrow 0$ by [10, Proposition 3.6 (4)] and lower bounded by the zero operator. Hence $P_t(\omega_j; A_j) = \inf_{\varepsilon > 0} P_t(\omega_j; A_j + \varepsilon)$ exists and

$$P_t(\omega_j; A_j) = \text{s-}\lim_{\varepsilon \downarrow 0} P_t(\omega_j; A_j + \varepsilon)$$

in the strong operator topology and so $P_t(\omega_j; A_j)$ is a solution of the power mean equation

$$X = \sum_j \omega_j (X \#_t A_j) \tag{1}$$

for $t \in (0, 1]$.

Similarly to the Karcher equation for positive operators, the power mean equation (1) always has a trivial solution $X = 0$. Thereby, we consider the following EPE (*Extended Power mean equation*) with the kernel condition:

$$\textbf{(EPE)} \qquad X = \sum_j \omega_j X \#_t A_j, \qquad \text{with} \quad \ker X = \bigcap_{w_j > 0} \ker A_j.$$

**Theorem 4.1.** *Let $A_j$ be positive operators and $\{\omega_j\}$ a weight. Then the power means $P_t(\omega_j; A_j)$ for $t \in (0,1]$ satisfy* (EPE) *and*

$$P_t(\omega_j; A_j) \searrow G_K(\omega_j; A_j) \qquad \text{as } t \downarrow 0.$$

**Theorem Y** (Yamazaki [14]). *Let $A_j$ and $X$ be positive operators for $j = 1, 2, \ldots, n$ and $\{\omega_j\}$ a weight. Then*

$$\sum_j \omega_j S(X|A_j) \geq 0 \quad \text{implies} \quad G_K(\omega_j; A_j) \geq X.$$

*Moreover, if $A_j$ and $X$ are invertible, then*

$$\sum_j \omega_j S(X|A_j) \leq 0 \quad \text{implies} \quad G_K(\omega_j; A_j) \leq X.$$

The following result follows from the uniqueness of the power means and is an extension of Theorem Y:

**Theorem 4.2.** *Let $A_j$ and $X$ be positive operators for $j = 1, 2, \ldots, n$ and a weight $\{\omega_j\}$. Then*

$$X \leq \sum_j \omega_j X \#_t A_j \quad \text{implies} \quad X \leq P_t(\omega_j; A_j).$$

*Moreover, if $A_j$ and $X$ are invertible, then*

$$X \geq \sum_j \omega_j X \#_t A_j \quad \text{implies} \quad X \geq P_t(\omega_j; A_j).$$

**Remark 4.1.** Theorem 4.2 is an extension of Theorem Y. Indeed, suppose $\sum_j \omega_j S(X|A_j)) \geq 0$. Then

$$0 \leq \sum_j \omega_j S(X|A_j)) \leq \frac{\sum_j \omega_j X \#_t A_j - X}{t}$$

and it follows from Theorem 4.2 that $X \leq P_t(\omega_j; A_j)$ for all $0 < t \leqq 1$. Taking limit as $t \downarrow 0$, we have $X \leq G_K(\omega_j; A_j)$. Another part is obtained by this result.

**Remark 4.2.** Finally, considering the power means for negative parameters as the adjoint in the sense of Kubo-Ando theory, for invertible case, we can show that the uniqueness of the Karcher equation follows from the case of power means without the implicit function theorem. See [8] for more detail.

Though the properties for power means are supplied, the following general problems are not still answered:

**Conjecture.** *For non-invertible positive operators on a Hilbert space, the Karcher mean satisfies* (EKE) *and it is a unique solution of* (EKE).

**Conjecture 2.** *For non-invertible positive operators on a Hilbert space, each power mean $P_t(\omega_j; A_j)$ for $t \in (0,1]$ is a unique solution of* (EPE).

## References

[1] J.I.Fujii, On Izumino's view of operator means, Math. Japon. **33**(1988), 671–675.
[2] J.I.Fujii, Operator means and the relative operator entropy. Operator theory and complex analysis (Sapporo, 1991), 161–172, Oper. Theory Adv. Appl., **59**, Birkhäuser, Basel, 1992.
[3] J.I.Fujii, Operator means and range inclusion. Linear Algebra Appl., **170** (1992), 137–146.
[4] J.I.Fujii, Relative operator entropy (Japanese) (Kyoto, 1994). Surikaiseki-kenkyusho Kokyuroku **903** (1995), 49–56.
[5] J.I.Fujii and E.Kamei, Rrelative operator entropy in noncommutative information theory, Math. Japon. **34** (1989), 341–348.
[6] J.I.Fujii and E.Kamei, Uhlmann's interpolational method for operator means, Math. Japon. **34** (1989), 541–547.
[7] J.I.Fujii and E.Kamei, Interpolational paths and their derivatives, Math. Japon. **39** (1993), 557–560.
[8] J.I.Fujii and Y.Seo, The relative operator entropy and the Karcher mean, preprint.
[9] F.Kubo and T.Ando, Means of positive linear operators, Math. Ann. **246** (1980), 205–224.
[10] J.Lawson and Y.Lim, Karcher means and Karcher equations of positive definite operators. Trans. Amer. Math. Soc., Ser. B **1** (2014), 1–22.
[11] M.Nakamura and H.Umegaki, A note on the entropy for operator algebras, Proc. Jap. Acad., **37** (1961), 149–154.
[12] A.Uhlmann, Relative entropy and the Wigner-Yamase-Dyson-Lieb concavity in an interpolation theory, Commun. Math. Phys. **54** (1977), 22–32.
[13] H.Umegaki, Conditional expectation in an operator algebra IV, Kodai Math. Sem. Rep. **14** (1962), 59–85.
[14] T.Yamazaki, *Riemannian mean and matrix inequalities related to the Ando-Hiai inequality and chaotic order*, Oper. Matrices, **6** (2012), 577–588.

[1] Department of Art and Sciences (Information Science), Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
*E-mail address*: fujii@cc.osaka-kyoiku.ac.jp

[2] Department of Mathematics Education, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan
*E-mail address*: yukis@cc.osaka-kyoiku.ac.jp