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On operational convex combinations

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Abstract

We introduce the notion of operational convex combinations together with operational exetreme points for linear maps of $M_n(\mathbb{C})$ and show that automorphisms are the operational extreme points of the set of the unital completely positive maps on $M_n(\mathbb{C})$.

1 Introduction

In the paper [1], we gave some characterization for unital positive Tr-preserving maps of the algebra of $n\times n$ conplex matrices $M_n(\mathbb{C})$ from a view point of von Neumann entropy for states of $M_n(\mathbb{C})$, That is, a positive unital Tr-preserving map Φ of $M_n(\mathbb{C})$ preserves the von Neumann entropy of a given state ϕ if and only if Φ plays a role of an automorphism for ϕ.

In this note, we pick up the set of unital completely positive (called "ucp" for short) maps of $M_n(\mathbb{C})$. That is, for a unital linear map of $M_n(\mathbb{C})$, we replace the property "Tr-preserving and positive " to the property "completely positive", and investigate that what kind of position the automorphisms stand in ucp maps.

Here, we shall consider the notion of convexity not only scalar multiplication but also the multiplication via operators and generalize the notion of convexity, i.e., we introduce the notion of operational convex conbination.

The motivation for the terminology "operational convex conbinations" comes from the following two definitions: One is Lindblad's "operational partition" in [6] (see [7] or [8]) and the other is Cuntz's canonical endomorphism Φ_n in [5]. It seems to be natural for treating the set of ucp maps as the set of operational convex conbinations of automorphisms of $M_n(\mathbb{C})$.

We also introduce the notion of operational extreme point, and we show that automorphisms are operational extreme points in ucp maps of $M_n(\mathbb{C})$, which implies that automorphisms are extreme points in ucp maps of $M_n(\mathbb{C})$ in the usual sense.

2 Operational partitions

What we need to define a convex sum ? In usual we need a probability vector $\lambda = (\lambda_1, \cdots, \lambda_n)$:

$$\lambda_i \geq 0, \quad \sum_i \lambda_i = 1.$$

Given a finite subset $x = \{x_1, ..., x_n\}$ of a vector space X, the vector $\sum_i \lambda_i x_i$ is a convex sum of x via λ.

Now, we consider such a λ as a "finite partition of 1".

Two generalized notions of finite partition of 1 are given in the framework of the non-commutative entropy as follows:

Let A be a unital C^*-algebra.

(1) A finite subset $\{x_1, ..., x_k\}$ of A is called a *finite partition of unity* by Connes-Størmer ([4]) if they are nonnegative operators which satisfy that $\sum_{i=1}^n x_i = 1_A$,

(2) A finite subset $\{x_1, ..., x_k\}$ of A is called a *finite operational partition in A of unity of size k* by Lindblad ([6]) if $\sum_i^k x_i^* x_i = 1_A$.

Our main target in this note is a finite subset $\{v_1, ..., v_k\}$ of non-zero elements in A such that $\{v_1^*, ..., v_k^*\}$ is a finite operational partition of unity so that $\sum_i^k v_i v_i^* = 1_A$. We call such a set $\{v_1, ..., v_k\}$ a *finite operational partition of unity of size k* in A, and denote the set of all finite operational partition of unity in A by $FOP(A)$:

$$FOP(A) = \{\{v_1, ..., v_k\} \mid 0 \neq v_i \in A, \forall i, \sum_i^k v_i v_i^* = 1_A, \ k = 1, 2, \cdots\} \quad (2.1)$$

We denote by $U(A)$ the set of all unitaries in A. Clearly, $U(A)$ is the set of the most trivial finite operational partition of unity with the size 1.

2.1 Unital completely positive (ucp) map Φ

Let $M_n(\mathbb{C})$ be the C^*-algebra of $n \times n$ matrices over the complex field $\mathbb{C}$. A linear map Φ on a unital C^*-algebra A is positive iff $\Phi(a)$ is positive for all positive $a \in A$ and *completely positive* iff $\Phi \otimes 1_k$ is positive for all positive integer k, where the map $\Phi \otimes 1_k$ is the map on $A \otimes M_k(\mathbb{C})$ defined by $\Phi \otimes 1_k(x \otimes y) = \Phi(x) \otimes y$ for all $x \in A$ and $y \in M_k(\mathbb{C})$.

We restrict the unital C^*-algebra A to $M_n(\mathbb{C})$.

In [3, Theorem 2], Choi gave the following characterization: a linear map Φ of $M_n(\mathbb{C})$ is completely positive iff Φ is of the form $\Phi(x) = \sum_{i=1}^m v_i x v_i^*$ for all $x \in M_n(\mathbb{C})$ by some $\{v_i\}_{i=1}^m \subset M_n(\mathbb{C})$. Moreover, for $\{v_i\}_{i=1}^m$ inducing the form $\Phi(x) = \sum_{i=1}^m v_i x v_i^*$, we may require that $\{v_i\}_i$ is linearly independent so that in the form the number m is uniquely determind. Such a form was called a 'canonical' expression for Φ (see [3, Remark 4]).

Let us call the uniquely determind number m the *size of the* Φ.

Now we pick up the case where Φ is a unital completely positive (called "ucp" for short) map of $M_n(\mathbb{C})$. Then the $\{v_1, \cdots, v_m\} \subset M_n(\mathbb{C})$ used in the form $\Phi(x) = \sum_{i=1}^m v_i x v_i^*, (x \in M_n(\mathbb{C}))$ satisfies that $\sum_{i=1}^m v_i v_i^* = 1$. This means that each ucp map Φ of $M_n(\mathbb{C})$ is induced some $\{v_1, \cdots, v_m\}$ in $FOP(M_n(\mathbb{C}))$.

Given an operator $v \in M_n(\mathbb{C})$, the map $\mathrm{Ad}v$ on $M_n(\mathbb{C})$ is given by $\mathrm{Ad}v(x) = vxv^*, (x \in M)$. Then the group $Aut(M_n(\mathbb{C}))$ of all automorphisms of $M_n(\mathbb{C})$ is written by the form $Aut(M_n(\mathbb{C})) = \{\mathrm{Ad}u \mid u \in U(M_n(\mathbb{C}))\}$, where $U(M_n(\mathbb{C}))$ is the group of all unitaries in $M_n(\mathbb{C})$. Similarly, the set $UCP(M_n(\mathbb{C}))$ of all ucp maps on $M_n(\mathbb{C})$ is written by the following form:

$$UCP(M_n(\mathbb{C})) = \{\sum_{i=1}^m \mathrm{Ad}v_i \mid \{v_i\}_{i=1}^m \in FOP(M_n(\mathbb{C})),\ m = 1, 2, \cdots\} \quad (2.2)$$

3 Operational Convex Combination

3.1 Operational convexity

Definition 3.1. Let $\{\Phi_i\}_{i=1}^m$ be a set of linear maps on $M_n(\mathbb{C})$ and $\{v_i\}_{i=1}^m \in FOP(M_n(\mathbb{C}))$. We call $\sum_{i=1}^m \mathrm{Ad}v_i \circ \Phi_i$ an *operational convex combination* of $\{\Phi_i\}_{i=1}^m$ with an *operational coefficients* $\{v_i\}_{i=1}^m$. We also say that a subset S of linear maps on $M_n(\mathbb{C})$ is *operational convex* if it is closed under all operational convex combinations.

We can consider $UCP(M_n(\mathbb{C}))$ as the set of all operator convex combinations of the group $Aut(M_n(\mathbb{C}))$. Moreover $UCP(M_n(\mathbb{C}))$ is represented as the set of all operational convex combinations of the identity id of $M_n(\mathbb{C})$. We give some characterization for a role of $Aut(M_n(\mathbb{C}))$ in $UCP(M_n(\mathbb{C}))$ from a view point of extreme points.

3.1.1 Cuntz's canonical endomorphism as an example

The Cuntz's canonical endomorphism Φ_n ([3]) is an interesting example in unital completely positive maps of infinite dimensional simple C^*-algebras, which is given as an operational convex combination of the identity: Let $\{S_1, S_2, \cdots, S_n\}$ be isometries on an infinite dimensional Hilbert space H such that $\sum_i S_i S_i^* = 1$. The Cuntz algebra O_n is the C^*-algebra generated by $\{S_1, S_2, \cdots, S_n\}$. The map Φ_n is given as $\Phi_n(x) = \sum_i S_i x S_i^*$ for all $x \in O_n$. So, in our notation, $\{S_1, S_2, \cdots, S_n\} \in FOP(O_n)$ and $\Phi_n \in UCP(O_n)$.

The left inverse Ψ of Φ_n plays an inportant role in the theory of Cuntz algebras and it is given by the form $\Psi(x) = (1/n)\sum_i S_i^* x S_i, (x \in O_n)$.

We remark that Ψ is also an operational convex combination of the identity and $\Psi \in UCP(O_n)$.

Later we discuss in another paper on the case of unital infinite dimensional C^*-algebras represented by O_n.

3.1.2 Operational extreme point

Now let us remember the notion of extreme points. Let S be a convex set. Then a $z \in S$ is an extreme point in S if z cannot be the convex combination $\lambda x + (1-\lambda)y$ of two points $x, y \in S$ with $x \neq y$ and $\lambda \in (0,1)$, i.e., if $z = \lambda x + (1-\lambda)y, (x, y \in S)$ then $x = y = z$.

In this note, we say this notion of extreme points an extreme point *in the usual sense.*

In the usual sense, any automorphism of $M_n(\mathbb{C})$ can not be expressed as a convex combination of two automorphisms (see [10]). However if we replace a convex combination to an oparational convex combination, then it is possible as in the following example:

Example 3.2. Let v, w be unitaries in $M_n(\mathbb{C})$, and let $\lambda \in (0,1)$.

Let $a = \lambda^{1/2}v^*$ and $b = (1-\lambda)^{1/2}w^*$. Then $\{a, b\} \in FOP(M_n(\mathbb{C}))$, and the operational convex conbination of the automorphisms Φ and Ψ of $M_n(\mathbb{C})$ with $\Phi = \mathrm{Ad}v$ and $\Psi = \mathrm{Ad}w$ with the operational coefficients $\{a, b\}$ is the identity map, i.e., $a\Phi(x)a^* + b\Psi(x)b^* = x$ for all $x \in M_n(\mathbb{C})$.

Moreover, for a given $\Theta \in Aut(M_n(\mathbb{C}))$ with $\Theta = Adu$, $(u \in U(M_n(\mathbb{C}))$, if we let $a' = \lambda^{1/2}uv^*$ and $b' = (1-\lambda)^{1/2}uw^*$ then $\{a', b'\} \in FOP(M_n(\mathbb{C}))$, and the operational convex conbination of Φ and Ψ with the operational coefficients $\{a', b'\}$ is the automorphism Θ, i.e., $a'\Phi(x)a'^* + b'\Psi(x)b'^* = \Theta(x)$ for all $x \in M_n(\mathbb{C})$.

In the case of operational convex combinations for linear maps Φ and Ψ on $M_n(\mathbb{C})$ with an operational coefficient $\{a, b\} \in FOP(M_n(\mathbb{C})$, the map $\mathrm{Ad}a \circ \Phi$ corresponds λx and the $\mathrm{Ad}b \circ \Psi$ does $(1-\lambda)y$. Putting this in mind, let us define as follows and show that an automorphism of $M_n(\mathbb{C})$ (i.e., the ucp maps with the size 1) is an operational extreme point.

Definition 3.3. Let S be an operational convex subset of linear maps on $M_n(\mathbb{C})$. We say that a $\Phi \in S$ is an *operational extreme point* of S if a reprensentation of Φ that $\Phi = \mathrm{Ad}a \circ \Phi_1 + \mathrm{Ad}b \circ \Phi_2$, $(\Phi_i \in S, (i = 1, 2), \{a, b\} \in FOP(M_n(\mathbb{C}))$ implies that $aa^* = \lambda 1_{M_n(\mathbb{C})}$, $bb^* = (1-\lambda)1_{M_n(\mathbb{C})}$ for some $\lambda \in (0, 1)$ so that $\lambda^{-1}\mathrm{Ad}a \circ \Phi_1 = \{1-\lambda\}^{-1}\mathrm{Ad}b \circ \Phi_2 = \Phi$.

Remark 3.4. (1) If $\{a, b\} \in FOP(M_n(\mathbb{C}))$, then, of course, $\{aa^*, bb^*\}$ is always a finite partition of unity in the Connes-Størmer's sense. This definition says that if an operational extreme point $\Phi \in S$ is represented as an operational convex combination of $\Phi_i \in S, (i = 1, 2), \{a, b\} \in FOP(M_n(\mathbb{C}))$, then the Connes-Størmer partition $\{aa^*, bb^*\}$ is a probability vector.

(2) If a $\Phi \in S$ is an operational extreme point of a convex subset S of linear maps on $M_n(\mathbb{C})$, then Φ is an extreme point of S in the usual sense.

In fact, assume that $\Phi = \lambda\Phi_1 + (1-\lambda)\Phi_2$, $(\Phi_i \in S)$ and Φ is an operational extreme point of S. Then $\Phi_1 = \lambda^{-1}\lambda\Phi_1 = \Phi$ and $\Phi_2 = \{1-\lambda\}^{-1}(1-\lambda)\Phi_2 = \Phi$ so that Φ is an extreme point of S in the usual sense.

The following lemma plays a key role to prove our main theorem:

Lemma 3.5. *Assume that* $\sum_{i=1}^m \mathrm{Ad}v_i = \mathrm{Ad}u$ *for* $\{v_i\}_{i=1}^m \in FOP(M_n(\mathbb{C}))$ *and* $u \in U(M_n(\mathbb{C}))$. *Then* v_i *is a scalar multiple of* u *for all* $i = 1, 2, \cdots, m$.

Theorem 3.6. *If an automorphism* Θ *of* $M_n(\mathbb{C})$ *is decomposed into an operational convex combination that* $\Theta = \mathrm{Ad}a \circ \Phi + \mathrm{Ad}b \circ \Psi$ *via* $\{a, b\} \in$

$FOP(M_n(\mathbb{C}))$ and $\Phi, \Psi \in UCP(M_n(\mathbb{C}))$, then there exist unitaries $v, w \in M_n(\mathbb{C})$ and a $\lambda \in (0,1)$ such that

$$a = \sqrt{\lambda} uv^*, \ b = \sqrt{1-\lambda} uw^* \quad \textit{and} \quad \Phi = \mathrm{Ad}v, \ \Psi = \mathrm{Ad}w \tag{3.1}$$

where u is a unitary with $\Theta = Adu$.

As a direct concequence of this theorem, we have the following:

Corollary 3.7. *The automorphism group is a subset of the operational extreme points of the unital completely positive linear maps on $M_n(\mathbb{C})$.*

As we remarked in the above, an operational extreme point is an extreme point in the usual sense, we have the following another consequence:

An automorphism of $M_n(\mathbb{C})$ is an extreme point of the set of ucp maps on $M_n(\mathbb{C})$ in the usual sense.

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