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A method of diminishing intervals and semiclosed subspaces

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1. INTRODUCTION

Let H be an infinite dimensional separable complex Hilbert space and let $\mathcal{B}(H)$ be the set of all bounded operators on H. A bounded operator on H means a linear bounded operator from H to H. This report is treated with the Invariant Subspace Problem on H. That is, for any bounded operator T, does there exist nontrivial closed subspace $M(\neq \{0\}, H)$ such that $TM \subseteq M$? This problem has not been completely solved yet. But it is known that, under additional conditions with T (normal, subnormal, having rich spectrum and so on), there exists a nontrivial T-invariant closed subspace.

In our study for ISP, we do not set additional conditions on T. However, we set hypothesises on something which are not concerned with T. Something is 'a choice function' which assigns a Hilbert norm to each semiclosed subspace. Among of all, we paticularly consider a choice function satisfying hypothesises (h1), (h2) and (h3) as stated in the following section. We do not know whether the choice funtion truely exists or not. A reason why we consider hypothesises is that, if a proposition 'any bounded operator T on H has nontrivial T-invariant closed subspace' is independent of ZFC (+CH), then we are disconsolate.

Our aim of this report is to state an outline of our approach to ISP, which does not mean that ISP can be solved by this way. There are some key words in our approach, that is, *a linear dimension*, *a Hilbert dimension*, *a choice function with hypothesises*, *T-invariant semiclosed subspaces*, *a codimension* and so on. The following Theorem1.1 is one of motivations for our study. It says that, under CH (continuous hypothesis), a separable Hilbert space is the only case the Hilbert dimension and the linear dimension are different. We feel a kind of corresponding between Theorem1.1 and ISP from the fact that a separable Hilbert space is the only case ISP is not solved.

Theorem 1.1 ([2]). *Let H be a Hilbert space. Then,*

(1) $\dim H < \infty \iff \dim_l H < \infty$. *In this case,* $\dim H = \dim_l H$.
(2) *If* $\aleph_0 \leq \dim H < 2^{\aleph_0}$, *then* $\dim_l H = 2^{\aleph_0}$. *Hence,* $\dim H < \dim_l H$.
(3) *If* $2^{\aleph_0} \leq \dim H$, *then* $\dim H = \dim_l H$.

2. From semiclosed subspaces to a closed subspace

It is well known ([3]) that any bounded operator T on H have many T-invariant semiclosed subspaces. If ISP is affirmatively solved, then the set of all nontrivial T-invariant semiclosed subspaces necessarily contains a closed subspace. In the following, we seek (but now, can not obtain) a nontrivial closed subspace among the set of all T-invariant semiclosed subspaces by a method of diminishing intervals of semiclosed subspaces, which are based on a choice function as stated above.

Definition 2.1. *A subspace M in H is said to be semiclosed if there exists a Hilbert norm $\|\cdot\|_M$ on M such that $(M, \|\cdot\|_M) \hookrightarrow H$ (continuously embedded).*

It is easily shown that a semiclosed subspace is equivalent to an operator range. Clearly, a closed subspace is semiclosed.

The following theorem plays a central role for applying a method of diminishing intervals to ISP.

Theorem 2.1 ([4]). *Let $T \in \mathcal{B}(H)$ and let M_1 and M_2 be nontrivial T-invariant semiclosed subspaces such that $M_1 \subsetneq M_2$ in H. Suppose that $\dim_l M_2/M_1 > 1$. Then, there exists T-invariant semiclosed subspace M_3 such that*

$$M_1 \subsetneq M_3 \subsetneq M_2.$$

3. A choice function α

By Definition 2.1, we can consider a concept of the following function in Definition 3.1.

Definition 3.1. *We choose a Hilbert norm from each semiclosed subspace. Denote such a choice function by α.*

Given a choice function α. For a semiclosed subspace M, there is the Hilbert norm $\|\cdot\|_M$ by α such that $(M, \|\cdot\|_M) \hookrightarrow H$. It follows from [1] that $(M, \|\cdot\|_M)$ is isometrically isomorphic to de Branges space $(\mathcal{M}(A), \|\cdot\|_A)$ for a unique positive operator $A \in \mathcal{B}(H)$. Then, we denote $M \overset{\alpha}{=} AH$. A corresponding between $\|\cdot\|_M$ and A is one to one. Hence, we consider a choice function α as a choice of positive operators. Summing up, the notation α has two meanings such as a choice of Hilbert norms or a choice of positive operators.

We explain an example of α in $L^2(\mathbb{R}^d)$ $(d \geq 1)$. A subspace

$$M^\sigma := \{f \in L^2(\mathbb{R}^d) : (1+|\xi|^2)^{\frac{\sigma}{2}} \hat{f}(\xi) \in L^2(\mathbb{R}^d)\} \quad (\sigma > 0)$$

of $L^2(\mathbb{R}^d)$ is semiclosed. Because, the inclusion mapping $(M^\sigma, \|\cdot\|_\sigma) \hookrightarrow L^2(\mathbb{R}^d)$ is contraction, where $\|f\|_\sigma := \|(1+|\xi|^2)^{\frac{\sigma}{2}} \hat{f}\|_{L^2}$ ($\hat{f}$ is the Fourier transform of f). As you know, $H^\sigma(\mathbb{R}^d) := (M^\sigma, \|\cdot\|_\sigma)$ is known as Sobolev spaces of the order σ. Then, we choose Sobolev norm $\|\cdot\|_\sigma$ for each semiclosed subspace M^σ for $\sigma > 0$, and we choose the appropriate Hilbert norm for other semiclosed subspaces.

On the other hand, it is proved ([5]) that Sobolev space $H^\sigma(\mathbb{R}^d)$ is isometrically isomorphic to de Branges space $\mathcal{M}(A_\sigma)$, where $A_\sigma = (I - \Delta)^{-\frac{\sigma}{2}}$ (the Bessel potential of the order σ). Hence we can also say that we choose the Bessel potential A_σ from Sobolev space $H^\sigma(\mathbb{R}^d)$, and we choose the appropriate positive operator from each semiclosed subspace except for Sobolev spaces.

4. Hypothesises of α

From this section, we handle a paticular choice function for constructing a method of diminishing intervals. A 'paticular' choice function means that it satisfies some hypothesises (h1), (h2) and (h3).

In the following statements (h2) and (h3), we use the notation α as a subset of $\mathcal{B}(H)$.

(h1) For semiclosed subspaces M, N such that $M \subseteq N$,

$$(M, \|\cdot\|_M) \hookrightarrow (N, \|\cdot\|_N) \quad \text{(contractively)}$$

(h2) $\alpha = \alpha^2$, where $\alpha^2 := \{A^2 : A \in \alpha\}$.

(h3) A set α is closed in $\mathcal{B}(H)$, that is, $\alpha = \overline{\alpha}$.

Remark 4.1.

(i) *A set α consists of positive contractions A $(0 \leq A \leq I)$.*

(ii) *For $M \overset{\alpha}{=} AH$ and $N \overset{\alpha}{=} BH$ $(M \subseteq N)$,* (h1) *if and only if $A^2 \leq B^2$.*

(iii) *For a closed subspace $M \overset{\alpha}{=} AH$, it deduces $AH = A^{\frac{1}{2}}H$. Since $A \in \alpha$ implies $A^{\frac{1}{2}} \in \alpha$ by* (h2)*, we see $A = A^{\frac{1}{2}}$. That is, A is the orthogonal projection.*

5. A metric on semiclosed subspaces

Let $\mathcal{S}$ be the set of all semiclosed subspaces in H. For semiclosed subspaces $M \overset{\alpha}{=} AH$ and $N \overset{\alpha}{=} BH$, we define a metric ρ_α on $\mathcal{S}$ by

$$\rho_\alpha(M, N)\big(= \rho(M, N)\big) := \|A - B\| \quad \text{(the operator norm)}$$

Example 5.1 ([5]). *The metric between Sobolev spaces are given by the following. For $0 < \sigma_1 < \sigma_2$,*

$$\rho((H^{\sigma_1}(\mathbb{R}^d), H^{\sigma_2}(\mathbb{R}^d)) = (\frac{\sigma_1}{\sigma_2})^{\frac{\sigma_1}{\sigma_2 - \sigma_1}} - (\frac{\sigma_1}{\sigma_2})^{\frac{\sigma_2}{\sigma_2 - \sigma_1}}.$$

In particular, $\rho(H^1(\mathbb{R}^d), H^2(\mathbb{R}^d)) = 0.25$.

Under the hypothesises (h1), (h2) and (h3), we have the following propositions.

Proposition 5.1. *The metric space $(\mathcal{S}, \rho_\alpha)$ is complete.*

We define an interval $[M, N]$ for semiclosed subspaces $M \subset N$.

$$[M, N] := \{L \in \mathcal{S} : M \subseteq L \subseteq N\}.$$

Proposition 5.2. *An interval $[M, N]$ is closed in $(\mathcal{S}, \rho_\alpha)$.*

We define the diameter of $I = [M, N]$ by a usual way.

$$\mathrm{diam} I := \sup_{L, L' \in I} \rho_\alpha(L, L').$$

Proposition 5.3. *Let $I_n := [M_n, N_n]$ $(n = 1, 2, \cdots)$. Then, as $n \to \infty$,*

$$\mathrm{diam} I_n \to 0 \Longleftrightarrow \rho_\alpha(M_n, N_n) \to 0$$

From previous propositions, we have 'a method of diminishing intervals' of semiclosed subspaces.

Theorem 5.4. *Let $I_n := [M_n, N_n]$ $(n = 1, 2, \cdots)$ such that $I_n \supset I_{n+1}$. If $\rho_\alpha(M_n, N_n) \to 0$ $(n \to \infty)$, then there exists a unique semiclosed subspace $M_\infty \in \bigcap_{n=1}^\infty I_n$.*

6. A THINKING FOR ISP

Given any $T \in \mathcal{B}(H)$. By [3], there exists many T-invariant semiclosed subspaces. We want to find a closed subspace among the set of all non-trivial T-invariant semiclosed subspaces $\{M_\lambda\}_{\lambda \in \Lambda}$. Without a loss of generality, we may assume that the linear dimension $\dim_l H/M_\lambda = \infty$ for all λ. For, $\dim_l H/M_\lambda < \infty$ for some λ implies that M_λ is closed. (It is known that a semiclosed subspace which has finite codimension in H is closed in H.) Roughly explaining, an idea related with a codimension is the following.

· The first step.

Pick non-trivial T-invariant semiclosed subspaces M_1 and M_2 such that

$$(*) \quad \cdots\cdots \quad \begin{cases} M_1 \subsetneq M_2 \\ \dim_l M_2/M_1 = \infty. \end{cases}$$

We put the interval $I_1 := [M_1, M_2]$.

· The second step.

Since $\dim_l M_2/M_1 = \infty$, there exists ([4]) a T-invariant semiclosed subspace M_3 such that

$$M_1 \subsetneq M_3 \subsetneq M_2.$$

Then, we see that $\dim_l M_3/M_1 = \infty$ or $\dim_l M_2/M_3 = \infty$. Now we suppose that $\dim_l M_3/M_1 = \infty$. Then we put the interval $I_2 := [M_1, M_3]$.

· The third step.

Since $\dim_l M_3/M_1 = \infty$, there exists a T-invariant semiclosed subspace M_4 such that

$$M_1 \subsetneq M_4 \subsetneq M_3.$$

Then, we see that $\dim_l M_4/M_1 = \infty$ or $\dim_l M_3/M_4 = \infty$. Now we suppose that $\dim_l M_3/M_4 = \infty$. Then we put the interval $I_3 := [M_4, M_3]$. By an inductive way, we get a sequence $\{I_n\}_n$ of intervals. Note that semiclosed subspaces M_i $(i = 1, 2, \cdots)$ are T-invariant.

$$\underbrace{[M_1, M_2]}_{I_1} \supset \underbrace{[M_1, M_3]}_{I_2} \supset \underbrace{[M_4, M_3]}_{I_3} \supset \cdots$$

'If' diam $I_n \to 0$, then, by Theorem 5.4, we get the semiclosed subspace

$$\exists!\ M_\infty \in \bigcap_{n=1}^{\infty} I_n.$$

We want to expect that M_∞ is the T-invariant closed subspace, so that it is necesarry to check the following questions.

Q1. Does there exist M_1 and M_2 satisfying $(*)$?
Q2. Does there exist a sequence $\{I_n\}_n$ such that 'diam $I_n \to 0$' ?
Q3. Is M_∞ T-invariant ?
Q4. Is M_∞ closed ?

7. About Q1

There exist semiclosed subspaces M_1 and M_2 such that $(*)$. Pick any T-invariant M_1. Put $M_1 \overset{\alpha}{=} A_1 H$. Since $A_1 \in \alpha$, $A_1^{\frac{1}{2}} \in \alpha$ by (h2). Hence, we define $M_1^{\frac{1}{2}}$ to be the semiclosed subspace $A_1^{\frac{1}{2}} H$, and let $M_1^{\frac{1}{2}} \overset{\alpha}{=} A_1^{\frac{1}{2}} H$. If $M_1 = M_1^{\frac{1}{2}}$, then M_1 is a (T-invariant) closed subspace. This attains our goal. Thus, without a loss of generality, we assume that $M_1 \subsetneq M_1^{\frac{1}{2}}$. Since M_1 is T-invariant, $M_1^{\frac{1}{2}}$ is also T-invariant. Since $\dim_l H/M_1^{\frac{1}{2}} = \infty$, there exists T-invariant M_2 satisfying $M_1^{\frac{1}{2}} \subsetneq M_2 \subsetneq H$. Moreover we can prove that $\dim_l M_1^{\frac{1}{2}}/M_1 = \infty$. Then we get the interval $[M_1, M_2]$ which satisfies the condition $(*)$.

8. About Q2, Q3 and Q4

We do not know that how do we try to solve these questions Q2, Q3 and Q4. But we think that these questions are linked each other. In the following, we state an idea in our brain.

Let $M \overset{\alpha}{=} AH$ and $N \overset{\alpha}{=} BH$. By (h1) and Douglas's majorization theorem, the interval

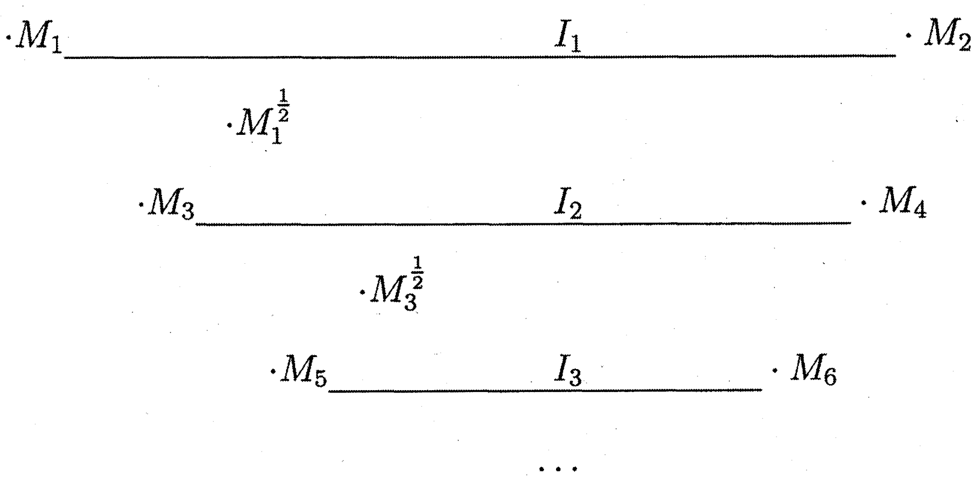
$$[M,N] = \{L \in \mathcal{S} : M \subseteq L \subseteq N\}$$

is order isomorphic to the operator interval

$$[A^2, B^2] := \{X^2 : A^2 \leq X^2 \leq B^2\},$$

where $X \in \mathcal{B}(H)$ is a positive operator.

Let $I_1 = [M_1, M_2]$ be the interval satisfying the condition $(*)$ in the section 6. Since $\dim_l M_1^{\frac{1}{2}}/M_1 = \infty$ and $M_1^{\frac{1}{2}}$ is T-invariant, there exists a T-invariant semiclosed subspace M_3 such that $M_1 \subsetneq M_3 \subsetneq M_1^{\frac{1}{2}}$. In the same way, we have a T-invariant semiclosed subspace M_5 such that $M_3 \subsetneq M_5 \subsetneq M_3^{\frac{1}{2}}$. Hence, we have inductively M_{2k+1} $(k = 1, 2, \cdots)$ such that $M_{2k-1} \subset M_{2k+1} \subset M_{2k-1}^{\frac{1}{2}}$. On the other hand, with respect to indexes of even numbers, we choose a T-invariant semiclosed subspace M_{2k+2} so that $\rho_\alpha(M_{2k+1}, M_{2k+2}) \to 0$ $(k \to \infty)$, and let $I_{k+1} = [M_{2k+1}, M_{2k+2}]$ $(k = 0, 1, 2, \cdots)$.



By $\rho_\alpha(M_{2k+1}, M_{2k+2}) \to 0$, that is, $\mathrm{diam} I_k \to 0$ $(k \to \infty)$, there uniquely exists the semiclosed subspace $M_\infty \in \cap_{k=1}^{\infty} I_k$. For intervals $[M_n, \ \cdot\ ]$ or $[\ \cdot\ , M_n]$, $\rho_\alpha(M_n, M_\infty) = \|A_n - A_\infty\| \to 0$ $(n \to \infty)$, where $M_n \overset{\alpha}{=} A_n H$, $M_\infty \overset{\alpha}{=} A_\infty H$. Therefore, since $A_{2k+1}^2 \leq A_{2k-1} \leq A_{2k+2}^2$ $(k = 1, 2, \cdots)$, we have that $A_\infty^2 \leq A_\infty \leq A_\infty^2$. This means that A_∞ is the orthogonal projection, i.e., $M_\infty \overset{\alpha}{=} A_\infty H$ is closed, which is a thinking for Q2 and Q4. For Q3, we note that M_{2k+1} and M_{2k+2} are T-invariant and $M_{2k+1} \subsetneq M_\infty \subsetneq M_{2k+2}$. Now $\rho_\alpha(M_{2k+1}, M_{2k+2}) \to 0$ $(k \to \infty)$, is M_∞ T-invariant?

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REFERENCES

[1] T.Ando, *De Branges Spaces and Analytic Operator Functions*, Lecture note, Hokkaido University, Sapporo, Japan, 1990.
[2] 高村多賀子, 復刊　基礎数学シリーズ　関数解析入門　朝倉書店 (Japanese)
[3] E.Nordgren, M.Radjabalpour, H.Radjavi and P.Rosenthal, *On Invariant Operator Ranges*, Trans. Amer. Math. Soc. vol.251 (1979), 389-398.
[4] L.Rodman and N.Zobin, *Linear Preservers of Isomorphic Types of Lattices of Invariant Operator Ranges* , Proc. Amer. Math. Soc. vol.129 (2001), 2981-2986.
[5] G.Hirasawa, *A Metric for Unbounded Linear Operators in a Hilbert Space*, Integr. Equ. Oper. Theory vol.70 (2011), 363-378.