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COMPOSITION OPERATORS IN L^2-SPACES, WEIGHTED SHIFTS ON DIRECTED TREES, AND INDUCTIVE LIMITS

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1. Introduction

In this note we survey some recent results concerning unbounded composition operators induced by matrices and unbounded weighted shifts on directed trees which were obtained by methods originated from the notions of the inductive limit of Hilbert spaces and the inductive limit of Hilbert space operators. It is not surprising that there are fundamental differences between studying bounded and unbounded operators. This applies also to the aforementioned classes of operators. As it turns out, in particular when dealing with the subnormality, dense definiteness and boundedness, using inductive limits can be helpful. Employing these versatile methods bridges nicely highly developed theories of bounded composition operators and classical weighted shifts with new and still developing theories of unbounded composition operators and weighted shifts on directed trees.

Composition operators in L^2-spaces can be found in many areas of mathematics. They are basic objects in classical mechanics (in the operatorial model due to B. O. Koopman and J. von von Neumann), ergodic theory, theory of dynamical systems and more. They are also very appealing from the operator theory point of view (see the monograph [31] and references therein). They belong to a larger class of operators composed of weighted composition operators in L^2-spaces. Besides composition operators, the class contains also multiplication operators in L^2-spaces and weighted shifts on directed trees. Multiplication operators are classical and well-known objects of operator theory, they can be found in any textbook concerning the subject due to their relevance to the spectral theorem. Weighted shifts on directed trees have been introduced recently in [21] but they generalize in a natural way classical weighted shifts in ℓ^2-spaces.

Unbounded composition operators in L^2-spaces have become objects of intensive studies quite recently. They proved to be extremely interesting ([14, 19, 7, 8, 9, 10, 13]). Composition operators induced by linear transformations of $\mathbb{R}^\kappa$ have been investigated in [29, 32, 17, 33] (in bounded case) and in [12, 13, 3, 4] (in unbounded case). The class of weighted shifts on directed trees, introduced in [21], generalizes that of classical weighted shifts on directed trees and weighted adjacency operators. Studying them has brought many highly nontrivial and interesting results (cf. [2, 5, 6, 21, 22, 23, 24, 25]).

Subnormal operators have been introduced by Halmos. Theory of subnormal operators turned out to be highly successful and it led to numerous problems in functional analysis, operator theory, and mathematical physics. The theory of bounded operators is well-developed now (see the monograph [15] and references therein). Theory of unbounded subnormal operators, having much shorter history, brought plenty of interesting results and problems as well (see [1, 18, 34, 35, 36] for the foundations). Subnormal operators and their relatives play a vital role in operator theory nowadays.

2. Preliminaries

2.1. Basic terminology. In all what follows $\mathbb{Z}_+$ stands for the set of nonnegative integers and $\mathbb{N}$ for the set of positive integers; $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. If X is a topological space, then $\mathfrak{B}(X)$ stands for the family of Borel subsets of X. For $n \in \mathbb{N}$, m_n denotes the n-dimensional Lebesgue measure on $\mathfrak{B}(\mathbb{R}^n)$. If X is a set, then $\operatorname{card}(X)$ stands for the cardinal number of X.

Let $\mathcal{H}$ be a (complex) Hilbert space and A be an operator in $\mathcal{H}$ (all operators are linear in this paper). By $\mathcal{D}(A)$, $\overline{A}$, and A^* we denote the domain, the closure, and the adjoint of A, respectively (if they exists). The set of C^∞-vectors of A is defined by $\mathcal{D}^\infty(A) := \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$. A is said to be *subnormal* if $\mathcal{D}(A)$ is dense in $\mathcal{H}$ and there exist a complex Hilbert space $\mathcal{K}$ and a normal operator N in $\mathcal{K}$ (i.e., N is closed, densely defined and satisfies $N^*N = NN^*$) such that $\mathcal{H}$ is isometrically embedded in $\mathcal{K}$ and $Ah = Nh$ for all $h \in \mathcal{D}(A)$. If A is densely defined and A^* is subnormal, then A is *cosubnormal*. A is *symmetric* whenever A is densely defined, $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $Af \subseteq A^*f$ for every $f \in \mathcal{D}(A)$. In turn, if A is densely defined, $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$, and $\|A^*f\| \leqslant \|Af\|$ for every $f \in \mathcal{D}(A)$, then A is said to be *hyponormal*. A linear subspace $\mathcal{F}$ of $\mathcal{D}(A)$ is called a *core* of A if $\mathcal{F}$ is dense in $\mathcal{D}(A)$ in the graph norm induced by A, i.e, the norm $\|\cdot\|_A$ given by $\|f\|_A^2 = \|Af\|^2 + \|f\|^2$, for $f \in \mathcal{D}(A)$. If $\mathcal{F}$ is a subspace of $\mathcal{H}$, then $A|_{\mathcal{F}}$ is the operator in $\mathcal{H}$ acting on the domain $\mathcal{D}(A|_{\mathcal{F}}) = \mathcal{F} \cap \mathcal{D}(A)$ according to the formula $A|_{\mathcal{F}}f = Af$.

Let $\mathcal{H}$ and $\{\mathcal{H}_k\}_{k=1}^\infty$ be Hilbert spaces. If $\mathcal{H} \subseteq \mathcal{H}_{k+1} \subseteq \mathcal{H}_k$ for every $k \in \mathbb{N}$, where "$\subseteq$" means inclusion of vector spaces, and $\|f\|_{\mathcal{H}} = \lim_{k\to\infty} \|f\|_{\mathcal{H}_k}$ for every $f \in \mathcal{H}$, then we write $\mathcal{H}_k \downarrow \mathcal{H}$ as $k \to \infty$.

2.2. Weighted composition operators in L^2-spaces. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $w\colon X \to \mathbb{C}$ be an $\mathcal{A}$-measurable function and $\mathbf{A}\colon X \to X$ be an $\mathcal{A}$-measurable transformation of X i.e., $\mathbf{A}$ is a self-map of X such that $\mathbf{A}^{-1}(\mathcal{A}) \subseteq \mathcal{A}$. Define the σ-finite measure $\mu_w\colon \mathcal{A} \to \overline{\mathbb{R}}_+$ by $\mu_w(\varDelta) = \int_\varDelta |w|^2 \mathrm{d}\mu$ for $\varDelta \in \mathcal{A}$. Let $\mu_w \circ \mathbf{A}^{-1}\colon \mathcal{A} \to \overline{\mathbb{R}}_+$ be the measure given by $\mu_w \circ \mathbf{A}^{-1}(\varDelta) = \mu_w(\mathbf{A}^{-1}(\varDelta))$ for $\varDelta \in \mathcal{A}$. Assume that $\mu_w \circ \mathbf{A}^{-1}$ is absolutely continuous with respect to μ. Then the operator $C_{\mathbf{A},w}$ in $L^2(\mu)$ given by

$$\mathcal{D}(C_{\mathbf{A},w}) = \{f \in L^2(\mu)\colon w \cdot (f \circ \mathbf{A}) \in L^2(\mu)\},$$
$$C_{\mathbf{A},w} f = w \cdot (f \circ \mathbf{A}), \quad f \in \mathcal{D}(C_{\mathbf{A},w}),$$

is well-defined (cf. [11, Proposition 7]) and closed. We call $C_{\mathbf{A},w}$ a *weighted composition operator*.

The class of weighted composition operators contains some important subclasses:

- multiplication operators in L^2-spaces,
- composition operators in L^2-spaces,
- weighted shifts on directed trees.

The reader interested in unbounded weighted composition operators in L^2-spaces is referred to [14] and [11]. Below we provide further information on the classes of composition operators in L^2-spaces and weighted shifts on directed trees.

2.3. **Composition operators.** If $w \equiv 1$, then $C_{\mathbf{A}} := C_{\mathbf{A},1}$ is called a *composition operator*. Assuming that the Radon-Nikodym derivative

$$\mathsf{h}_{\mathbf{A}} = \frac{\mathrm{d}\mu \circ \mathbf{A}^{-1}}{\mathrm{d}\mu}$$

belongs to $L^\infty(\mu)$, the space of all $\mathbb{C}$-valued and essentially bounded functions on X, we can show that $C_{\mathbf{A}}$ is bounded on $L^2(\mu)$ and $\|C_{\mathbf{A}}\| = \|\mathsf{h}_{\mathbf{A}}\|_{L^\infty(\mu)}^{1/2}$. The reverse implication is also true. It known that $C_{\mathbf{A}}$ is closed. Moreover, if $\mathsf{h}_{\mathbf{A}} < \infty$ a.e. $[\mu]$, then $C_{\mathbf{A}}$ is densely defined. Classical reference on bounded composition operators is the monograph [31]. For up-to-date information on unbounded composition operators we refer the reader to [7] and [9]

Composition operators induced by transformations having additional properties are particularly interesting. In this paper we focus on composition operators induced by linear transformations of $\mathbb{R}^\kappa$. Such operators were first investigated in [29] and [32].

Denote by $\mathscr{E}_+$ the set of all entire functions γ on $\mathbb{C}$ of the form $\gamma(z) = \sum_{n=0}^\infty a_n z^n$, for $z \in \mathbb{C}$, where a_n are nonnegative real numbers and $a_k > 0$ for some $k \geqslant 1$. For a given positive integer κ, a function $\gamma \in \mathscr{E}_+$ and a norm $|\cdot|$ on $\mathbb{R}^\kappa$ induced by an inner product we define the σ-finite measure $\mu_\gamma = \mu_\gamma^{|\cdot|}$ on $\mathfrak{B}(\mathbb{R}^\kappa)$ by

$$\mu_\gamma(\mathrm{d}x) = \gamma(|x|^2)\mathrm{m}_\kappa(\mathrm{d}x).$$

If $\mathbf{A}$ is a linear transformation of $\mathbb{R}^\kappa$ (clearly, such an $\mathbf{A}$ is $\mathfrak{B}(\mathbb{R}^\kappa)$-measurable), we can verify that the composition operator $C_{\mathbf{A}}$ in $L^2(\mu_\gamma)$ is well-defined if and only if $\mathbf{A}$ is invertible. If this is the case, then (cf. [32, equation (2.1)])

$$\mathsf{h}_{\mathbf{A}}(x) = \frac{1}{|\det \mathbf{A}|}\frac{\gamma(|\mathbf{A}^{-1}x|^2)}{\gamma(|x|^2)}, \quad x \in \mathbb{R}^\kappa \setminus \{0\}.$$

(Here, and later on, $|\det \mathbf{A}|$ stands for the modulus of the determinant of $\mathbf{A}$.) Hence, by [7, Proposition 6.2], each well-defined composition operator $C_{\mathbf{A}}$ is automatically densely defined and injective. The following theorem solves the question of boundedness of $C_{\mathbf{A}}$.

Theorem 2.1 ([32, Proposition 2.2]). *Let γ be in $\mathscr{E}_+$ and $|\cdot|$ be a norm on $\mathbb{R}^\kappa$ induced by an inner product. Let $\mathbf{A}$ be an invertible linear transformation of $\mathbb{R}^\kappa$. Then the following assertions hold:*

(1) *If γ is a polynomial, then $\mathbf{A}$ induces bounded composition operator on $L^2(\mu_\gamma)$ and on $L^2(\mu_{1/\gamma})$.*

(2) *If γ is not a polynomial, then $\mathbf{A}$ induces bounded composition operator on $L^2(\mu_\gamma)$ (resp. on $L^2(\mu_{1/\gamma})$) if and only if $\|\mathbf{A}^{-1}\| \leqslant 1$ (resp. $\|\mathbf{A}\| \leqslant 1$).*

2.4. **Weighted shifts on directed trees.** Let $\mathscr{T} = (V, E)$ be a directed tree (V and E stand for the sets of vertices and edges of $\mathscr{T}$, respectively). Denote by root the root of $\mathscr{T}$ (provided it exists) and write $\mathsf{Root}(\mathscr{T}) = \{\mathsf{root}\}$ if $\mathscr{T}$ has a root and $\mathsf{Root}(\mathscr{T}) = \varnothing$ otherwise. Define $V^\circ = V \setminus \mathsf{Root}(\mathscr{T})$. Set $\mathsf{Chi}(u) = \{v \in V\colon (u, v) \in E\}$ for $u \in V$. A member of $\mathsf{Chi}(u)$ is called a *child* of u. Denote by par the partial function from V to V which assigns to each vertex $u \in V^\circ$ its parent $\mathsf{par}(u)$ (i.e. a unique $v \in V$ such that $(v, u) \in E$). We refer the reader to [5, 21] for all facts about directed trees needed in this paper.

Denote by $\ell^2(V)$ the Hilbert space of all square summable complex functions on V with the inner product $\langle f, g\rangle = \sum_{u\in V} f(u)\overline{g(u)}$. For $u \in V$, we define $e_u \in \ell^2(V)$

to be the characteristic function of the one-point set $\{u\}$. Then $\{e_u\}_{u\in V}$ is an orthonormal basis of $\ell^2(V)$. Set $\mathscr{E}_V = \text{LIN}\{e_u : u \in V\}$, where $\text{LIN}\, X$ is a linear span of a set X. By a *weighted shift* on $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ} \subseteq \mathbb{C}$ we mean the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ defined by

$$\begin{aligned}\mathcal{D}(S_{\boldsymbol{\lambda}}) &= \{f \in \ell^2(V) \colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\},\\ S_{\boldsymbol{\lambda}} f &= \varLambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\boldsymbol{\lambda}})\end{aligned}$$

where $\varLambda_{\mathscr{T}}$ is the mapping defined on functions $f \colon V \to \mathbb{C}$ via

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ,\\ 0 & \text{if } v = \mathsf{root}.\end{cases}$$

The following result gives a connection between weighted shifts on rootless directed trees and composition operators.

Theorem 2.2 ([22, Lemma 4.3.1]). *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T} = (V, E)$ with positive weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$. Assume that $\mathscr{T}$ is rootless and countably infinite. Then $S_{\boldsymbol{\lambda}}$ is unitarily equivalent to a composition operator $C_{\mathbf{A}}$ in an L^2-space over a σ-finite measure space. Moreover, if the directed tree $\mathscr{T}$ is leafless, then $C_{\mathbf{A}}$ can be made injective.*

In fact, if the tree $\mathscr{T}$ is countably infinite, then any weighted shift is a weighted composition operator. Indeed, if $S_{\boldsymbol{\lambda}}$ is a weighted shift on a directed tree $\mathscr{T} = (V, E)$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$, then we set $X = V$ and $\mathscr{A} = 2^V$. The measure $\mu \colon \mathscr{A} \to \overline{\mathbb{R}}_+$ is the counting measure on X (it is σ-finite because V is countable). Now, define the weight function $w \colon X \to \mathbb{C}$ and the transformation $\mathbf{A} \colon X \to X$ by

$$w(x) = \begin{cases} \lambda_x & \text{if } x \in V^\circ\\ 0 & \text{if } x = \mathsf{root}\end{cases} \quad \text{and} \quad \mathbf{A}(x) = \begin{cases} \mathsf{par}(x) & \text{if } x \in V^\circ\\ \mathsf{root} & \text{if } x = \mathsf{root}.\end{cases}$$

Using the above definitions it is easy to observe that $S_{\boldsymbol{\lambda}} = C_{\mathbf{A},w}$ (the observation has already been used in [2]).

3. Inductive limits

In this section we show how methods inspired by inductive limits of Hilbert spaces and inductive limits of Hilbert space operators can be used when investigating the subnormality of composition operators with matrix symbols and weighted shifts on directed trees, and also dense definiteness and boundedness of composition operators with infinite matrix symbols.

Let us begin by recalling the notions of inductive limits of Hilbert spaces and Hilbert space operators. Suppose $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ is a sequence of Hilbert spaces. We say that a Hilbert space $\mathcal{H}$ is the *inductive limit* of $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ if there are isometries $\Lambda_k^l \colon \mathcal{H}_k \to \mathcal{H}_l$, $k \leqslant l$, and $\Lambda_k \colon \mathcal{H}_k \to \mathcal{H}$ such that the following conditions are satisfied:

(i) Λ_k^k is the identity operator on $\mathcal{H}_k$,
(ii) $\Lambda_k^m = \Lambda_l^m \circ \Lambda_k^l$ for all $k \leqslant l \leqslant m$,
(iii) $\Lambda_k = \Lambda_l \circ \Lambda_k^l$ for all $k \leqslant l$,
(iv) $\mathcal{H} = \overline{\bigcup_{n\in\mathbb{N}} \Lambda_n \mathcal{H}_n}$.

We write $\mathcal{H} = \mathtt{LIM}\,\mathcal{H}_n$ then.

Assume that $\mathcal{H} = \mathtt{LIM}\,\mathcal{H}_n$. For $n \in \mathbb{N}$, let C_n be an operator in $\mathcal{H}_n$. Consider the subspace $\mathrm{D}_\infty = \mathrm{D}_\infty(\{C_n\}_{n\in\mathbb{N}})$ of $\mathcal{H}$ given by

$$\mathrm{D}_\infty = \bigcup_{k\in\mathbb{N}} \{\Lambda_k f \mid \exists M \geqslant k\colon \Lambda_k^m f \in \mathrm{D}(C_m) \text{ for all } m \geqslant M\}$$

and define the operator $\mathtt{lim}\,C_n$ in $\mathcal{H}$ by

$$\mathrm{D}(\mathtt{lim}\,C_n) = \{\Lambda_k f \in \mathrm{D}_\infty\colon \lim_{m\to\infty} \Lambda_m C_m \Lambda_k^m f \text{ exists}\}$$

$$(\mathtt{lim}\,C_n)\Lambda_k f = \lim_{m\to\infty} \Lambda_m C_m \Lambda_k^m f, \quad \Lambda_k f \in \mathrm{D}(\mathtt{lim}\,C_n).$$

We call $\mathtt{lim}\,C_n$ the *inductive limit* of $\{C_n\}_{n\in\mathbb{N}}$.

Inductive limits have proved to be effective tools in operator theory. They have been used to study operators of a specific type (see [28] for the application of inductive limits to differential operators, and [26] for the application to classical weighted shifts) but also in a more general context (see [20] for the application to unbounded hyponormal operators, and [26] for the application to other hyponormality classes). The following two general ideas support using inductive limits (actually they are two in a sense opposite points of view on the same matter).

(1) Suppose that C is an operator in a Hilbert space $\mathcal{H}$ whose properties are to be verified. It may happen, especially if C is an unbounded operator, that handling C is difficult whence handling restrictions or some parts of C is relatively easy. If this is a case, then it seems natural to approximate the operator C with a sequence $\{C_n\}_{n\in\mathbb{N}}$ composed of operators related to the parts of C that are handleable. If this approximation is rigid enough, then we can expect that properties of C_n's are transferred to C. Assuming that $C = \mathtt{lim}\,C_n$ gives quite rigid approximation for many operators and many properties, so using inductive limits enables us to investigate properties of C by looking at properties of appropriately chosen C_n's.

(2) Suppose one is interested in providing an example of a Hilbert space operator say C having some particular property. If the property is transferable to some extent onto inductive limits of operators, then a natural solution to the problem is to construct a sequence $\{C_n\}_{n\in\mathbb{N}}$ of operators possessing the property in question such that the inductive limit $\mathtt{lim}\,C_n$ exists. Then $\mathtt{lim}\,C_n$ may serve as a basis for constructing the example.

Of course, in some cases using inductive limits in a strict sense is not possible, however some small deviations from the definition are sometimes acceptable. In particular, as we will see in further parts of the paper, the assumption that $\mathcal{H} = \mathtt{LIM}\,\mathcal{H}_n$ can be relaxed on some occasions. This is shown to be the case for subnormality (see the following section).

3.1. **Subnormality via inductive methods – general case.** The following theorem can be used as a basis for inductive limit approach.

Theorem 3.1 ([5, Theorem 3.1.1]). *Let $\{S_\omega\}_{\omega\in\Omega}$ be a net of subnormal operators in a complex Hilbert space $\mathcal{H}$ and let S be a densely defined operator in $\mathcal{H}$. Suppose that there is a subset $\mathcal{X}$ of $\mathcal{H}$ such that*

(i) $\mathcal{X} \subseteq \mathcal{D}^\infty(S) \cap \bigcap_{\omega\in\Omega} \mathcal{D}^\infty(S_\omega)$,

(ii) $\mathcal{F} := \mathrm{LIN}\bigcup_{n=0}^\infty S^n(\mathcal{X})$ *is a core of S,*

(iii) $\langle S^m x, S^n y\rangle = \lim_{\omega\in\Omega}\langle S_\omega^m x, S_\omega^n y\rangle$ *for all $x, y \in \mathcal{X}$ and $m, n \in \mathbb{Z}_+$.*

Then S is subnormal.

The above theorem has a version which in some cases is more effective.

Theorem 3.2 ([3, Lemma 3.7]). *Let S be a densely defined operator in a complex Hilbert space $\mathcal{H}$. Suppose that there are a family $\{\mathcal{H}_k\}_{k\in\mathbb{N}}$ of Hilbert spaces such that $\mathcal{H}_k \downarrow \mathcal{H}$ as $k \to \infty$, and a set $\mathcal{X} \subseteq \mathcal{H}$ such that*

(i) $\mathcal{X} \subseteq \mathcal{D}^\infty(S)$,
(ii) $\mathcal{F} := \operatorname{LIN} \bigcup_{n=0}^\infty S^n(\mathcal{X})$ *is a core of S,*
(iii) $\mathcal{F}$ *is dense in $\mathcal{H}_k$ for every $k \in \mathbb{N}$,*
(iv) $S|_{\mathcal{F}}$ *is a subnormal operator in $\mathcal{H}_k$ for every $k \in \mathbb{N}$.*

Then S is subnormal.

The two above theorems can be proved in a very much similar fashion by using the following criterion for subnormality invented in [16].

Theorem 3.3 ([16, Theorem 21]). *Let S be a densely defined linear operator in a complex Hilbert space $\mathcal{H}$ such that $S(\mathcal{D}(S)) \subseteq \mathcal{D}(S)$. Then, the following conditions are equivalent:*

(1) *S is subnormal*
(2) *for every $m \in \mathbb{N}$ and every $\{a_{p,q}^{i,j}\}_{p,q=0,\ldots,n}^{i,j=1,\ldots,m} \subset \mathbb{C}$,*

$$\sum_{i,j=1}^{m} \sum_{p,q=0}^{n} a_{p,q}^{i,j} \lambda^p \bar{\lambda}^q z_i \bar{z}_j \geqslant 0, \quad \lambda, z_1, \ldots, z_m \in \mathbb{C},$$

implies

$$\sum_{i,j=1}^{m} \sum_{p,q=0}^{n} a_{p,q}^{i,j} \langle S^p f_i^l, S^q f_j^k \rangle \geqslant 0, \quad f_1, \ldots, f_m \in \mathcal{D}(S).$$

Let us mention that there is also one handy tool, provided in [26, Proposition 1.5]. It allows to verify subnormality of a bounded operator, by studying subnormality of its restrictions to closed linear subspaces.

3.2. **Composition operators with matrix symbols and their subnormality.** The subnormality of bounded composition operators with matrix symbols has been completely characterized in terms of the symbol.

Theorem 3.4 ([32, Theorem 2.5]). *Let γ be in $\mathscr{E}_+$ and $|\cdot|$ be a norm on $\mathbb{R}^\kappa$ induced by an inner product. Let $\mathbf{A}$ be an invertible linear transformation of $\mathbb{R}^\kappa$ such that $C_{\mathbf{A}}$ is bounded on $L^2(\mu_\gamma)$. Then $C_{\mathbf{A}}$ is subnormal if and only if $\mathbf{A}$ is normal in $(\mathbb{R}^\kappa, |\cdot|)$*

The question about the subnormality of unbounded composition operators with matrix symbols arises naturally. There are at least two different approaches to this question. One relies on the so-called consistency condition. This approach is by far the most general one when it comes to studying the subnormality of unbounded composition operators in L^2-spaces (it has been used with a great success for example in [9, 10, 11]). However, in case of composition operators with matrix symbols an inductive limit based approach is as effective as the consistency condition approach, and it seems a bit simpler. Both the methods give the following criterion.

Theorem 3.5 ([9, Theorem 32], [3, Proposition 3.8]). *Let γ be in $\mathscr{E}_+$, $|\cdot|$ be a norm on $\mathbb{R}^\kappa$ induced by an inner product, and $\mathbf{A}$ be an invertible linear transformation of $\mathbb{R}^\kappa$. Then $C_{\mathbf{A}}$ is subnormal whenever $\mathbf{A}$ is normal in $(\mathbb{R}^\kappa, |\cdot|)$.*

The inductive limit based argument leading to the above result is as follows. We start with $\gamma \in \mathscr{E}_+$ and $\mathbf{A}$ as in Theorem 3.5. Then $C_{\mathbf{A}}$ is well-defined in $L^2(\mu_\gamma)$. Also, by Theorem 2.1, it is a densely defined operator. Now, since $\mathbf{A}$ is normal in $\mathbb{R}^\kappa, |\cdot|)$, the composition operator induced by $\mathbf{A}$ is a subnormal operator on $L^2(\mu_\beta)$, with $\beta \in \mathscr{E}_+$, whenever it is bounded (this follows from Theorem 3.4). Therefore, it seems natural to approach the subnormality of $C_{\mathbf{A}}$ in $L^2(\mu_\gamma)$ by approximating $C_{\mathbf{A}}$ with a sequence of composition operators induced by the same symbol $\mathbf{A}$ but acting in different spaces $L^2(\mu_{\gamma_n})$, $n \in \mathbb{N}$. The most natural choice of functions $\{\gamma_n\}_{n\in\mathbb{N}}$ is of course

$$\gamma_n(z) = \sum_{k=0}^{n} a_k z^k, \quad z \in \mathbb{C},\ n \in \mathbb{N},$$

where $\gamma(z) = \sum_{k=0}^{\infty} a_k z^k$. This choice has two advantages. Firstly, γ_n's are polynomials and this implies, by Theorem 2.1, that

$$\mathbf{A} \text{ induces a bounded composition operator on } L^2(\mu_{\gamma_n}),\ n \in \mathbb{N}.$$

Secondly, $\{\gamma_n\}_{n\in\mathbb{N}}$ approximates γ, which yields that $L^2(\mu_{\gamma_n})$ can be recovered as a limit of $\{L^2(\mu_{\gamma_n})\}_{n\in\mathbb{N}}$ in a sense of Theorem 3.2, i.e.,

$$L^2(\mu_{\gamma_n}) \downarrow L^2(\mu_\gamma) \text{ as } n \to \infty.$$

Now, it suffices to use Theorem 3.2 and deduce the subnormality of $C_{\mathbf{A}}$ in $L^2(\mu_\gamma)$.

3.3. **Weighted shifts on directed trees and their subnormality.** Applying the Lambert characterization of subnormality (see Section 4), using determinacy of the Stieltjes moment sequences generated by bounded subnormal operators, and employing some properties of weighted shifts on directed trees lead to the following characterization of the subnormality of weighted shifts on directed trees.

Theorem 3.6 ([21, Theorem 6.1.3 & Lemma 6.1.10], [5, Lemma 4.1.3]). *Let $S_{\boldsymbol\lambda}$ be a bounded weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then $S_{\boldsymbol\lambda}$ is subnormal if and only if there exist a system $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and a system $\{\varepsilon_v\}_{v\in V}$ of nonnegative real numbers that satisfy*

$$\mu_u(\sigma) = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{s}\,\mathrm{d}\mu_v(s) + \varepsilon_u \delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+), \tag{1}$$

for every $u \in V$.

It is natural to ask whether there exist an unbounded counterpart of the above theorem. It is hard to expect that conditions like (1) would be necessary for the subnormality of $S_{\boldsymbol\lambda}$ if the operator is unbounded (see Section 4 for some explanation). However, we can show them to be sufficient whenever the set of C^∞-vectors of $S_{\boldsymbol\lambda}$ is big enough. The idea of proving this relies on Theorem 3.1. It follows from this result that finding a sequence of subnormal operators approximating $S_{\boldsymbol\lambda}$ (in the sense of the condition (iii) of Theorem 3.1) will do the job. To construct such an approximating sequence we assume that $S_{\boldsymbol\lambda}$ is weighted shift on a directed tree $\mathscr{T}$ such that $\mathscr{E}_V \subseteq \mathcal{D}^\infty(S_{\boldsymbol\lambda})$ and that there exist a system $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and a system $\{\varepsilon_v\}_{v\in V}$ of nonnegative real numbers satisfying (1) for every $u \in V$. Then, for every fixed positive integer i, we define the system $\boldsymbol{\lambda}^{\langle i\rangle} = \{\lambda_v^{\langle i\rangle}\}_{v\in V^\circ}$ of complex numbers, the system $\{\mu_v^{\langle i\rangle}\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and the system $\{\varepsilon_v^{\langle i\rangle}\}_{v\in V}$ of nonnegative real numbers by

$$\lambda_v^{\langle i\rangle} = \begin{cases} \lambda_v\sqrt{\dfrac{\mu_v([0,i])}{\mu_{\mathsf{par}(v)}([0,i])}} & \text{if } \mu_{\mathsf{par}(v)}([0,i]) > 0, \\ 0 & \text{if } \mu_{\mathsf{par}(v)}([0,i]) = 0, \end{cases} \quad v \in V^\circ,$$

$$\mu_v^{\langle i\rangle}(\sigma) = \begin{cases} \dfrac{\mu_v(\sigma\cap[0,i])}{\mu_v([0,i])} & \text{if } \mu_v([0,i]) > 0, \\ \delta_0(\sigma) & \text{if } \mu_v([0,i]) = 0, \end{cases} \quad \sigma \in \mathfrak{B}(\mathbb{R}_+),\ v \in V,$$

$$\varepsilon_v^{\langle i\rangle} = \begin{cases} \dfrac{\varepsilon_v}{\mu_v([0,i])} & \text{if } \mu_v([0,i]) > 0, \\ 1 & \text{if } \mu_v([0,i]) = 0, \end{cases} \quad v \in V.$$

It can be showed that for all $u \in V$ and $i \in \mathbb{N}$,

$$\mu_u^{\langle i\rangle}(\sigma) = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v^{\langle i\rangle}|^2 \int_\sigma \frac{1}{s}\mathrm{d}\mu_v^{\langle i\rangle}(s) + \varepsilon_u^{\langle i\rangle}\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}_+).$$

Using the above one can deduce that $S_{\boldsymbol{\lambda}^{\langle i\rangle}}$, the weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda}^{\langle i\rangle}$, is a bounded operator on $\ell^2(V)$. In turn, by Theorem 3.6, the operator $S_{\boldsymbol{\lambda}^{\langle i\rangle}}$ is subnormal. Noting that

$$\lim_{i\to\infty} \|S^n_{\boldsymbol{\lambda}^{\langle i\rangle}} e_u\|^2 = \int_0^\infty s^n \mathrm{d}\mu_u(s) = \|\mathsf{S}^n_{\boldsymbol{\lambda}} e_u\|^2, \quad n \in \mathbb{Z}_+,\ u \in V,$$

using the fact that $\mathscr{E}_V$ is a core of $\mathsf{S}_{\boldsymbol{\lambda}}$, and applying Theorem 3.1 to the operators $\{S_{\boldsymbol{\lambda}^{\langle i\rangle}}\}_{i=1}^\infty$ and $\mathsf{S}_{\boldsymbol{\lambda}}$ with $\mathcal{X} := \{e_u\colon u \in V\}$ we deduce the required conclusion, i.e., the following.

Theorem 3.7 ([5, Theorem 5.1.1]). *Let $\mathsf{S}_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$ such that $\mathscr{E}_V \subseteq \mathcal{D}^\infty(\mathsf{S}_{\boldsymbol{\lambda}})$. Suppose that there exist a system $\{\mu_v\}_{v\in V}$ of Borel probability measures on $\mathbb{R}_+$ and a system $\{\varepsilon_v\}_{v\in V}$ of nonnegative real numbers that satisfy (1) for every $u \in V$. Then $\mathsf{S}_{\boldsymbol{\lambda}}$ is subnormal.*

3.4. **Composition operators with infinite matrix symbols, their dense definiteness, and boundedness.** As we showed above, some properties of composition operators with matrix symbols could be characterized entirely in terms of the matrices inducing the operators, which makes them very interesting. Substituting finite matrices by infinite ones as symbols inducing composition operators seems to be a natural idea. Below we show that this is doable and we present some recent results concerning dense definiteness and boundedness of composition operators induced with infinite matrices as symbols. The best reference for this subject is [13]. The idea for considering composition operators with infinite matrix symbols comes from [29] and [32].

We begin by setting the framework of our considerations, i.e., the measure space $(X, \mathcal{A}, \mu)$. The most natural choice is

$$X = \mathbb{R}^\infty, \quad \mathcal{A} = \mathfrak{B}(\mathbb{R}^\infty), \quad \mu = \mu_{\mathsf{G}},$$

where $\mathfrak{B}(\mathbb{R}^\infty)$ stands for the σ-algebra generated by cylinder sets, i.e., sets of the form $\sigma \times \mathbb{R}^\infty$ where σ is a Borel subset of $\mathbb{R}^k$ for some $k \in \mathbb{N}$, and μ_{G} is the gaussian measure on $\mathbb{R}^\infty$, i.e., the tensor product measure

$$\mu_{\mathrm{G}} = \mathrm{g}\mathrm{d}\mathrm{m}_1 \otimes \mathrm{g}\mathrm{d}\mathrm{m}_1 \otimes \ldots,$$

where

$$\mathrm{g}(x) = \frac{1}{\sqrt{2\pi}} \exp\big(-\frac{x^2}{2}\big), \quad x \in \mathbb{R}.$$

The advantage of choosing μ in this way is that we have

$$L^2(\mathbb{R}^\infty, \mathfrak{B}(\mathbb{R}^\infty), \mu_{\mathrm{G}}) = \mathrm{LIM}\, L^2(\mathbb{R}^n, \mathfrak{B}(\mathbb{R}^n), \mu_{\mathrm{G},n}),$$

where $\mu_{\mathrm{G},n}$ denotes the n-dimensional gaussian measure

$$\mathrm{d}\mu_{\mathrm{G},n} = \frac{1}{(\sqrt{2\pi})^n} \exp\big(-\frac{x_1^2 + \ldots + x_n^2}{2}\big)\mathrm{d}\mathrm{m}_n.$$

Let $(a_{ij})_{i,j\in\mathbb{N}}$ be a matrix with real entries. We say that a transformation $\mathbf{A}$ of $\mathbb{R}^\infty$ is induced by $(a_{ij})_{i,j\in\mathbb{N}}$ if the following condition holds

$$\mathbf{A}(x_1, x_2, \ldots) = \big(\sum_{j\in\mathbb{N}} a_{1j}\, x_j, \sum_{j\in\mathbb{N}} a_{2j}\, x_j, \ldots\big), \quad (x_1, x_2, \ldots) \in \mathbb{R}^\infty.$$

(We assume that all the series $\sum_{j\in\mathbb{N}} a_{kj}\, x_j$, $k \in \mathbb{N}$, are convergent.) It is easy to see that $\mathbf{A}$ is $\mathfrak{B}(\mathbb{R}^\infty)$-measurable, hence there arises a question of whether $\mathbf{A}$ induces a composition operator, and if so, whether the operator is densely defined or even bounded. Since $L^2(\mu_{\mathrm{G}})$ is the inductive limit of $\{L^2(\mu_{\mathrm{G},n})\}_{n\in\mathbb{N}}$ it is tempting to use inductive methods to address these problems. On the other hand, problems concerning infinite matrices are often solvable by finite section argument combined with appropriate approximation. This suggests considering composition operators $C_{\mathbf{A}_n}$ acting in $L^2(\mu_{\mathrm{G},n})$ and induced by transformations

$$\mathbf{A}_n(x_1, \ldots, x_n) = \Big(\sum_{j=1}^n a_{1j}\, x_j, \ldots, \sum_{j=1}^n a_{nj}\, x_j\Big), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

In view of Theorem 2.1, this makes no problem whenever every such transformation $\mathbf{A}_n$ is invertible. To ensure that $\mathbf{A}$ is a transformation defined on the whole of $\mathbb{R}^\infty$ and that inductive limit approximation is possible we may assume that all the rows in matrix $(a_{ij})_{i,j\in\mathbb{N}}$ are finite, i.e.,

for every $j \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that $a_{jk} = 0$ for every $k \geqslant K$.

This implies that all the series $\sum_{j\in\mathbb{N}} a_{kj}\, x_j$, $k \in \mathbb{N}$, are convergent and thus $\mathbf{A}$ is a well-defined transformation of $\mathbb{R}^\infty$. This also implies that the action of any $\mathbf{A}_n$ is closely related to the action of $\mathbf{A}$. Having all this assumed we can consider composition operators $C_{\mathbf{A}_n}$ acting in $L^2(\mu_{\mathrm{G},n})$ and their inductive limit $\mathrm{LIM}\, C_{\mathbf{A}_n}$. Now, the last thing in our approach is to determine conditions under which $C_{\mathbf{A}}$ is well defined, the domain of $\mathrm{LIM}\, C_{\mathbf{A}_n}$ is sufficiently big, and $C_{\mathbf{A}}$ and $\mathrm{LIM}\, C_{\mathbf{A}_n}$ are somehow related to each other. It turns out that assuming that all the Radon-Nikodym derivatives $\mathsf{h}_{\mathbf{A}_n}$ are uniformly in $L^{1+\varepsilon}(\mu_{\mathrm{G}})$ does the job. As a result we get the following criterion for the dense definiteness of composition operators with infinite matrix symbols. Below, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$ (for simplicity we do not make the dependence of $|\cdot|$ on n explicit).

Theorem 3.8 ([13, Corollary 5.1]). *Let $\mathbf{A}$ be a transformation of $\mathbb{R}^\infty$ induced by a matrix $(a_{ij})_{i,j\in\mathbb{N}}$. Let $\mathbf{A}_n$, $n \in \mathbb{N}$, be the linear transformation of $\mathbb{R}^n$ induced by the matrix $(a_{ij})_{i,j=1}^n$. If the following conditions are satisfied:*

(i) *for every $n \in \mathbb{N}$, $\mathbf{A}_n$ is invertible,*
(ii) *for every $j \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that $a_{jk} = 0$ for all $k \geqslant K$,*
(iii) *there exists $\varepsilon > 0$ such that*

$$\sup_{n\in\mathbb{N}} \Big\| |\det \mathbf{A}_n^{-1}| \exp \tfrac{1}{2}\big(|\cdot|^2 - |A_n^{-1}(\cdot)|^2\big)\Big\|^2_{L^{1+\varepsilon}(\mu_{\mathrm{G},n})} < \infty,$$

then C_A is densely defined operator in $L^2(\mu_{\mathrm{G}})$ and $C_A = \overline{\operatorname{lim} C_{A_n}|_{\mathscr{F}}}$, where $\mathscr{F}$ denotes the linear span of the set of characteristic functions of cylinder sets.

In a similar fashion, by approximating $C_{\mathbf{A}}$ again by composition operators $C_{\mathbf{A}_n}$ induced by finite sections of the matrix $(a_{ij})_{i,j\in\mathbb{N}}$, we may investigate the boundedness of $C_{\mathbf{A}}$. Assumptions have to be stronger but as a bonus we get nice description of $C_{\mathbf{A}}$ as a strong operator topology limit of tensor products of $C_{\mathbf{A}_n}$ and the identity operator. The criterion reads as follows.

Theorem 3.9 ([13, Corollary 5.5]). *Let $\mathbf{A}$ be a transformation of $\mathbb{R}^\infty$ induced by a matrix $(a_{ij})_{i,j\in\mathbb{N}}$. Let $\mathbf{A}_n$, $n \in \mathbb{N}$, be the linear transformation of $\mathbb{R}^n$ induced by the matrix $(a_{ij})_{i,j=1}^n$. If the following conditions are satisfied:*

(i) $\inf_{n\in\mathbb{N}} |\det \mathbf{A}_n| > 0$,
(ii) *for every $j \in \mathbb{N}$ there is $K \in \mathbb{N}$ such that $a_{jk} = 0$ for all $k \geqslant K$,*
(iii) $\sup_{n\in\mathbb{N}} \|\mathbf{A}_n\| \leqslant 1$,

then $C_{\mathbf{A}} \in \mathcal{B}(L^2(\mu_{\mathrm{G}}))$. Moreover, $C_{\mathbf{A}}$ is the limit in the strong operator topology of $\{C_{\mathbf{A}_n} \otimes I_n\}_{n\in\mathbb{N}}$, where I_n is the identity operator on $L^2(\mu_{\mathrm{G}})$.

A particular case of the above theorem, when $\mathbf{A}$ is induced by a diagonal matrix, was proved (by different methods) in [29, Theorem 3.1]; in turn, in [32, Theorem 4.1] it was shown that if $\mathbf{A}$ is diagonal and $C_{\mathbf{A}}$ is bounded, then $C_{\mathbf{A}}$ is cosubnormal. The latter result was generalized in [4, Theorem 5.1], again with help of inductive methods.

4. Subnormality of unbounded operators

We finish the paper with a selection of results concerning the subnormality of unbounded operators. For a comprehensive account on the subnormality of bounded operators we refer the reader to the monograph [15]. The subnormality of unbounded operators is treated in great detail in the papers [34, 35, 36].

Let A in be an operator in a Hilbert space $\mathcal{H}$. We say that A *generates Stieltjes moment sequences* if for every $f \in \mathcal{D}^\infty(A)$ there exists a positive Borel measure μ_f on $\mathbb{R}_+$ such that

$$\|A^n f\|^2 = \int_0^\infty t^n \mathrm{d}\mu_f, \quad n \in \mathbb{Z}_+.$$

The following theorem due to A. Lambert characterizes bounded subnormal operators in terms of Stieltjes moment sequences.

Theorem 4.1 ([27]). *Let A be a bounded operator on a Hilbert space $\mathcal{H}$. Then A is subnormal if and only if A generates Stieltjes moment sequences.*

Employing the spectral theorem it is fairly easy to show that unbounded subnormal operators also generates Stieltjes moment sequences.

Theorem 4.2 ([5, Proposition 3.2.1]). *If A is subnormal, then A generates Stieltjes moment sequences.*

However, the property of generating Stieltjes moment sequences and the subnormality are not equivalent. Indeed, since symmetric operators are subnormal, the following theorem due to M. Naimark shows that there are operators whose subnormality cannot be recovered from the property itself.

Theorem 4.3 ([30]). *There exist a symmetric operator A such that $\mathcal{D}(A^2) = \{0\}$.*

It is worth noting that examples of operators as in the Naimark's result are excluded from some notable classes of operators. These are for example weighted shifts on directed trees or composition operators. In both the mentioned classes symmetric operators are automatically self-adjoint and have a dense set of C^∞-vectors. Nevertheless, one still can found examples of operators in these particular classes showing that the property of generating Stieltjes moment sequences is by no means sufficient for the subnormality. The first such example was given in [2, Example 1] where subnormal and non-symmetric weighted shifts on directed trees and composition operators in L^2-spaces that have non-densely defined nth power (for any prescribed natural $n \geqslant 2$). Very recently even a more pathological example was invented.

Theorem 4.4 ([10, Theorem 3.1]). *There exist a subnormal non-symmetric operator A such that $\mathcal{D}(A^2) = \{0\}$.*

In view of the above theorems it makes sense to ask whether the property of generating Stieltjes moment sequences and density of C^∞-vectors implies subnormality. That the answer is negative was shown first in [22]. Let us recall here that subnormal operators are hyponormal.

Theorem 4.5 ([22, Example 4.2.1]). *There exist a non-hyponormal operator A which generates Stieltjes moment sequences and satisfies $\overline{\mathcal{D}^\infty(A)} = \mathcal{H}$.*

The examples provided in [22] were from the class of weighted shifts on directed trees and composition operators. Recently, the example was improved in [12], where (among other things) a non-hyponormal composition operator with a dense set of C^∞-vectors over a locally finite directed graph was constructed.

All the results presented above show that even in the case of operators belonging to reasonable classes of operators subnormality cannot be studied by means of Stieltjes moment sequences exclusively. This means that other methods should also be engaged. As shown in the previous section in case of composition operators with matrix symbols or weighted shifts on directed trees techniques relying on inductive limits might come in handy.

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