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Asymptotic profiles of solutions to the semilinear wave equation with time-dependent damping

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1 Introduction

In this note we consider the Cauchy problem of the semilinear wave equation with time-dependent damping

$$\begin{cases} u_{tt}-\Delta u+b(t)u_t=|u|^p, & t>0, x\in\mathbb{R}^n, \\ u(0,x)=\varepsilon u_0(x),\ u_t(0,x)=\varepsilon u_1(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.1}$$

Here $u=u(t,x)$ is a real-valued unknown function and the coefficient of the damping term $b=b(t)$ behaves as $b(t)\sim(1+t)^{-\beta}$ with some $\beta\in[-1,1)$.

Our aim is to prove that when $p>1+2/n$, there exists a unique global solution for small initial data and the asymptotic profile of the global solution is given by the scaled Gaussian.

As an introduction, we give a short survey about the study of the asymptotic behavior of solutions to the semilinear wave equation with time-dependent dissipation. In what follows, unless specifically mentioned, the initial data is sufficiently regular and rapidly decays at the infinity. The asymptotic behavior of solutions to the damped wave equation has been studied for a long time. It is well known that the solution of the wave equation with classical damping

$$u_{tt}-\Delta u+u_t=0 \tag{1.2}$$

is approximated by a constant multiple of the Gaussian as time tends to infinity. Here we shall give an intuitive observation about the asymptotic behavior of solutions via Fourier transform. For simplicity, we consider the initial data $(u,u_t)(0,x)=(0,g)(x)$. Applying the Fourier transform to (1.2), we have

$$\hat{u}_{tt}+|\xi|^2\hat{u}+\hat{u}_t=0,\quad (\hat{u},\hat{u}_t)(0,\xi)=(0,\hat{g})(\xi).$$

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Solving this ordinary differential equation, we easily deduce

$$\hat{u}(t,\xi) = \frac{1}{\sqrt{1-4|\xi|^2}} \left(e^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2})} - e^{-\frac{t}{2}(1+\sqrt{1-4|\xi|^2})} \right) \hat{g}(\xi).$$

When $|\xi|$ is sufficiently large, $\hat{u}(t,\xi)$ decays exponentially. On the other hand, when $|\xi|$ is sufficiently small, we observe

$$\frac{1-\sqrt{1-4|\xi|^2}}{2} \sim |\xi|^2$$

and hence,

$$\hat{u}(t,\xi) \sim e^{-t|\xi|^2}\hat{g}(\xi).$$

The right-had side is nothing but the Fourier transform of the solution of the heat equation

$$v_t - \Delta v = 0, \quad v(0,x) = g(x). \tag{1.3}$$

Therefore, we expect that the solution of the damped wave equation (1.2) behaves as that of (1.3). In fact, Matsumura [15] showed the estimates

$$\|\partial_t^i \partial_x^\alpha u(t)\|_{L^\infty} \le C(1+t)^{-\frac{n}{2m}-i-\frac{|\alpha|}{2}} \left(\|g\|_{L^m} + \|g\|_{H^{[n/2]+i+|\alpha|}} \right),$$
$$\|\partial_t^i \partial_x^\alpha u(t)\|_{L^2} \le C(1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-i-\frac{|\alpha|}{2}} \left(\|g\|_{L^m} + \|g\|_{H^{i+|\alpha|-1}} \right),$$

where $m \in [1,2]$. Here we remark that the decay rates above is the same as that of the heat equation (1.3):

$$\|\partial_t^i \partial_x^\alpha v(t)\|_{L^p} \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-i-\frac{|\alpha|}{2}} \|g\|_{L^q},$$

where $1 \le q \le p \le \infty$.

The precise asymptotic profile of dissipative hyperbolic equations is firstly studied by Hsiao and Liu [11]. They studied the hyperbolic conservation law with damping and the asymptotic profile of the solution is given by a solution of a system given by Darcy's law. After that, Nishihara [17] considered a quasilinear hyperbolic equation with linear damping and proved that the solution has the diffusion phenomena, that is, the solution approaches to that of the corresponding quasilinear parabolic equation (see also Yang and Milani [31] and Karch [12] for higher dimensional cases).

More precise information about the asymptotic behavior of solutions to (1.2) is given by [10, 14, 16, 18]. They proved the L^p-L^q estimate

$$\|u(t) - v(t) - e^{-t/2}w(t)\|_{L^p} \le Ct^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-1}\|g\|_{L^q},$$

where $1 \le q \le p \le \infty$. Here, when $n \le 3$, $w(t,x)$ is the solution of the free wave equation

$$w_{tt} - \Delta w = 0, \quad (w, w_t)(0,x) = (0,g)(x)$$

(when $n \geq 4$, w behaves like a solution of the free wave equation but does not coincides with it).

Next, we consider the semilinear wave equation with classical damping

$$u_{tt} - \Delta u + u_t = |u|^p, \quad (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x). \tag{1.4}$$

For the corresponding parabolic problem

$$v_t - \Delta v = |v|^p, \quad v(0, x) = \varepsilon v_0(x),$$

Fujita [5] discovered that $p = 1 + 2/n$ is the critical exponent, that is, if $p > 1 + 2/n$, then for any $v_0 \in L^1 \cap L^\infty$, there exists a unique global-in-time solution, provided that ε is sufficiently small; if $1 < p \leq 1 + 2/n$, then for any $\varepsilon > 0$, the local-in-time solution blows up in finite time, provided that $v_0 \geq 0$ and $v_0 \not\equiv 0$. In other words, the number $1 + 2/n$ is the threshold between the existence and nonexistence of global solutions for small initial data.

In view of the diffusion phenomena for the linear problem stated before, we expect that the semilinear damped wave equation (1.4) also has the same critical exponent $p = 1 + 2/n$. Indeed, Todorova and Yordanov [22] and Zhang [32] gave an affirmative answer. About the asymptotic profile of the global solution, Gallay and Raugel [6] considered one-dimensional case and proved the diffusion phenomena. After that, Hayashi, Kaikina and Naumkin [9] extended to higher dimensional cases.

One of the generalization of the diffusion phenomena is for wave equation with time-dependent dissipation

$$u_{tt} - \Delta u + b(t)u_t = 0, \quad (u, u_t)(0, x) = (u_0, u_1). \tag{1.5}$$

Wirth [25, 26, 27, 28, 29] studied the asymptotic behavior of solutions via the Fourier transform. For simplicity we assume that $b(t)$ is a positive, monotone function satisfying

$$\left| \frac{d^k}{dt^k} b(t) \right| \leq C_k (1 + t)^{-k} b(t)$$

for any nonnegative integer k. A typical example is $b(t) = (1 + t)^{-\beta}$ with $\beta \in \mathbb{R}$. We also put

$$\lambda(t) = \exp\left(\frac{1}{2} \int_0^t b(\tau) d\tau \right), \quad B(t) = \int_0^t \frac{d\tau}{b(\tau)}.$$

Wirth classified the asymptotic behavior of solutions by the strength of the damping in the following way:

- (scattering) If $b \in L^1(0, \infty)$, then the solution is asymptotically free. Namely, the solution approaches to that of the wave equation without damping in the energy sense.

- (non-effective dissipation) If $\limsup_{t\to\infty} tb(t) < 1$, then the solution satisfies the L^p-L^q estimate
$$\|(\nabla u, u_t)\|_{L^p} \le \frac{C}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\|u_0\|_{W^{s+1,q}} + \|u_1\|_{W^{s,q}}\right)$$
for $p \in [2,\infty)$, $q = p/(p-1)$ and $s > n(1/q - 1/p)$.

- (scale-invariant weak dissipation) If $b(t) = \mu/(1+t)$ with $\mu > 0$, then the solution u satisfies the L^p-L^q estimate
$$\|(\nabla u, u_t)\|_{L^p} \le C(1+t)^{\max\{-\frac{n-1}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{\mu}{2}, -n\left(\frac{1}{q}-\frac{1}{p}\right)-1\}} \left(\|u_0\|_{W^{s+1,q}} + \|u_1\|_{W^{s,q}}\right)$$
for $p \in [2,\infty)$, $q = p/(p-1)$ and $s > n(1/q - 1/p)$.

- (effective dissipation) If $tb(t) \to \infty$ as $t \to \infty$, then the solution satisfies the L^p-L^q estimate
$$\|(\nabla u, u_t)\|_{L^p} \le C(1+B(t))^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \left(\|u_0\|_{W^{s+1,q}} + \|u_1\|_{W^{s,q}}\right)$$
for $p \in [2,\infty)$, $q = p/(p-1)$ and $s > n(1/q - 1/p)$.

- (overdamping) If $b(t)^{-1}$, then the solution with $(u_0, u_1) \in L^2 \times H^{-1}$ converges to the asymptotic state
$$u_\infty(x) = \lim_{t\to\infty} u(t,x)$$
in L^2. Moreover, this limit is non-zero for non-trivial initial data. In particular, in general the L^2-norm of the solution does not decay to zero as time tends to infinity.

Moreover, for the damping term satisfying $b(t) \sim (1+t)^{-\beta}$ $(0 < \beta < 1)$, Yamazaki [30] studied the asymptotic profile of solutions to the abstract damped wave equation $u_{tt} + Au + b(t)u_t = 0$. As a corollary of her result, we have the diffusion phenomena for (1.5):
$$\|u(t) - v(t)\|_{H^1} \le C(1+t)^{\beta-1}(\|u_0\|_{H^1} + \|u_1\|_{L^2}),$$
where v is the solution of the corresponding heat equation
$$b(t)v_t - \Delta v = 0, \quad v(0,x) = u_0 + \frac{u_1}{b(0)} - u_1 \int_0^\infty \frac{b'(\tau)}{\lambda(\tau)^2 b(\tau)^2} d\tau.$$

Next, we consider the semilinear wave equation with time-dependent damping
$$u_{tt} - \Delta u + \mu(1+t)^{-\beta} u_t = N(u), \quad (u, u_t)(0,x) = \varepsilon(u_0, u_1)(x).$$
When $N(u) = |u|^p$, as in the case $\beta = 0$, there exists the critical exponent. Indeed, when $\beta \in (-1, 1)$, Nishihara [19] and Lin, Nishihara and Zhai [13] proved that the critical exponent is given by $1 + 2/n$. After that, D'Abbicco, Lucente and Reissig [3] (see also [2]) extended

to more general effective damping term and initial data. When $\beta = 1$, the situation becomes complicated. D'Abbicco [1] proved that if $\mu \geq n+2$ and $p > 1+2/n$, then there exists a unique global solution for small initial data. D'Abbicco, Lucente and Reissig [4] studied the special case $\mu = 2$ and proved that the critical exponent is $\max\{1+2/n, p_0(n+2)\}$ when $n \leq 3$, where $p_0(n)$ is the positive root of $(n-1)p^2 - (n+1)p - 2 = 0$, that is the Strauss critical exponent for the nonlinear wave equation. Furthermore, recently, Wakasa [23] obtained the optimal estimate of the lifespan of solutions in one dimensional case.

On the other hand, for the absorbing nonlinearity $N(u) = -|u|^{p-1}u$, when $\beta \in (-1,1)$, Nishihara and Zhai [21] obtained the global existence of solutions for any $1 < p < \frac{n+2}{[n-2]_+}$. Moreover, when $n = 1$ and $p > 3$, Nishihara [20] proved that the asymptotic profile of solutions is given by the scaled Gaussian. However, there are no results about the asymptotic profile for $n \geq 2$. In this note, we give the asymptotic profile of solutions in the case $N(u) = |u|^p$ with supercritical condition $p > 1+2/n$ and for small initial data. We also extend the results of [13] and [3] to more general initial data and our result includes the case $\beta = -1$.

2 Main result

First, we explain the notations used in the following. The letter C indicates a generic constant, which may change from line to line. For a function $\alpha = \alpha(s)$ defined on an interval in $\mathbb{R}$, we denote $\dot{\alpha}(s) = \alpha'(s) = \frac{d\alpha}{ds}(s)$. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote by $\hat{f}$ the Fourier transform of f, that is,

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

Also, $\mathcal{F}^{-1}$ stands for the inverse Fourier transform. Let L^p and $H^{k,m}$ be the Lebesgue space and the weighted Sobolev spaces, respectively, equipped with the norms defined by

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx\right)^{1/p} \quad (1 \leq p < \infty), \quad \|f\|_{L^\infty} = \operatorname{ess\,sup} |f(x)|,$$
$$\|f\|_{H^{k,m}} = \sum_{|\alpha| \leq k} \|(1+|x|)^m \partial_x^\alpha f\|_{L^2}.$$

We put the following assumptions on the damping term, the initial data and the nonlinearity. The coefficient of the damping term $b = b(t)$ is a smooth function satisfying

$$C^{-1}(1+t)^{-\beta} \leq b(t) \leq C(1+t)^{-\beta}, \quad |b'(t)| \leq C(1+t)^{-1} b(t) \tag{2.1}$$

with some $C > 0$ and $\beta \in [-1, 1)$. Next, the initial data (u_0, u_1) belong to $H^{1,m} \times H^{0,m}$ with $m = 1$ $(n = 1)$, $m > n/2 + 1$ $(n \geq 2)$. Finally, the exponent of the nonlinearity p satisfies

$$1 + \frac{2}{n} < p < \infty \ (n = 1, 2), \quad 1 + \frac{2}{n} < p \leq \frac{n}{n-2} \ (n \geq 3).$$

Let $G(t,x) = (4\pi t)^{-n/2}\exp\left(\frac{|x|^2}{t}\right)$ be the Gaussian and let $B(t) = \int_0^t b(\tau)^{-1}d\tau$. The main result of this note is the following:

Theorem 2.1 ([24]). *Under the assumptions stated above, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, the equation (1.1) admits a unique global solution*

$$u \in C([0,\infty); H^{1,m}(\mathbb{R}^n)) \cap C^1([0,\infty); H^{0,m}(\mathbb{R}^n)).$$

Moreover, the global solution u satisfies

$$\|u(t,\cdot) - \alpha^* G(1+B(t),\cdot)\|_{L^2} \le C(1+B(t))^{-n/4-\lambda/2+\kappa}\|(u_0,u_1)\|_{H^{1,m}\times H^{0,m}},$$

where $\alpha^* = \lim_{t\to\infty}\int_{\mathbb{R}^n} u(t,x)dx$, $\kappa > 0$ is an arbitrary small number and λ is

$$\lambda = \min\left\{1, m - \frac{n}{2}, \frac{2(1-\beta)}{1+\beta}, \frac{n}{2}\left(p - 1 - \frac{2}{n}\right)\right\} \tag{2.2}$$

(if $\beta = -1$, the term $\frac{2(1-\beta)}{1+\beta}$ is omitted from the minimum).

Remark 2.1. *The number α^* can be explicitly written. For example, when $b = (1+t)^{-\beta}$, we have*

$$\alpha^* = \int_{\mathbb{R}^n}(u_0 + u_1)\,dx + \beta(1-\beta)\int_0^\infty (1+t)^{-(2-\beta)}\int_{\mathbb{R}^n} u\,dxdt + \int_0^\infty\int_{\mathbb{R}^n}|u|^p\,dxdt$$

(see Section 1 of [20] for general cases).

3 Idea of the proof: scaling variables and fractional integrals

The purpose of this section is to explain the idea of the proof of our main theorem. The proof is based on the method of Gallay and Raugel [6], in which the one-dimensional semilinear wave equation with classical damping is considered. To generalize their method to higher dimensional cases, we use the fractional derivative of the form $F(s,y) = \mathcal{F}^{-1}[|\xi|^{-n/2-\delta}\hat{f}(s,\cdot)](y)$. In order to explain this idea, for simplicity, we consider the linear heat equation

$$\begin{cases} u_t - \Delta u = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

Following [6], we apply the scaling variables (self-similar transformation)

$$s = \log(1+t), \quad y = \frac{x}{\sqrt{1+t}}$$

and we put

$$u(t,x) = (1+t)^{-n/2} v\left(\log(1+t), \frac{x}{\sqrt{1+t}}\right).$$

Then, the new unknown function $v = v(s,y)$ satisfies

$$v_s - \frac{y}{2} \cdot \nabla_y v - \frac{n}{2} v = \Delta_y v. \tag{3.2}$$

Next, we decompose v as

$$v(s,y) = \alpha\varphi_0(y) + f(s,y),$$

where

$$\alpha = \int_{\mathbb{R}^n} v(s,y)dy, \quad \varphi_0(y) = (4\pi)^{-n/2} \exp\left(-\frac{|y|^2}{4}\right).$$

We note that α is independent of s and φ_0 satisfies $\int_{\mathbb{R}^n} \varphi_0(y)dy = 1$ and $\Delta\varphi_0 = -\frac{y}{2}\cdot\nabla\varphi_0 - \frac{n}{2}\varphi_0$. We prove that the asymptotic profile of v is given by $\alpha\varphi_0$. To this end, it suffices to show that f decays to zero as time goes to infinity. We easily see that f satisfies the equation

$$f_s - \frac{y}{2} \cdot \nabla_y f - \frac{n}{2} f = \Delta_y f$$

and $\int_{\mathbb{R}^n} f(s,y)\,dy = 0$. We define

$$F(s,y) = \begin{cases} \displaystyle\int_{-\infty}^{y} f(s,z)\,dz & (n=1), \\ \mathcal{F}^{-1}\left[|\xi|^{-n/2-\delta}\hat{f}(s,\cdot)\right](y) & (n \geq 2), \end{cases}$$

where $0 < \delta < 1$. From the following Hardy-type inequality, we note that F makes sense as an L^2-function

Lemma 3.1. *We have* $\|F\|_{L^2} \leq C\|f\|_{H^{0,m}}$ *if* $m = 1$ $(n=1)$, $m > n/2+1$ $(n \geq 2)$.

Proof. The case $n = 1$ is proved in [8, Section 9.9] and we omit the proof. When $n \geq 2$, from $\hat{f}(s,0) = \int_{\mathbb{R}^n} f(s,y)\,dy = 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\hat{F}(s,\xi)|^2\,d\xi &= \int_{\mathbb{R}^n} |\xi|^{-n-2\delta}|\hat{f}(s,\xi)|^2\,d\xi \\ &\leq \|\nabla_\xi \hat{f}(s)\|_{L^\infty}^2 \int_{|\xi|\leq 1} |\xi|^{2-n-2\delta}\,d\xi + \|\hat{f}(s)\|_{L^2(|\xi|\geq 1)}^2.\end{aligned}$$

This and

$$\|\nabla_\xi \hat{f}(s)\|_{L^\infty} \leq C\||y|f\|_{L^1} \leq C\|f\|_{H^{0,m}}$$

with $m > n/2+1$ imply the conclusion. □

The following interpolation inequality enables us to control the bad term $\|f\|_{L^2}$ appearing in the energy estimate.

Lemma 3.2. *For any $\eta > 0$, there exists a constant $C > 0$ such that*

$$\|f\|_{L^2} \le \eta\|\nabla f\|_{L^2} + C\|\nabla F\|_{L^2}.$$

This lemma is easily proved by decompose the integral region in the Fourier space and we omit the detail.

In what follows, we consider the case $n \ge 2$. By the definition of F, we have the equation of $\hat{F}$:

$$\hat{F}_s + \frac{\xi}{2} \cdot \nabla_\xi \hat{F} + \frac{1}{2}\left(\frac{n}{2} + \delta\right)\hat{F} = -|\xi|^2 \hat{F}.$$

By the energy method, we prove that $\|f(s)\|_{L^2}$ decays to zero. First, we obtain

Lemma 3.3. *We have*

$$\frac{d}{ds}\left[\frac{1}{2}\int_{\mathbb{R}^n} |F(s,y)|^2\, dy\right] = -\frac{\delta}{2}\int_{\mathbb{R}^n} |F(s,y)|^2\, dy - \int_{\mathbb{R}^n} |\nabla_y F(s,y)|^2\, dy.$$

Proof. We calculate

$$\begin{aligned}
\frac{d}{ds}\left[\frac{1}{2}\int_{\mathbb{R}^n} |\hat{F}(s,\xi)|^2\, d\xi\right] &= \mathrm{Re}\int_{\mathbb{R}^n} \hat{F}_s \overline{\hat{F}}\, d\xi \\
&= \mathrm{Re}\int_{\mathbb{R}^n} \left(-\frac{\xi}{2}\cdot\nabla_\xi \hat{F} - \frac{1}{2}\left(\frac{n}{2}+\delta\right)\hat{F} - |\xi|^2 \hat{F}\right)\overline{\hat{F}}\, d\xi \\
&= -\frac{\delta}{2}\int_{\mathbb{R}^n} |\hat{F}(s,\xi)|^2\, d\xi - \int_{\mathbb{R}^n} |\xi|^2 |\hat{F}(s,\xi)|^2\, d\xi.
\end{aligned}$$

Thus, the Plancherel theorem completes the proof. □

Similarly, we can obtain the following:

Lemma 3.4. *We have*

$$\frac{d}{ds}\left[\frac{1}{2}\int_{\mathbb{R}^n} |f|^2\, dy\right] = \frac{n}{4}\int_{\mathbb{R}^n} |f|^2\, dy - \int_{\mathbb{R}^n} |\nabla_y f|^2\, dy.$$

Lemmas 3.2 and 3.4 imply

$$\frac{d}{ds}\left[\frac{1}{2}\int_{\mathbb{R}^n} |f|^2\, dy\right] \le -\frac{\delta}{2}\int_{\mathbb{R}^n} |f|^2\, dy + C\int_{\mathbb{R}^n} |\nabla_y F|^2\, dy.$$

From this, Lemma 3.3 and taking sufficiently large $C_0 > 0$, we obtain

$$\frac{d}{ds}\left[\int_{\mathbb{R}^n} (C_0|F|^2 + |f|^2)\, dy\right] \le -\delta\int_{\mathbb{R}^n} (C_0|F|^2 + |f|^2)\, dy.$$

We multiply this by $e^{\delta s}$ to obtain

$$\frac{d}{ds}\left[e^{\delta s}\int_{\mathbb{R}^n}(C_0|F|^2+|f|^2)\,dy\right]\le 0.$$

Integrating over $[0,s]$, we have

$$\int_{\mathbb{R}^n}(C_0|F(s,y)|^2+|f(s,y)|^2)\,dy \le e^{-\delta s}\int_{\mathbb{R}^n}(C_0|F(0,y)|^2+|f(0,y)|^2)\,dy.$$

From Lemma 3.1 and that $\alpha \le \|u_0\|_{H^{0,m}}$, we deduce that the right-hand side of the above inequality is estimated by $Ce^{-\delta s}\|u_0\|_{H^{0,m}}$. Finally, by rewriting f by $v-\alpha\varphi_0$, we have

$$\|v(s)-\alpha\varphi_0\|_{L^2}\le Ce^{-\delta s/2}\|u_0\|_{H^{0,m}}.$$

Changing variables leads to

$$\|u(t)-\alpha G(1+t)\|_{L^2}\le C(1+t)^{-n/4-\delta/2}\|u_0\|_{H^{0,m}}.$$

Therefore, the asymptotic profile of u is a constant multiple of the Gaussian.

4 Outline of the proof of Theorem 2.1

In this section, we turn back to our problem

$$\begin{cases} u_{tt}-\Delta u+b(t)u_t=|u|^p, & t>0, x\in\mathbb{R}^n,\\ u(0,x)=\varepsilon u_0(x),\ u_t(0,x)=\varepsilon u_1(x), & x\in\mathbb{R}^n \end{cases}$$

and give an outline of the proof of Theorem 2.1. For simplicity, we consider only the case $n\ge 2$ and $\beta\in(-1,1)$, because the case $\beta=-1$ is treated in the same way. Similarly to the previous section, we apply the scaling variables with the scaling function $B(t)$:

$$s=\log(1+B(t)),\quad y=(1+B(t))^{-1/2}x.$$

We also put

$$\begin{aligned} u(t,x)&=(1+B(t))^{-n/2}v\left(\log(1+B(t)),(1+B(t))^{-1/2}x\right),\\ u_t(t,x)&=b(t)^{-1}(1+B(t))^{-n/2-1}w\left(\log(1+B(t)),(1+B(t))^{-1/2}x\right). \end{aligned}$$

Then, we obtain the first order system

$$\begin{aligned} v_s-\frac{y}{2}\cdot\nabla_y v-\frac{n}{2}v&=w,\\ \frac{e^{-s}}{b(t)^2}\left(w_s-\frac{y}{2}\cdot\nabla_y w-\frac{n+2}{2}w\right)+w&=\Delta_y v+\frac{b'(t)}{b(t)^2}w+e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s}|v|^p. \end{aligned}$$

By a standard argument, we can prove that the above system admits a unique solutions $(v, w) \in C([0, S); H^{1,m} \times H^{0,m})$ with some $S > 0$. Therefore, to obtain the global existence of solutions, it suffices to show an a priori estimate of the solution. To obtain an a priori estimate, we may assume that $\|(v, w)\|_{H^{1,m}\times H^{0,m}} \leq 1$.

Since b satisfies (2.1) and $B(t) \sim (1+t)^{-\frac{1}{1+\beta}}$, we have

$$\frac{e^{-s}}{b(t)^2} \leq Ce^{-\frac{1-\beta}{1+\beta}s}, \quad \frac{|b'(t)|}{b(t)^2} \leq C(1+t)^{-1+\beta} \leq Ce^{-\frac{1-\beta}{1+\beta}s}.$$

Also, the supercritical condition $p > 1+2/n$ implies that the nonlinear term $e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s}|v|^p$ decays exponentially. Therefore, letting $s \to \infty$ formally, we obtain the equation (3.2) as the limiting equation of the above system. Hence, we expect that the asymptotic behavior of the solution is determined from the equation (3.2). In view of this, we decompose the solutions v, w as

$$v(s,y) = \alpha(s)\varphi_0(y) + f(s,y),$$
$$w(s,y) = \dot{\alpha}(s)\varphi_0(y) + \alpha(s)\Delta_y\varphi_0(y) + g(s,y),$$

where $\alpha(s) = \int_{\mathbb{R}^n} v(s,y)\,dy$ and $\varphi_0(y) = (4\pi)^{-n/2}\exp(-\frac{|y|^2}{4})$. Note that in this case $\alpha(s)$ depends on s. By the above system, we easily obtain

$$\dot{\alpha}(s) = \int_{\mathbb{R}^n} w(s,y)\,dy,$$
$$\frac{e^{-s}}{b(t)^2}\ddot{\alpha}(s) = \frac{e^{-s}}{b(t)^2}\dot{\alpha}(s) - \dot{\alpha}(s) + \frac{b'(t)}{b(t)^2}\dot{\alpha}(s) + e^{-\frac{n}{2}(p-(1+\frac{2}{n}))}\int_{\mathbb{R}^n}|v|^p\,dy.$$

A straightforward calculation shows that f and g satisfy the first order system

$$\begin{cases} f_s - \dfrac{y}{2}\cdot\nabla_y f - \dfrac{n}{2}f = g, \\ \dfrac{e^{-s}}{b(t)^2}\left(g_s - \dfrac{y}{2}\cdot\nabla g - \dfrac{n+2}{2}g\right) + g = \Delta_y f + h, \end{cases}$$

where

$$h(s,y) = \frac{e^{-s}}{b(t)^2}\left(-2\dot{\alpha}(s)\Delta_y\varphi_0(y) + \alpha(s)\left(\frac{y}{2}\cdot\nabla_y\Delta\varphi_0(y) + \frac{n+2}{2}\Delta_y\varphi_0(y)\right)\right)$$
$$+\frac{b'(t)}{b(t)^2}w + e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s}|v|^p + \left(\int_{\mathbb{R}^n}\frac{b'(t)}{b(t)^2}w + e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s}|v|^p\,dy\right)\varphi_0(y).$$

By the definition of f and g, we easily see that $\int_{\mathbb{R}^n} f(s,y)\,dy = \int_{\mathbb{R}^n} g(s,y)\,dy = 0$. This and the above system also imply that $\int_{\mathbb{R}^n} h(s,y)\,dy = 0$. Using this property, we prove that f, g decay to zero as time goes to infinity. To this end, as in the previous section, we define

$$F(s,y) = \mathcal{F}^{-1}[|\xi|^{-n/2-\delta}\hat{f}(s,\cdot)](y), \quad G(s,y) = \mathcal{F}^{-1}[|\xi|^{-n/2-\delta}\hat{g}(s,\cdot)](y)$$

and $H(s,y) = \mathcal{F}^{-1}[|\xi|^{-n/2-\delta}\hat{h}(s,\cdot)](y)$. We also define the following three energies:

$$E_0(s) = \int_{\mathbb{R}^n} \frac{1}{2}\left(|\nabla F|^2 + \frac{e^{-s}}{b(t)^2}G^2\right) + \frac{1}{2}F^2 + \frac{e^{-s}}{b(t)^2}FG\,dy,$$
$$E_1(s) = \int_{\mathbb{R}^n} \frac{1}{2}\left(|\nabla f|^2 + \frac{e^{-s}}{b(t)^2}g^2\right) + \frac{n+4}{4}\left(\frac{1}{2}f^2 + \frac{e^{-s}}{b(t)^2}fg\right)dy,$$
$$E_2(s) = \int_{\mathbb{R}^n} |y|^{2m}\left[\frac{1}{2}\left(|\nabla f|^2 + \frac{e^{-s}}{b(t)^2}g^2\right) + \frac{1}{2}f^2 + \frac{e^{-s}}{b(t)^2}fg\right]dy.$$

Then, as in the previous section, we can obtain the following energy estimates for the above energies.

Lemma 4.1. *We have*

$$\frac{d}{ds}E_0(s) + \delta E_0(s) + L_0 = R_0,$$

where

$$L_0(s) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla F|^2\,dy + \int_{\mathbb{R}^n} |G|^2\,dy,$$
$$R_0(s) = \frac{3}{2}\frac{e^{-s}}{b(t)^2}\int_{\mathbb{R}^n} |G|^2\,dy - \frac{b'(t)}{b(t)^2}\int_{\mathbb{R}^n} (2F+G)G\,dy + \int_{\mathbb{R}^n} (F+G)H\,dy.$$

Lemma 4.2. *We have*

$$\frac{d}{ds}E_1(s) + \delta E_1(s) + L_1(s) = R_1(s),$$

where

$$L_1(s) = \frac{1-\delta}{2}\int_{\mathbb{R}^n} |\nabla f|^2\,dy + \int_{\mathbb{R}^n} |g|^2\,dy - \frac{n+4}{4}\left(\frac{n}{4}+\frac{\delta}{2}\right)\int_{\mathbb{R}^n} |f|^2\,dy,$$
$$R_1(s) = \frac{n+4}{4}\left(\frac{n}{2}+\delta\right)\frac{e^{-s}}{b_0(t)^2}\int_{\mathbb{R}^n} fg\,dy + \frac{n+3+\delta}{2}\frac{e^{-s}}{b_0(t)^2}\int_{\mathbb{R}^n} g^2\,dy$$
$$- \frac{b_0'(t)}{b_0(t)^2}\int_{\mathbb{R}^n}\left(\frac{n+4}{2}f+g\right)g\,dy + \int_{\mathbb{R}^n}\left(\frac{n+4}{4}f+g\right)h\,dy.$$

Lemma 4.3. *Let $m > n/2$. Then, for any $\kappa \in (0, m-n/2)$, we have*

$$\frac{d}{ds}E_2(s) + \left(m - \frac{n}{2} - \kappa\right)E_2(s) + L_2(s) = R_2(s),$$

where

$$L_2(s) = \frac{\kappa}{2}\int_{\mathbb{R}^n}|y|^{2m}f^2dy + \frac{\kappa+1}{2}\int_{\mathbb{R}^n}|y|^{2m}|\nabla_y f|^2dy + \int_{\mathbb{R}^n}|y|^{2m}g^2dy$$
$$+ 2m\int_{\mathbb{R}^n}|y|^{2m-2}(y\cdot\nabla_y f)(f+g)dy,$$
$$R_2(s) = -\kappa\frac{e^{-s}}{b_0(t)^2}\int_{\mathbb{R}^n}|y|^{2m}fgdy - \frac{\kappa+1}{2}\frac{e^{-s}}{b_0(t)^2}\int_{\mathbb{R}^n}|y|^{2m}g^2dy$$
$$- \frac{b_0'(t)}{b_0(t)^2}\int_{\mathbb{R}^n}|y|^{2m}(2f+g)gdy + \int_{\mathbb{R}^n}|y|^{2m}(f+g)hdy.$$

Let $\kappa > 0$ be an arbitrary number. We define

$$E_3(s) = C_0E_0(s) + C_1E_1(s) + E_2(s) + \frac{e^{-s}}{2b(t)^2}\dot{\alpha}(s)^2 + e^{-(\lambda-\kappa)s}\alpha(s)^2,$$

where λ is defined as (2.2) and C_0, C_1 are chosen so that $C_0 \gg C_1 \gg 1$. Taking δ so that $\lambda - \kappa < \delta$, we have the following.

Lemma 4.4. *We have*

$$\frac{d}{ds}E_3(s) + (\lambda-\kappa)E_3(s) + L_3(s) = R_3(s),$$

where

$$L_3(s) = (\delta - \lambda + \kappa)(C_0E_0(s) + C_1E_1(s)) + \left(m - \frac{n}{2} - \lambda\right)E_2(s)$$
$$+ C_0L_0(s) + C_1L_1(s) + L_2(s) + \dot{\alpha}(s)^2,$$
$$R_3(s) = C_0R_0(s) + C_1R_1(s) + R_2(s)$$
$$+ \frac{\lambda-\kappa+1}{2}\frac{e^{-s}}{b(t)^2}\dot{\alpha}(s)^2 + 2e^{-(\lambda-\kappa)s}\alpha(s)\dot{\alpha}(s) + e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s}\left(\int_{\mathbb{R}^n}|v|^p\,dy\right)\dot{\alpha}(s).$$

Finally, we define

$$E_4(s) = E_3(s) + \frac{1}{2}\alpha(s)^2 + \frac{e^{-s}}{b(t)^2}\alpha(s)\dot{\alpha}(s).$$

Then, we have the following estimate.

Lemma 4.5. *We have*

$$\frac{d}{ds}E_4(s) + (\lambda-\kappa)E_4(s) + L_3(s) = R_4(s),$$

where

$$R_4(s) = R_3(s) + \frac{e^{-s}}{b(t)^2}\dot{\alpha}(s)^2 - \frac{b'(t)}{b(t)^2}\alpha(s)\dot{\alpha}(s) + e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s}\left(\int_{\mathbb{R}^n}|v|^p\,dy\right)\alpha(s).$$

The remainder terms R_3, R_4 are estimated as

$$R_3, R_4 \leq \frac{1}{2} L_3(s) + Ce^{-\lambda s} E_4(s)$$

for sufficiently large $s > 0$. In fact, for example, the term $e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s} \left(\int_{\mathbb{R}^n} |v|^p \, dy \right) \alpha(s)$, which is in R_4, is estimated as

$$\begin{aligned} e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s} \left(\int_{\mathbb{R}^n} |v|^p \, dy \right) \alpha(s) &\leq Ce^{-\frac{n}{2}(p-(1+\frac{2}{n}))s} \left(\int_{\mathbb{R}^n} (1+|y|)^{2m} |v|^{2p} \, dy \right)^{1/2} |\alpha(s)| \\ &\leq Ce^{-\frac{n}{2}(p-(1+\frac{2}{n}))s} \|v\|_{H^{1,m}}^p |\alpha(s)|. \end{aligned}$$

Here we have used the Gagliardo-Nirenberg inequality (see [7, Section 6.6.1]) and we remark that to apply this inequality, we need the restriction $p \leq n/(n-2)$ when $n \geq 3$. Noting that we assume $\|v\|_{H^{1,m}} \leq 1$, we have $\|v\|_{H^{1,m}}^p \leq \|v\|_{H^{1,m}} \leq \|f\|_{H^{1,m}} + |\alpha|$ and hence, the right-hand side of the above inequality is bounded by

$$e^{-\frac{n}{2}(p-(1+\frac{2}{n}))s} (\|f\|_{H^{1,m}} + |\alpha(s)|) |\alpha(s)| \leq Ce^{-\lambda s} E_4(s)$$

and we obtain the desired estimate. The other terms can be estimated in a similar way.

Therefore, we have $\frac{d}{ds} E_4(s) \leq Ce^{-\lambda s} E_4(s)$ and hence, $E_4(s) \leq CE_4(s_0)$ with sufficiently large $s_0 > 0$ and $s \geq s_0$. This a priori estimate shows the existence of the global solution, provided that the amplitude of the initial data ε is sufficiently small. This a priori estimate also implies

$$\frac{d}{ds} E_3(s) + (\lambda - \kappa) E_3(s) + \frac{1}{2} L_3(s) \leq Ce^{-\lambda s} E_4(s_0).$$

Multiplying both sides by $e^{(\lambda-\kappa)s}$ and integrating over $[s_0, s]$, we have

$$e^{(\lambda-\kappa)s} E_3(s) + \frac{1}{2} \int_{s_0}^{s} e^{(\lambda-\kappa)\sigma} L_3(\sigma) \, d\sigma \leq e^{(\lambda-\kappa)s_0} E_3(s_0) + CE_4(s_0).$$

This and $L_3(s) \geq \dot{\alpha}(s)^2$ imply that $\alpha^* = \lim_{s \to \infty} \alpha(s)$ exists and $E_3(s) \leq Ce^{-(\lambda-\kappa)s} E_4(s_0)$. In particular, we obtain

$$\|v(s) - \alpha^* \varphi_0\|_{L^2}^2 \leq Ce^{-(\lambda-\kappa)s} (\|v(0)\|_{H^{1,m}}^2 + \|w(0)\|_{H^{0,m}}^2).$$

From this, we reach the conclusion.

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