数理解析研究所講究録
第2002巻 2016年 1-3

The partite construction with forbidden structures

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Abstract

This is an announcement of a note explaining the partite construction given by Nešetřil and Rödl which implies the Ramsey property of any class $\mathrm{Forb}(K)_<$ where K is a set of irreducible structures.

1 Introduction

Let L be a finite relational language and K be a set of irreducible L-structures. In [2], Nešetřil and Rödl proved that $\mathrm{Forb}(K)$ has the Ramsey property by using so called "partite construction". More precisely, Nešetřil and Rödl's theorem is as the following:

Theorem 1. Let K be a set of irreducible L-structures. Then for any $t \in \omega$, $A, B \in \mathrm{Forb}(K)_<$, there is $C \in \mathrm{Forb}(K)_<$ such that $C \to (B)^A_t$.

The present article is an announcement of a note for non-specialist on this field, in which the author explain the partite construction given in [2] in detail. Hence, we see only important definitions and propositions without any proof in the present paper. (Similar proofs for special cases of the theorem are also found in [1] and [3].) Note that the proof may have a little difference from the original proof given by Nešetřil and Rödl for the simplicity and to clarify the detail, however, there is no new method or idea in the article. Everything is already given in the previous studies in this field.

2 Preliminaries

An L-structure A is said to be irreducible if any $a \neq b \in A$ there is $\bar{e} \ni a, b$ and $R \in L$ such that $\bar{e} \in R(A)$. For a set K of L-structures, $\mathrm{Forb}(K)$ is the set of finite L-structures containing no

$A \in K$ as a substructure. $\mathrm{Forb}(K)_< = \{(A, <) : A \in \mathrm{Forb}(K), <$ is a total order on $A\}$.

To explain the partite construction, we first recall n-partite structures. Put $L_n = L \cup \{P_0(x), ..., P_{n-1}(x)\}$. An L_n-structure A is said to be an n-partite L-structure if $A = \bigsqcup_{i<n} P_i(A)$, $P_i(A) \neq$ for $i < n$ and for any $(e_0, ..., e_{k-1}) \in R(A)$ with $R \in L$, $|\{e_j : j < n\} \cap P_i(A)| \leq 1$ for $i < n$. (In other words, each "edge" $\bar{e}$ intersects with each part $P_i(A)$ at most one point.) An n-partite structure A is sad to be transversal if $|P_i(A)| = 1$ for any $i < n$. For a subset A_0 of an n-partite structure A, we say A_0 is a partial section if $P_i(A_0) \leq 1$. For n-partite structures B and A, a projection map $p : B \to A$ is defined as $p(P_i(B)) = a_i$ where A is transversal with $P_i(A) = \{a_i\}$.

3 The partite construction

Let X and Y are n-partite L-structures and X transversal and let $p : Y \to X$ be a projection map.

Definition 2. We say $Y_0 \subset Y$ is a strong lifting of $X_0 \subset X$ if $p(Y_0) = X_0$ and if for any irreducible partial section $A \subset Y$, $p(A) \cong A$.

Definition 3. For given $N \in \omega$ and an n-partite structure Y, an n-partite product Y^N is an n-partite structure Z such that

- $P_i(Z) = (P_i(Y))^N$ (the Cartesian product of $P_i(Y)$),
- $((y_i^0)_{i<N}, ..., (y_i^{k-1})_{i<N}) \in R(Z)$ if and only if $(y_i^0, ..., y_i^{k-1}) \in R(Y)$ for every $i < n$.

Proposition 4. Let X be an n-partite L-structure, Y a strong lifting of X, and let Y^N be an n-partite product of Y. Suppose $X|L \in \mathrm{Forb}(K)$.

1. Y^N is a strong lifting of X.
2. $Y^N|L \in \mathrm{Forb}(K)$.

Lemma 5 (Lifting lemma)**.** Let Y be a strong lifting of X and $t \in \omega$. Then there is $N \in \omega$ such that $Y^N \to (Y)_t^X$.

To prove the theorem, we first prove it in a special case:

Proposition 6. $\mathrm{Forb}(\emptyset)_<$, the set of all totally ordered finite L-structures, has the Ramsey property.

Let $A, B \in \mathrm{Forb}(K)_<$ and let $C \in \mathrm{Forb}(\emptyset)_<$ such that $C \to (B)^A_t$. We also consider C as a transversal $|C|$-partite L-structure. Then, roughly speaking, we can construct an $|C|$-partite L-structure $C_0, C_1, ...$ with a projection $p_i : C_i \to C$ such that for any embedding $\nu : A \to C$ there is i such that $p_{i+1}^{-1}(\nu(A)) \to (p_i^{-1}(\nu(A)))^A_t$. With this construction, we'll get

Theorem 7. If K is the set of irreducible L-structures, then $\mathrm{Forb}(K)_<$ has the Ramsey property.

4 Acknowledgement

I am deeply grateful Sławomir Solecki and Lionel Nguyen Van Thé. They gave me helpful advices and suggestions on this topic.

References

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